LOOP, THRESHOLD, AND RESIDUE: THE ALGEBRAIC STRUCTURE OF THE $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$ TRANSITION

GIOVANNI LEVRATTI

Abstract. Three interconnected consequences of the Cayley–Dickson algebraic threshold established in Papers 1–9 are developed.

Objective (i) (pre-dynamical structure of the $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$ loop): The feedback loop $\mathcal{A}_3 \to \mathcal{A}_4 \to \mathcal{A}_3$ is formalised as a conditional discrete dynamical system. The algebraic skeleton — the tower map $T\colon \varrho(n) \mapsto \varrho(n+1)$ — is derived unconditionally. It has no fixed point; its unique asymptote is $\varrho = 0$; its aeon-to-aeon rate is $(2^n - 1)/[(2^{n-3} - 1) \cdot 16]$, converging to $1/2$ from above. The cosmological 5%/95% ratio is shown to be *not derivable* — a precise negative result. A conditional conjecture characterises exactly what a future QFT over $\mathcal{A}_4$ must supply to close this gap.

Objective (ii) (Black Circle metric ansatz): The Schwarzschild radius r_s is identified as the $\mathcal{A}_3$ projection of the $\mathcal{A}_3 \to \mathcal{A}_4$ algebraic transition. A metric ansatz is constructed: exact Schwarzschild for $r \geq r_s$, de Sitter interior for $r < r_s$ with de Sitter radius $L = r_s$. The Kretschmann scalar at the origin is $K(0) = 6\Delta N / r_s^4 = 24/r_s^4$, finite and algebraically fixed by the norm-gap $\Delta N = 4$. The first-derivative jump at r_s is derived from the Frobenius identity [4].

Objective (iii) (sedeonic residue between aeons): The residual algebra level of aeon k is defined algebraically as $n_{\mathrm{res}}(k) = 3{+}k$. The ZD content satisfies $|ZD_{\mathrm{dark}}(k)| = |ZD(3{+}k)|$, proved from the Fano filtration theorem. The sign-parity constraint $\pi_A \pi_B \pi_C = -1$ is inherited by the residue across every CCC crossing. The first-aeon problem of Penrose CCC [7] does not arise.

All three objectives are governed by the single map $T\colon n \mapsto n{+}1$, and by the algebraic chain $S_4 \subset \mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2 \subset SO(7) \leftarrow \mathrm{Cliff}(0,7)$, verified by `GAP` 4.15.1 (proved in this paper, §7). $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ is already present in $\mathcal{A}_3$ as a finite subgroup of $G_2 = \mathrm{Aut}(\mathcal{A}_3)$, with exactly two conjugacy classes; the $\mathcal{A}_3 \to \mathcal{A}_4$ transition breaks the continuous G_2 symmetry while preserving its discrete subgroup. A unifying result (§6) shows that N is the unique algebraic structure satisfying simultaneously: universal persistence through all doublings, gravitational coupling at every level, and coincidence with the Born Rule probability measure exactly at $n = 3$. At the IIB threshold these two roles split: gravity persists into $\mathcal{A}_4$; the Born Rule does not. $\mathcal{A}_3$ is the unique level of the tower where the algebra can measure itself.

Epistemic key. *[Derived]* — follows from Papers 1–9, no additional assumptions. *[Conjecture]* — precise statement; additional physical assumptions stated in full. *[Speculation]* — structurally motivated; requires QFT over $\mathcal{A}_4$.

1. Introduction

1.1. **The three objectives.** Papers 1–9 have established a precise algebraic picture of the Cayley–Dickson tower. The threshold $\mathcal{A}_3 \to \mathcal{A}_4$ is characterised from three independent directions: the Born Rule fails for $n \geq 4$ [1]; the norm loses multiplicativity [3]; the unique quartic contact interaction appears with a single coupling constant [6]. Paper 2 develops physical conjectures from this threshold: IIB-I through IIB-IV [2].

Three questions emerge naturally from this foundation.

Objective (i). The $\mathcal{A}_4$ sector is proposed to curve spacetime (IIB-II), which in turn affects the distribution of $\mathcal{A}_3$ matter — creating a feedback loop $\mathcal{A}_3 \to \mathcal{A}_4 \to \mathcal{A}_3$. Can this loop be formalised

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as a discrete dynamical system? Does it have a fixed point? Is the cosmological 5%/95% ratio derivable?

Objective (ii). If r_s is the $\mathcal{A}_3$ projection of the algebraic transition, what replaces the classical singularity inside? Can a metric ansatz be written connecting Schwarzschild (outside r_s) to $\mathcal{A}_4$ structure (inside r_s) using $\Delta N = 4$ as the regulator?

Objective (iii). What survives between aeons in the CCC tower? Can the residual algebra level n_{res} be defined precisely, and what is the ZD content of the dark sector it carries?

1.2. The common thread.

All three objectives are governed by the *tower map* $T: n \mapsto n+1$. Objective (i) uses T to describe the aeon-to-aeon decrease in dark fraction. Objective (ii) locates T at $n = 4$, where r_s is the $\mathcal{A}_3$ projection of the transition. Objective (iii) uses T to track $n_{\text{res}}(k) = 3+k$ across aeons. The three objectives are not independent physical proposals — they are three projections of the same algebraic structure.

1.3. What this paper does not attempt.

This paper does not construct a QFT over $\mathcal{A}_4$. It does not derive the 5%/95% cosmological ratio. It does not solve the Einstein equations in the $\mathcal{A}_4$ interior. These are explicitly declared gaps, carried as open problems.

2. Algebraic Foundation

All results in this section are proved in Papers 1–9 and are cited without re-derivation.

2.1. The tower and the threshold.

The Cayley–Dickson algebras over $\mathbb{R}$ form the tower

$$\mathcal{A}_0 = \mathbb{R} \subset \mathcal{A}_1 = \mathbb{C} \subset \mathcal{A}_2 = \mathbb{H} \subset \mathcal{A}_3 = \mathbb{O} \subset \mathcal{A}_4 = \mathbb{S} \subset \cdots$$

At $n = 4$ three properties are lost simultaneously: norm multiplicativity, alternativity, and zero-divisor freeness. Three independent threshold results converge on $n = 3$ as the stable minimum:

Result	Threshold	Source
Born Rule: unique solution for $n \leq 3$, none for $n \geq 4$	$n = 3$	[1, Thm 2.7]
Norm multiplicative iff $n \leq 3$	$n = 3$	[3]
Integral domain iff $n \leq 3$	$n = 3$	[3]
Unique single coupling λ in ZD-active interaction	$n = 4$ only	[6, Cor. 3.20]

The observer is in $\mathcal{A}_3$ by necessity, not selection. The Born Rule theorem is a theorem, not an anthropic principle [1].

2.2. Zero-divisor structure.

Theorem 2.1 ([1])**.** *For all $n \geq 4$:*

$$|ZD(n)| = 336 \cdot \binom{n-1}{3}_2,$$

where $\binom{n-1}{3}_2$ is the Gaussian binomial coefficient. The density $\varrho(n) = |ZD(n)|/4^{2n}$ satisfies $\varrho(n+1) < \varrho(n)$ for all $n \geq 4$, and $\varrho(n) \cdot 2^n \to 1/4$.

Theorem 2.2 (Fano filtration, [1])**.** *$ZD(n) = \bigsqcup_{j=4}^{n} F_j$, where $|F_k| = 336 \cdot (C(k-1, 3)_2 - C(k-2, 3)_2)$ and $ZD_{\text{new}}(k) = F_k$.*

2.3. Key algebraic facts.

Fact	Formula	Source
Norm-gap	$\Delta N(a,b) = N(a)N(b) - N(ab) = 4$, exact	[3, 6]
2+2 splitting	$\Delta N = F_{m_1} + F_{m_2} = 2+2$	[6, Thm 1.12]
Frobenius identity	$\|L_a\|_F^2 = 2^n N(a)$, all n	[4]
AMT	$\mathbb{E}_v[N(av)] = N(a)$, all n	[2, Thm 2.4]
Sign-parity	$\pi_A \pi_B \pi_C = -1$ for every 2-flat	[1, Thm 7.3/7.17]
Kernel dimension	$\dim\ker(L_a) = 2^{n-2}$ for ZD unit a	[1]
Lagrangian	$\mathcal{L}_{\text{int}} = \lambda \cdot \Delta N(\varphi, \psi)$, unique λ	[6]
$\mathrm{Aut}(\mathcal{A}_4)$	$C_2 \times ((C_2^3)\,.\,\mathrm{PSL}(3,2))$, order 2688, non-split	[6]

2.4. Physical conjectures from Paper 2.

The following are *[Conjecture]* or *[Speculation]* from [2]. Paper 10 builds on them without modifying or deriving them.

- **IIB-I**: $\mathcal{A}_3/\mathcal{A}_4$ is the Born Rule boundary.
- **IIB-II**: N is the algebraic correlate of gravity.
- **IIB-III**: $\mathcal{A}_4 \setminus \mathcal{A}_3$ is gauge-invisible and gravitationally active (dark sector).
- **IIB-IV** *[Speculation]*: CCC sedenionic residue: terminal Hawking evaporation leaves $\mathcal{A}_4 \setminus \mathcal{A}_3$ content that crosses the conformal boundary and constitutes dark matter of the next aeon.

3. Objective (i): Pre-Dynamical Structure of the Loop

3.1. Feasibility demarcation.

The loop $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$ has three legs.

Leg 1 ($\mathcal{A}_3 \to \mathcal{A}_4$): *[Derived]*. Algebraic necessity; $\Delta N = 4$ exact.

Leg 2 ($\mathcal{A}_4$ curves space): *[Conjecture]*, depends on IIB-II. N has not been shown to be the gravitational field in any dynamical sense [2, §8.2].

Leg 3 (curvature guides $\mathcal{A}_3$): not yet formalisable. No mechanism defined.

The normalisation gap [2, Rem. 3.19]: $\varrho(n)$ is a combinatorial density; Ω_{DM} is a cosmological energy density ratio. The map between them requires QFT over $\mathcal{A}_4$ (Open Problem 7.4 of [2]; Open Problem 4 of [6]).

Conclusion: The loop is formalisable as an algebraic pre-dynamical structure with a conditional physical extension. The 5%/95% ratio is not derivable.

3.2. The algebraic skeleton.

Definition 3.1. For $n \geq 4$, the *aeon state at level* n is $\sigma_n = (n, \varrho(n))$. The *algebraic tower map* is $T\colon \sigma_n \mapsto \sigma_{n+1}$.

Proposition 3.2 (*[Derived]*).

$$\frac{\varrho(n+1)}{\varrho(n)} = \frac{1}{16} \cdot \frac{2^n - 1}{2^{n-3} - 1}. \tag{1}$$

Proof. $|ZD(n{+}1)|/|ZD(n)| = C(n,3)_2/C(n{-}1,3)_2 = (2^n-1)/(2^{n-3}-1)$; $|C(n{+}1)|^2/|C(n)|^2 = 4^2 = 16$. ∎

Transition	Exact ratio	Decimal
$n = 4 \to 5$	15/16	0.9375
$n = 5 \to 6$	31/48	0.6458
$n = 6 \to 7$	9/16	0.5625
$n = 7 \to 8$	127/240	0.5292
$n \to \infty$	1/2	0.5000

Theorem 3.3 (*[Derived]*)**.** *T has no fixed point. A fixed point requires $2^n = 15$ — no integer solution.*

Theorem 3.4 (*[Derived]*)**.** *$\varrho(n) \to 0$ with $\varrho(n) \cdot 2^n \to 1/4$. The asymptote $\varrho^* = 0$ is not attained in finite aeons.*

Theorem 3.5 (*[Derived]*)**.** *$\varrho(n+1)/\varrho(n) \to 1/2$ as $n \to \infty$. The aeon-to-aeon dark fraction is halved in the long-run limit, regardless of any free parameter.*

Remark 3.6 (Slow start, accelerating decay)**.** The ratio $15/16 \approx 0.94$ at the first transition means the current aeon loses almost no dark fraction. The decay accelerates and converges to halving — a structural prediction depending on no free parameter.

3.3. **The conditional physical extension.**

Definition 3.7. An *energy-algebra map* is a function $E\colon \bigsqcup_{n\geq 4} ZD^*(n) \to \mathbb{R}_+$. No such map is derived from Papers 1–9.

Definition 3.8 (Minimal properties $P_{\min}$)**.** E satisfies $P_{\min}$ if:

(P.1) $\mathrm{Aut}(\mathcal{A}_n)$**-invariance.** E is constant on each $\mathrm{Aut}(\mathcal{A}_n)$-orbit of $ZD^*(n)$. Forced by the same symmetry that forces λ to be unique [6, Cor. 3.20].

(P.2) ΔN**-proportionality.** $E(a,b) = \kappa \cdot \Delta N(a,b)$, $\kappa > 0$ dimensional. ΔN is the unique Aut-invariant scalar on ZD pairs [6].

(P.3) Extensivity. $E_{\mathrm{dark}}(n) = \kappa \cdot |ZD^*(n)| \cdot \overline{\Delta N}(n)$, where $\overline{\Delta N}(n)$ is the average over $ZD^*(n)$.

(P.4) Candidate-pair normalisation. $E_{\mathrm{tot}}(n) = \kappa \cdot |C(n)| = \kappa \cdot 4^{2n}$, so $\Omega_{\mathrm{dark}}(n) = \frac{1}{2}\varrho(n)\overline{\Delta N}(n)$.

P.4 is the load-bearing assumption — the precise locus where QFT over $\mathcal{A}_4$ enters.

Definition 3.9. Under $P_{\min}$, define the *physical tower map* $G\colon \Omega_{\mathrm{dark}}(n) \mapsto \Omega_{\mathrm{dark}}(n+1)$. If $\overline{\Delta N}(n) = 4$ for all n, then G is isomorphic to T up to a constant factor.

Theorem 3.10 (*[Derived]* — closes Open Problem 1)**.** $\overline{\Delta N}(n) = 4$ *for every* $n \geq 4$.

Proof. By definition, $ZD(n) \subseteq C(n) \times C(n)$ where $C(n) = \{e_i \pm e_j : 0 \leq i < j < 2^n\}$. Every $a \in C(n)$ satisfies $N(a) = N(e_i) + N(e_j) = 1 + 1 = 2$, since $N(e_k) = 1$ for all basis vectors and $\langle e_i, e_j\rangle = 0$ for $i \neq j$. For any $(a,b) \in ZD(n)$: $ab = 0$, so $N(ab) = 0$. Therefore $\Delta N(a,b) = N(a)N(b) - N(ab) = 2 \cdot 2 - 0 = 4$ for every ZD pair at every $n \geq 4$, and $\overline{\Delta N}(n) = 4$ exactly. □

Corollary 3.11 (*[Derived]*)**.** *Under $P_{\min}$, since $\overline{\Delta N}(n) = 4$ for all n:*

$$\Omega_{\mathrm{dark}}(n) = \tfrac{1}{2} \cdot \varrho(n) \cdot 4 = 2\,\varrho(n).$$

The physical map G satisfies

$$\frac{G(\Omega_{\mathrm{dark}}(n))}{\Omega_{\mathrm{dark}}(n)} = \frac{\varrho(n+1)}{\varrho(n)} = \frac{1}{16} \cdot \frac{2^n - 1}{2^{n-3} - 1}$$

*for **all** $n \geq 4$. The rate formula Q.3 is universal, not asymptotic.*

Proposition 3.12 (*[Derived]*, AMT normalisation)**.** *The AMT provides a natural normalisation of* (P.4)*. Define $E^{\mathrm{AMT}}_{\mathrm{tot}}(n) = 2\kappa|C(n)|$. Then $\Omega^{\mathrm{AMT}}_{\mathrm{dark}}(n) = \varrho(n)$ exactly.*

Proof. By the AMT [2, Thm 2.4]: $\mathbb{E}_v[N(av)] = N(a)$ for all a, all n. Summing over all $a \in C(n)$: $\sum_{a\in C(n)} \mathbb{E}_v[N(av)] = \sum_{a \in C(n)} N(a) = 2|C(n)|$ (since $N(a) = 2$ for every $a \in C(n)$). Setting $E^{\mathrm{AMT}}_{\mathrm{tot}}(n) = 2\kappa|C(n)|$: $\Omega^{\mathrm{AMT}}_{\mathrm{dark}}(n) = \kappa|ZD^*(n)| \cdot 4/(2\kappa|C(n)|) = \varrho(n)$. □

3.4. **Derived characteristics under $P_{\min}$.** Under $P_{\min}$ the following hold *[Derived]*:

Q.1 $G(\Omega(n)) < \Omega(n)$ for all $n \geq 4$.

Q.2 G has no fixed point in $(0, 1)$.

Q.3 $\Omega_{\text{dark}}(n{+}1)/\Omega_{\text{dark}}(n) = (2^n - 1)/[(2^{n-3} - 1) \cdot 16]$. Parameter-free.

Q.4 $\Omega_{\text{dark}}(n{+}1) \to \frac{1}{2}\Omega_{\text{dark}}(n)$ for large n.

Q.5 $E_{\text{dark}}(n{+}1) = E_{\text{dark}}(n) + E_{\text{new}}(n{+}1)$, with the inherited component carried forward by $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$.

3.5. **Negative result: the 5%/95% ratio.**

Proposition 3.13 (*[Derived]*)**.** *Under $P_{\min}$, $\Omega_{\text{dark}}(4) \approx 1.03\%$. The observed $\Omega_{\text{DM}}/\Omega_{\text{tot}} \approx 27\%$ is not derivable from $P_{\min}$ or from any extension that does not introduce a physical energy scale linking ZD pairs to measured energy densities.*

Proof sketch. Under (P.4), κ cancels in the ratio. The gap — roughly one order of magnitude — cannot be closed by κ alone. Closing it requires identifying E_{tot} with the total cosmological energy budget, which presupposes QFT over $\mathcal{A}_4$. □

The 5%/95% **ratio is a free parameter of the conditional system under $P_{\min}$.**

3.6. **Conditional conjecture for QFT normalisation.**

Conjecture 3.14 (*[Conjecture]*)**.** *Let $\mathcal{Q}$ be any QFT over $\mathcal{A}_4$ satisfying Open Problem 7.4* [2]*, supplying states, observables, and $T_{\mu\nu}$ for $\mathcal{A}_4$-valued fields, and inducing an energy-algebra map $E_{\mathcal{Q}}$ satisfying* (P.1)–(P.3)*.*

- (i) Direction preserved. $G_{\mathcal{Q}}(\Omega) < \Omega$ *iff* $E_{\text{tot}}(n{+}1)/E_{\text{tot}}(n) > |ZD(n{+}1)| \cdot \overline{\Delta N}(n{+}1)/[|ZD(n)| \cdot \overline{\Delta N}(n)]$.
- (ii) Fixed-point condition. *A fixed point Ω^* of $G_{\mathcal{Q}}$ requires* $E_{\text{tot}}(n{+}1)/E_{\text{tot}}(n) = (2^n - 1)/(2^{n-3} - 1)$*. This is a* ***falsifiable constraint on any candidate QFT over** $\mathcal{A}_4$.*
- (iii) Asymptotic universality. *For any $\mathcal{Q}$ satisfying* (P.1)–(P.3) *with* $E_{\text{tot}}(n)/E_{\text{tot}}(n{+}1) \to c \in (0, 1)$*: $\Omega_{\text{dark}} \to 0$ as $n \to \infty$, regardless of c.*

Parts (i) and (iii) are conditional theorems; part (ii) is the genuine conjecture. It converts "is 5%/95% derivable?" into a precise algebraic condition on any future theory.

4. Objective (ii): The Black Circle Metric Ansatz

4.1. **Feasibility demarcation. Available:** $\Delta N = 4$ exact [3]; Frobenius identity $\|L_a\|_F^2 = 2^n N(a)$ [4]; AMT universality [2]; ZD-active masslessness [6]; isotropic amplitude [6]; IIB-I (r_s as algebraic transition) [2].

Not available: The Einstein equations are not derivable from N [2, §8.2]. $T_{\mu\nu}$ for $\mathcal{A}_4$ matter is not derived.

Verdict: A metric ansatz is feasible; a derivation is not.

4.2. **The Black Circle identification.** *[Conjecture]* (depends on IIB-I, IIB-II):

- **Claim 4.1.** $r_s = 2GM/c^2$ is the $\mathcal{A}_3$ projection of the algebraic transition $\mathcal{A}_3 \to \mathcal{A}_4$.
- **Claim 4.2.** The classical singularity $K \to \infty$ at $r = 0$ is an artifact of applying $\mathcal{A}_3$ language in a region that is intrinsically $\mathcal{A}_4$. In $\mathcal{A}_4$, $\Delta N = 4$ provides a strict finite bound on the norm-gap, which bounds the curvature.
- **Claim 4.3** (intentional conflict with GR). The Penrose–Hawking singularity theorems assume the strong energy condition. The $\mathcal{A}_4$ interior violates it (positive Λ_{eff}, de Sitter type). The algebraic justification: the strong energy condition requires a canonical probability assignment, but by the Born Rule theorem [1], no such assignment exists for $n \geq 4$. The condition is an $\mathcal{A}_3$-language statement; it does not apply in the $\mathcal{A}_4$ domain.

Remark 4.1 (Observational consistency — not confirmation, *[Conjecture]*)**.** A JWST measurement reported in Juodžbalis et al. [11] (*Nature*, May 2026) provides the first direct dynamical mass measurement of a black hole at z corresponding to 700 Myr post-Big Bang. The object Abell 2744-Qso1 has a black hole of 50 million solar masses constituting approximately $2/3$ of the total system mass, in a metal-poor environment ($< 0.5\%$ solar metallicity), indicating formation prior to significant stellar nucleosynthesis. The inferred formation mechanism is direct collapse.

Two structural consistencies with the Black Circle picture are noted.

(i) Direct collapse $\leftrightarrow$ discrete transition. The catastrophic direct-collapse mechanism is structurally consistent with the discrete algebraic jump $\ker = 0 \to \ker = 4$ at r_s (Remark 4.9) and with Proposition 4.10.

(ii) Formation before stars. Formation prior to stellar nucleosynthesis in a chemically minimal $(\mathrm{H} + \mathrm{He})$ environment is consistent with Black Circle formation on primordial $\mathcal{A}_3$ matter before SM gauge structure has produced complex nucleosynthetic products.

None of these consistencies constitutes a confirmation of the Black Circle ansatz. The data is compatible with the picture but does not distinguish it from other models of primordial black hole formation.

4.3. **Three algebraic inputs. Input I — Finite norm-gap bound.** *[Derived]* [3]. $\Delta N(a, b) = N(a)N(b) - N(ab) = 4$, exact, for every ZD-active pair. The norm-gap cannot exceed 4; this is the algebraic mechanism preventing $K \to \infty$.

Input II — Frobenius doubling. *[Derived]* [4]. $\|L_a\|_F^2 = 2^n N(a)$. At $n = 3$: $8N(a)$. At $n = 4$: $16N(a)$. **Ratio $= 2$, exact.** The gravitational operator content doubles at the algebraic transition.

Input III — Algebraic masslessness and isotropy. *[Conjecture]*/*[Derived]* [6]. ZD-active modes: no mass generation through $\mathcal{L}_{\text{int}}$ [6, Thm 3.15]; self-interaction amplitude VEV-independent $= 4\lambda$ [6, Cor. 3.16]; angular distribution isotropic [6, Cor. 3.10]. These are the algebraic signatures of an effective cosmological constant: massless, isotropic, constant coupling. The maximally isotropic spacetime is de Sitter.

4.4. **The metric ansatz.**

Definition 4.2 (*[Conjecture]*)**.** The *Black Circle metric* is

$$ds^2 = -f(r)\, c^2\, dt^2 + f(r)^{-1}\, dr^2 + r^2\, d\Omega^2,$$

with lapse function

$$f(r) = \begin{cases} 1 - \dfrac{r_s}{r} & r \geq r_s \quad [\mathcal{A}_3 \text{ domain — exact Schwarzschild}] \\[2ex] \dfrac{r^2}{r_s^2} - 1 & r < r_s \quad [\mathcal{A}_4 \text{ domain — de Sitter core}] \end{cases}$$

Remark 4.3. The inner metric contains *no free parameters* beyond r_s. The de Sitter radius $L = r_s$ is fixed by the single requirement $f(r_s) = 0$. No quantum length, no Planck scale, no additional constant is introduced.

4.5. **Properties of the ansatz.**

Proposition 4.4 (*[Derived]* from ansatz)**.** $f(r_s^+) = 0 = f(r_s^-)$. *No conical singularity.*

Proposition 4.5 (*[Derived]* + [4])**.**

$$\frac{f'_{\text{in}}(r_s)}{f'_{\text{out}}(r_s)} = \frac{2/r_s}{1/r_s} = 2 = \frac{\|L_a\|_F^2\big|_{n=4}}{\|L_a\|_F^2\big|_{n=3}} = \frac{16\, N(a)}{8\, N(a)}.$$

The ratio of metric derivatives at r_s equals the Frobenius doubling factor. The first-derivative jump $\Delta f' = (16/8 - 1) \cdot (1/r_s) = 1/r_s$ is algebraically fixed *by the Frobenius identity.*

The C^1 discontinuity generates, via Israel's junction conditions [9], a distributional surface energy-momentum at r_s. Its explicit computation is Open Problem 2.

Proposition 4.6 (*[Derived]* from ansatz)**.** $f_{\text{in}}(0) = -1$. *Finite. No divergence in the metric components.*

Proposition 4.7 (*[Derived]* from ansatz)**.** $f_{\text{in}} = r^2/r_s^2 - 1$ *is the static de Sitter metric with de Sitter radius* $L = r_s$*. Effective cosmological constant:*

$$\Lambda_{\text{eff}} = \frac{3}{r_s^2} = \frac{3c^4}{4G^2M^2}.$$

Larger black holes have smaller Λ_{eff} — inner geometry flatter for more massive holes.

Theorem 4.8 (*[Derived]* from ansatz)**.** *The Kretschmann scalar at the origin of a Black Circle is*

$$K_{\text{in}}(0) = \frac{24}{r_s^4} = \frac{6\,\Delta N}{r_s^4},$$

where $\Delta N = 4$ is the exact algebraic norm-gap for ZD-active pairs. ***The classical singularity $K \to \infty$ is replaced by the finite value $6\Delta N/r_s^4$.***

Proof. The de Sitter metric with $\Lambda = 3/r_s^2$ has constant curvature $K = 24/L^4 = 24/r_s^4 = 6{\cdot}4/r_s^4$. □

Remark 4.9 (*[Derived]*, [1])**.** The formation of a Black Circle has a precise algebraic signature. In $\mathcal{A}_3 = \mathbb{O}$, every $a \neq 0$ satisfies $\dim\ker(L_a) = 0$; there is no ceiling on $N(a)N(b)$. In $\mathcal{A}_4$, for every ZD unit: $\dim\ker(L_a) = 2^{n-2}\big|_{n=4} = 4$. The kernel appears *discretely* — from 0 to 4 — at the algebraic transition. This is a structural phase transition in the algebra, not a continuous deformation.

Physical reading *[Conjecture]* (under IIB-I): the formation of a Black Circle is the moment when the algebraic description transitions from $\mathcal{A}_3$ (ker $= 0$, fully invertible) to $\mathcal{A}_4$ (ker $= 4$, ZD-active, N bounded). The discontinuity in the metric first derivative (Proposition 4.5) is the geometric projection of this discrete algebraic jump.

Proposition 4.10 (*[Conjecture]* under IIB-I, IIB-II)**.** *Under IIB-I and IIB-II, the transition $\mathcal{A}_3 \to \mathcal{A}_4$ at $r = r_s$ is the unique mechanism in the Cayley–Dickson tower that preserves N for collapsing configurations.*

(i) No cap in $\mathcal{A}_3$. [Derived]. *$ZD(3) = \emptyset$ (Hurwitz); no algebraic ceiling on $N(ab)$.*
(ii) Exact cap in $\mathcal{A}_4$. [Derived] [3]. *$\Delta N = 4$ for every ZD-active pair; the AMT guarantees $\mathbb{E}_v[N(av)] = N(a)$ universally.*
(iii) Uniqueness. [Derived]. *The Cayley–Dickson construction is parameter-free; there is no alternative algebra at this step.*

Under IIB-I/II: the Black Circle is not one possible outcome — it is the unique algebraically consistent continuation of $\mathcal{A}_3$ matter beyond r_s.

Remark 4.11 (*[Derived]*, [2])**.** The AMT $\mathbb{E}_v[N(av)] = N(a)$ holds for all n, including $n \geq 4$. The transition from *local* conservation of N (multiplicative in $\mathcal{A}_3$) to *statistical* conservation ($\mathcal{A}_4$, via AMT) is the algebraic mechanism of Black Circle formation: the norm survives, but its distribution changes from multiplicative to ZD-structured.

4.6. **Curvature comparison.**

Region	Kretschmann scalar K	Language
$r \gg r_s$	$12r_s^2/r^6 \to 0$	$\mathcal{A}_3$
$r = r_s$ (outer)	$12/r_s^4$	$\mathcal{A}_3 \to \mathcal{A}_4$
$r_s > r \geq 0$	$24/r_s^4 = 6\Delta N/r_s^4$ (constant, finite)	$\mathcal{A}_4$
$r \to 0$, classical GR	$12r_s^2/r^6 \to \infty$	$\mathcal{A}_3$ artifact

Remark 4.12 (*[Derived]*)**.** The Frobenius factor 2 manifests at r_s in three independent ways:

Quantity	$\mathcal{A}_3$ side	$\mathcal{A}_4$ side	Ratio
$\|L_a\|_F^2$	$8N(a)$	$16N(a)$	2 [4]
Metric derivative f'	$1/r_s$	$2/r_s$	2 [ansatz]
Kretschmann scalar	$12/r_s^4$	$24/r_s^4$	2 [ansatz]

All three ratios derive from the same source: $2^{n=4}/2^{n=3} = 2$.

Remark 4.13 (Hawking evaporation as algebraic separation, *[Conjecture]* IIB-I/II + *[Speculation]* IIB-IV)**.** As a Black Circle evaporates (M decreasing, $r_s \to 0$), three quantities grow simultaneously from the ansatz:

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} \propto \frac{1}{r_s}, \qquad \Lambda_{\text{eff}} = \frac{3}{r_s^2} \propto \frac{1}{r_s^2}, \qquad K_{\text{in}} = \frac{6\Delta N}{r_s^4} \propto \frac{1}{r_s^4}.$$

The exponents $1, 2, 4$ are fixed by the ansatz. Evaporation compresses the $\mathcal{A}_4$ interior toward increasing curvature density.

The Black Circle contains two populations: $\mathcal{A}_3$*-recoverable*: degrees of freedom that can exit as Hawking radiation (Born-accessible, SM-coupled); $\mathcal{A}_4$*-irrecoverable*: degrees of freedom in $\ker(L_a)$, for which no canonical probability assignment exists in $\mathcal{A}_3$ language [1]. They cannot exit as Hawking radiation — not because the coupling is small, but because the algebraic structure for their emission does not exist in $\mathcal{A}_3$.

Conditional prediction. *[Conjecture]* pending Open Problem 5. Under identifications (a)–(c) of §5.6: exactly 1/4 of the Black Circle's information content is irrecoverable by Hawking radiation at every mass scale. This would be a specific, falsifiable prediction. The identifications are not established; the fraction 1/4 is recorded as a conditional result.

Terminal state. At $M \to M_{\text{Planck}}$, $r_s \to \ell_{\text{Planck}}$ and $K_{\text{in}} \to K_{\text{Planck}}$. The Black Circle has shed all Born-accessible content. The structured $\mathcal{A}_4$ residue — carrying $\pi_A\pi_B\pi_C = -1$ (Theorem 5.6) — crosses the CCC conformal boundary as the dark-sector seed of the next aeon (IIB-IV).

Remark 4.14 (Symmetry-breaking character of the de Sitter ansatz)**.** The precise symmetry-breaking chain induced by the de Sitter inner metric is established by the GAP computations of §7 (Open Problems 7 and 8). The de Sitter ansatz captures the three algebraic inputs ($\Delta N = 4$, masslessness, isotropy) while breaking the full Fano symmetry to one seventh. The correct inner geometry — once QFT over $\mathcal{A}_4$ is constructed — lives in an extended space of dimension ≥ 6.

5. Objective (iii): The Sedeonic Residue Between Aeons

5.1. **Feasibility demarcation.** Items (1) and (2) of Objective (iii) are formalisable from the Fano filtration theorem alone — no additional physical assumptions. Item (3) (connection to Penrose CCC) requires assumption (A3) — a physical assumption not algebraically derived.

The kernel dimension theorem ($\dim\ker(L_a) = 2^{n-2}$) is **not used** as supporting evidence; see §5.6.

5.2. **Algebraic definition of n_{res}.**

Definition 5.1 (*[Conjecture]* under IIB-IV)**.** Label CCC aeons as $k = 1, 2, 3, \ldots$ with $k = 1$ the current aeon. Each aeon k has: a visible sector in $\mathcal{A}_3$ (by necessity of the Born Rule theorem); and a dark sector which is the accumulated content of Fano filtration components from all previous aeons.

Definition 5.2. $n_{\text{res}}(k) := 3 + k$.

Justification: At $k = 1$: $n_{\text{res}} = 4$, dark sector is F_4, consistent with IIB-III. At each crossing $k \to k{+}1$: one new component F_{k+4} is added; the highest active algebra level increases by 1. **The tower map T from Objective (i) is identical to $k \to k{+}1$ on aeon labels.**

Remark 5.3. The dark fraction at aeon k is $\varrho(n_{\mathrm{res}}(k)) = \varrho(3{+}k)$. Objectives (i) and (iii) share the same algebraic variable.

5.3. **The ZD content formula.**

Theorem 5.4 (*[Derived]* — Fano filtration)**.**

$$|ZD_{\mathrm{dark}}(k)| = \bigsqcup_{j=4}^{3+k} |F_j| = |ZD(3+k)|.$$

Proof. By Theorem 2.2: $ZD(n) = \bigsqcup_{j=4}^{n} F_j$. Set $n = 3{+}k$. ☐

Aeon k	n_{res}	$\lvert ZD_{\mathrm{dark}}(k)\rvert$	New pairs
1 (current)	4	336	—
2	5	5 040	4 704
3	6	52 080	47 040
4	7	468 720	416 640
k	$3{+}k$	$336 \cdot C(2{+}k,3)_2$	$336 \cdot (C(2{+}k,3)_2 - C(1{+}k,3)_2)$

Corollary 5.5 (*[Derived]*)**.** *For $j \neq k$, elements of F_j and F_k do not form ZD pairs with each other.*

Physical reading [Conjecture] *(IIB-IV): dark-matter populations from different aeons are gravitationally coupled (N is universal) but algebraically non-communicating (no ΔN interaction between strata).*

5.4. **Sign-parity of the residue.**

Theorem 5.6 (*[Derived]*, [1])**.** *The sedenionic residue crossing the CCC boundary carries the constraint $\pi_A\pi_B\pi_C = -1$ at every level. This is preserved by the hereditary embedding.*

A purely thermal Hawking spectrum has no such global constraint. The ZD residue carries a mandatory non-local sign-parity structure distinguishing it from thermal radiation at every level of the tower.

Remark 5.7 (*[Conjecture]* under IIB-I, IIB-II, IIB-III)**.** The informal statement "matter becomes information" at Black Circle formation has a precise algebraic content:

Property	$\mathcal{A}_3$ (before r_s)	$\mathcal{A}_4$ (after r_s)
Born rule	✓ defined [1]	✗ fails for $n \geq 4$
Gauge-coupled	✓ SM charges	✗ Born failure forbids
N-active (gravitational)	✓	✓ AMT universal
ZD-structured	✗ N multiplicative	✓ $\Delta N = 4$ exact
$\pi_A\pi_B\pi_C$ constraint	✗	✓ structured

The degree of freedom is not destroyed. Its norm N is preserved on average (AMT). It continues to source gravity (IIB-II). But it becomes *measurement-inaccessible* in the $\mathcal{A}_3$ sense — no Born-rule-compatible probability assignment can be made for it.

This is the algebraic content of "matter becomes information at the Black Circle": the degrees of freedom become **Born-inaccessible but N-active**. They are not lost; they change algebraic stratum.

5.5. The first-aeon problem: algebraic resolution.

Proposition 5.8 (*[Derived]* + IIB-IV)**.** *There is no algebraically admissible "aeon 0". $ZD(n) = \emptyset$ for $n \leq 3$ (Hurwitz); therefore F_3, F_2, F_1 are empty. The IIB-IV identification $F_k \leftrightarrow$ dark-sector content of aeon k gives empty dark sector for $k < 1$. Aeon $k = 1$ is the algebraically forced minimum of the tower.* ***The first-cause problem of Penrose CCC* [7] *does not arise.***

Feature	Penrose CCC	IIB-IV + Fano tower
First aeon defined?	No — infinite regress	Yes — algebraic minimum ($ZD = \emptyset$, $n \leq 3$)
Starting point	Undefined	$n = 4$: minimum non-empty filtration
Residue structure	Not specified	Structured by $\pi_A \pi_B \pi_C = -1$
Successive dark fractions	Not constrained	$\varrho(n) \searrow 0$ monotonically *[Derived]*

5.6. The kernel dimension theorem: an honest demarcation.

Theorem 5.9 (*[Derived]*, [1])**.** $\dim\ker(L_a) = 2^{n-2} = \frac{1}{4}\dim(\mathcal{A}_n)$ *for every ZD unit at every $n \geq 4$.*

The fraction 1/4 is level-independent and might appear to correspond to a dark-matter fraction. **This paper deliberately does not make this identification.**

Three physical identifications are required, none currently defined: (a) black-hole states $\leftrightarrow$ elements of $\mathcal{A}_n$; (b) Hawking non-recovery $\leftrightarrow \ker(L_a)$ for specific a; (c) this kernel $\leftrightarrow$ CCC conformal-boundary residue.

Until (a)–(c) are established, using Theorem 5.9 as evidence would give a false impression of derivation.

6. The Unified Structure

6.1. Three facets of the tower map T.

Objective	Role of $T\colon n \mapsto n{+}1$
(i) Pre-dynamical loop	T on $\varrho(n)$: dark fraction *decreases* per aeon
(ii) Black Circle	T at $n = 4$: r_s is $\mathcal{A}_3$ projection of transition
(iii) Sedeonic residue	T on $n_{\mathrm{res}}(k)$: dark structure *accumulates*

The simultaneous decrease of dark fraction and accumulation of dark structure is not paradoxical: the absolute ZD count $|ZD_{\mathrm{dark}}(k)|$ grows, but the fraction $\varrho(3{+}k)$ decreases because the total candidate pairs grow faster. The algebra expands faster than the dark sector.

6.2. N as the unique dual instrument.

Proposition 6.1 (*[Derived]* + *[Conjecture]* IIB-II)**.** *N is the unique algebraic structure satisfying simultaneously:*

(i) Universal persistence. [Derived]*. N is preserved by every Cayley–Dickson doubling. No other algebraic property — commutativity, associativity, alternativity, norm-multiplicativity, zero-divisor freeness — survives all doublings. N has no threshold.*

(ii) Universal gravitational coupling. [Conjecture] *(IIB-II). N is the algebraic correlate of gravity — the unique interaction acting at every level. All gauge forces are confined to specific levels; none survives the crossing. N does not stop.*

(iii) Coincidence with quantum probability at $n = 3$. [Derived] [1, Thm 2.7]*. $f = N$ is the* unique *solution of the Born Rule axioms exactly at $n \leq 3$ — no solution exists for $n \geq 4$. At the level where the observer is algebraically necessary, the probability measure* is *N. The gravitational field and the quantum probability measure are the same object.*

Corollary 6.2 (*[Derived]* + *[Conjecture]* IIB-II)**.** *At $n = 3$, N simultaneously is the gravitational coupling and the canonical probability measure. At $n = 4$ these two roles* ***split****: N persists as gravity (AMT holds universally), but the Born Rule fails.*

The dark sector $\mathcal{A}_4 \setminus \mathcal{A}_3$ is precisely the content for which this splitting is physical: N-active (gravitationally coupled) but Born-inaccessible. The split at the IIB threshold is not a pathology — it is the mechanism that generates the dark sector.

Remark 6.3. For $n < 3$: the Born Rule exists but $ZD = \emptyset$ — nothing beyond the boundary to regulate. For $n > 3$: the Born Rule fails. At $n = 3$, and only at $n = 3$: the tower has both the capacity for probabilistic self-description and the first non-trivial boundary to describe.

$\mathcal{A}_3$ is the unique level where the tower can measure itself.

This is the conjunction of two theorems [1]: the Born Rule uniquely selects $n = 3$ as the last coherent level, and the zero-divisor structure selects $n = 4$ as the first non-trivial boundary. The observer at $n = 3$ is algebraically positioned between the last possible measurement and the first unmeasurable content. This is a structural fixed point of the tower, not a choice.

Remark 6.4 (*[Derived]* + *[Conjecture]* IIB-II + *[Speculation]* IIB-IV)**.** Between aeons, all gauge content is extinguished: SM degrees of freedom cannot cross the CCC conformal boundary in $\mathcal{A}_3$ language (Born Rule fails for the crossing configuration). Only N-active content survives.

N is not merely the gravitational field within an aeon. It is the **unique causal instrument available to the tower for transmitting structure from one aeon to the next**. The gravitational skeleton of aeon $k+1$ is literally all that aeon k can leave behind. The tower is self-generating not because it chooses to be, but because N is the only structure algebraically capable of crossing the boundary.

Remark 6.5 (Intra-aeon enrichment and the gravitational map, *[Conjecture]* under IIB-I, IIB-II, IIB-III)**.** The CCC crossing (Remark 6.4) is the inter-aeon mechanism. Within each aeon, a structurally parallel loop operates continuously.

At every Black Circle formation event inside aeon k (Proposition 4.10), $\mathcal{A}_3$ matter crossing r_s injects Born-inaccessible but N-active content into the dark sector — new ZD-active pairs enriching $F_4(k)$ continuously throughout the aeon, not only at the CCC boundary.

(i) The dark sector writes a gravitational map. Accumulated $\mathcal{A}_4$ content sources N (IIB-II), creating gravitational potential wells that determine where $\mathcal{A}_3$ matter subsequently clusters — where stars form, where galaxies condense, where new Black Circles arise.

(ii) Black Circles precede stars by algebraic structure, under IIB-I/II/III. Stars are $\mathcal{A}_3$ structures requiring gauge forces, quantum coherence, and matter clustering. Clustering requires pre-existing gravitational wells; wells require pre-existing N-active content; N-active content in $\mathcal{A}_4$ requires pre-existing Black Circle formation. The causal chain is one-directional and algebraically forced under the stated conjectures:

$$\text{Black Circles} \xrightarrow{\text{inject } N\text{-active content}} \text{gravitational wells} \xrightarrow{\text{guide clustering}} \text{stars and galaxies.}$$

(iii) The intra-aeon loop is T operating within a single aeon. Each Black Circle event is a local instance of the $\mathcal{A}_3 \to \mathcal{A}_4$ transition, contributing incrementally to the accumulation that T counts in aggregate at the aeon boundary.

The JWST observation of Remark 4.1 — a primordial black hole at 700 Myr post-Big Bang formed before stellar nucleosynthesis — is structurally consistent with point (ii). [Consistent, not confirmation.]

6.3. **The full structure in one statement.** *$\mathcal{A}_3$ matter collapsing past r_s cannot preserve N multiplicatively (no ZD cap in $\mathcal{A}_3$); the only algebraically consistent continuation is $\mathcal{A}_4$, where $\Delta N = 4$ bounds the norm product and the AMT guarantees N is conserved on average (Proposition 4.10). The transition is discrete — $\ker(L_a)$ jumps from 0 to 4 at r_s. The collapsed content becomes Born-inaccessible but N-active: it sources gravity, carries Fano structure $\pi_A\pi_B\pi_C = -1$, and at the CCC* *boundary seeds the dark sector of the next aeon at level $n_{\text{res}} = 4$. Across all subsequent aeons, the dark fraction $\varrho(3+k)$ decreases monotonically toward zero — governed by the same tower map T that forced the original $\mathcal{A}_3 \to \mathcal{A}_4$ transition.*

Every clause carries a precise epistemic label. The $\mathcal{A}_3 \to \mathcal{A}_4$ transition under collapse and the aeon-to-aeon decrease are *[Conjecture]* under IIB-I/II. The norm-gap bound, AMT, ker jump, sign-parity, and monotonicity are *[Derived]*. The CCC crossing is *[Speculation]*.

7. Open Problems

Stratum I: Accessible without new physical theory.

(1) $\overline{\Delta N}(n)$ **for** $n \geq 5$ **— RESOLVED.** Theorem 3.10 proves $\overline{\Delta N}(n) = 4$ for all $n \geq 4$ directly: every element of $C(n)$ has $N = 2$, every ZD pair has $N(ab) = 0$, hence $\Delta N = 4$ universally. The physical map G is exactly isomorphic to T for all n (Corollary 3.11). Rate formula Q.3 is universal.

(2) **Israel shell at** r_s **— RESOLVED.** *[Conjecture]* under IIB-I/II + ansatz §4; computation *[Derived]* [SymPy]

Theorem 7.1. *The Barrabès–Israel thin shell at $r = r_s$ of the Black Circle ansatz has*

$$\sigma = 0, \qquad p = \frac{1}{64\pi G^3 M^2}, \qquad T_{\text{total}} = 4\pi r_s^2\, p = \frac{1}{4G}.$$

Proof. In Eddington–Finkelstein coordinates the lapse jump is *Convention:* $[f'(r_s)] = f'_{\text{out}}(r_s) - f'_{\text{in}}(r_s)$ (outer minus inner); this gives $[f'] = 1/r_s - 2/r_s = -1/r_s$, so $p > 0$ (compression). SymPy: `delta_f_prime = df_out_rs - df_in_rs` $\to [f'] = -1/r_s$ (zero residual). The Barrabès–Israel formula for a null shell gives $\sigma = 0$ and

$$p = \frac{-[f'(r_s)]}{16\pi G\, r_s} = \frac{1}{16\pi G\, r_s^2} = \frac{1}{64\pi G^3 M^2}.$$

The factor 2 in $f'_{\text{in}} = 2/r_s$ is the Frobenius factor [4]. The universal total pressure $T_{\text{total}} = 4\pi r_s^2\, p = 1/(4G)$ is independent of M. Verified by SymPy. ☐

Properties: (i) $p > 0$: compression shell, not exotic matter. (ii) $p \propto 1/M^2$: shell pressure diverges as $M \to M_{\text{Planck}}$. (iii) $T_{\text{total}} = 1/(4G)$: universal constant — every Black Circle carries the same total shell pressure regardless of mass.

Status note: The computation is exact given the ansatz; the physical interpretation is *[Conjecture]* (the ansatz is not derived from GR).

(3) **Conformal invariance of the Fano residue.** Does the orbit-336 stratum of F_4 admit conformally invariant field configurations? Is $\pi_A \pi_B \pi_C = -1$ compatible with conformal invariance?

(4) **AMT as physical** E_{tot} **normalisation.** Does $\sum_a \mathbb{E}_v[N(av)]$ over $\mathcal{A}_n$ provide the natural normalisation for (P.4), replacing $|C(n)|$ with a norm-averaged quantity?

Stratum II: Require one new physical ingredient.

(5) **Three identifications for the kernel dimension.** Establish identifications (a)–(c) of §5.6. If established, Theorem 5.9 predicts the information fraction non-recoverable by Hawking radiation is exactly $1/4$.

(6) **Black Circle evaporation as level advancement.** Does each Planck-scale Black Circle collapse correspond to the generation of exactly one new Fano component F_{n+1}?

(7) $\mathrm{Aut}(\mathcal{A}_4)$ **as subgroup of de Sitter isometries — RESOLVED (negative).** *[Derived]* [`GAP` 4.15.1]

Theorem 7.2. $\mathrm{Aut}(\mathcal{A}_4)$ *does* not *embed as a subgroup of* $SO(4,1)$*.*

Proof. Every finite subgroup of $SO(4,1)$ is compact, hence conjugable into the maximal compact subgroup $SO(4)$. Therefore $\mathrm{Aut}(\mathcal{A}_4) \hookrightarrow SO(4,1)$ iff $\mathrm{Aut}(\mathcal{A}_4) \hookrightarrow SO(4)$, iff $\mathrm{Aut}(\mathcal{A}_4)$ has a faithful real representation of degree ≤ 4.

$\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$ is a quotient of $\mathrm{Aut}(\mathcal{A}_4)$; if $\mathrm{Aut}(\mathcal{A}_4) \hookrightarrow SO(4)$ then $\mathrm{PSL}(2,7) \hookrightarrow SO(4)$, requiring a faithful real representation of degree ≤ 4.

`GAP` 4.15.1 computation on the character table of $\mathrm{PSL}(2,7)$ (order 168):

Degree	FS indicator	Faithful?
1	real	no
3	complex	yes
3	complex	yes
6	real	yes
7	real	yes
8	real	yes

The minimum faithful real irreducible representation has degree 6. The two degree-3 faithful representations are complex type (Frobenius–Schur indicator 0): over $\mathbb{R}$ they become 6-dimensional. Therefore $\mathrm{PSL}(2,7)$ has no faithful real representation of degree ≤ 5, cannot embed in $O(5)$, cannot embed in $SO(4)$, and cannot embed in $SO(4,1)$. Since $\mathrm{PSL}(2,7)$ is a quotient of $\mathrm{Aut}(\mathcal{A}_4)$, neither can $\mathrm{Aut}(\mathcal{A}_4)$. □

Consequence: The de Sitter inner metric is a symmetry-breaking approximation of the correct $\mathcal{A}_4$ inner geometry. The exact inner geometry lives in a space of real dimension ≥ 6.

(8) **Surviving subgroup of $\mathrm{Aut}(\mathcal{A}_4)$ in $SO(4,1)$ — RESOLVED.** *[Derived]* [`GAP` 4.15.1]

Theorem 7.3. *The maximum subgroup of* $\mathrm{PSL}(2,7)$ *that embeds in* $SO(4,1)$ *is* S_4 *of order* 24*, with exactly* 14 *conjugate copies in* $\mathrm{PSL}(2,7)$*. The surviving fraction is* $|S_4|/|\mathrm{PSL}(2,7)| = 24/168 = 1/7$*.*

Proof. `GAP` 4.15.1 computed all conjugacy classes of subgroups of $\mathrm{PSL}(2,7)$ and their minimum faithful real representation degree. S_4 (order 24, $\mathrm{min\_R} = 3 \leq 4$) is the unique maximal surviving subgroup. The 14 conjugate copies are: 7 point-stabilisers + 7 line-stabilisers of $\mathrm{PG}(2,\mathbb{F}_2)$. □

Remark 7.4 (*[Derived]*)**.** Three consequences follow.

(i) Factor 7 is Fano. $\mathrm{PSL}(2,7)$ acts transitively on the 7 points of $\mathrm{PG}(2,\mathbb{F}_2)$; the stabiliser of any point has order $168/7 = 24 \cong S_4$. The de Sitter inner geometry preserves exactly the symmetry that *fixes one point of the Fano plane.*

(ii) 14 copies = projective duality. $\mathrm{PG}(2,\mathbb{F}_2)$ is self-dual: 7 points and 7 lines, each with stabiliser S_4. The 14 copies are the exact signature of Fano duality.

(iii) CSS bridge [5]. Paper 8 established: Fano points $\leftrightarrow$ X-stabilisers of the Steane $[[7,1,3]]$ code; Fano lines $\leftrightarrow$ Z-stabilisers. Therefore the 14 surviving copies of S_4 correspond to the **14 stabilisers of the Steane code** (7 X-type + 7 Z-type). Choosing the de Sitter inner metric is algebraically equivalent to **choosing a gauge in the CSS code structure**.

(iv) $C_2 \times D_8$ survives separately. The vertex stabiliser $C_2 \times D_8$ of $\mathcal{L}_{\mathrm{int}}$ [6] is not a subgroup of $\mathrm{PSL}(2,7)$ (confirmed by `GAP`: no order-16 subgroup with that structure exists in $\mathrm{PSL}(2,7)$). It lives in the C_2^3-extension part of $\mathrm{Aut}(\mathcal{A}_4)$. But $C_2 \times D_8$ has $\mathrm{min\_R} = 3 \leq 4$, hence *does* embed in $SO(4,1)$. The interaction symmetry of $\mathcal{L}_{\mathrm{int}}$ is preserved; the full Fano symmetry is broken.

(9) **Two conjugacy classes of $\mathrm{PSL}(2,7)$ in G_2 — RESOLVED.** *[Derived]* [`GAP` 4.15.1]

Theorem 7.5. $\mathrm{PSL}(2,7)$ *has exactly two conjugacy classes of embeddings in* $G_2 = \mathrm{Aut}(\mathcal{A}_3)$*.*

Proof. G_2 is the stabiliser of the Cayley 3-form $\varphi \in \wedge^3(\mathbb{R}^7)$ in $SO(7)$. A subgroup $H \subset SO(7)$ lies in G_2 iff the trivial representation appears in $\wedge^3(7)|_H$.

`GAP` 4.15.1 computes $\wedge^3$ of both candidate 7-dimensional representations of $\mathrm{PSL}(2,7)$:

Embedding	Rep. on $\mathbb{R}^7$	$\wedge^3$ decomp.	$m(\mathbf{1})$	In G_2?
Via 7D irrep	irr5 = 7 (irred.)	$\mathbf{1}+2\cdot\mathbf{6}+2\cdot\mathbf{7}+\mathbf{8}$	1	**yes**
Via SU(3)	**1**+(irr2+irr3) (red.)	$3\cdot\mathbf{1}+2\cdot\mathbf{3}+2\cdot\bar{\mathbf{3}}+2\cdot\mathbf{6}+\mathbf{8}$	3	**yes**
Via **1**+irr4	**1**+irr4 (red.)	—	0	no

The two valid embeddings give non-isomorphic representations on $\mathbb{R}^7$ (irreducible 7 vs. reducible $1+6$), hence are not conjugate in G_2. □

Remark 7.6 (*[Derived]*)**.** The two embeddings correspond to two distinct ways the $\mathcal{A}_3 \to \mathcal{A}_4$ transition can "see" the octonion structure:

(i) 7D irrep embedding (minimal G_2 path): $m(\mathbf{1}) = 1$ — PSL(2,7) acts irreducibly on $\mathrm{Im}(\mathcal{A}_3)$ and preserves exactly the G_2 structure. This is the direct Fano path, connected to the Clifford representation $\rho\colon \mathrm{ZD}^*(4) \to \mathrm{Cliff}(0,7)$ of [4].

(ii) SU(3) embedding (gauge path): $m(\mathbf{1}) = 3$ — three independent PSL(2,7)-invariant 3-forms: the Cayley form φ, $\mathrm{Re}(\Omega)$, and $\mathrm{Im}(\Omega)$ where $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ is the SU(3) holomorphic 3-form. This is the gauge path, connected to SU(3) colour symmetry via $\mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2$.

Stratum III: Require both new theory and new experiment.

(8) **QFT over $\mathcal{A}_4$.** Construct states, observables, and $T_{\mu\nu}$ for fields valued in $\mathcal{A}_4$. Central open problem of the IIB programme.

(9) **CMB signature of the structured residue.** A residue with $\pi_A\pi_B\pi_C = -1$ and orbit structure $\{2, 14, 84, 336\}$ differs from thermal dark matter and the Penrose–Gurzadyan Weyl signal.

(10) **Vaidya dynamic extension — RESOLVED.** *[Conjecture]* under IIB-I/II + ansatz; computation *[Derived]* [SymPy]

Theorem 7.7. *The Vaidya-BC dynamic metric* [8] *is* $f_{\mathrm{out}}(v,r) = 1 - r_s(v)/r$, $f_{\mathrm{in}}(v,r) = r^2/r_s(v)^2 - 1$, $r_s(v) = 2GM(v)$. *It has the following exact properties.*

(i) $\sigma = 0$ *in all three dynamic regimes: timelike* ($\dot{r}_s > 0$*, formation), null* ($\dot{r}_s = 0$*, static), spacelike* ($\dot{r}_s < 0$*, evaporation).*

(ii) Apparent horizon = algebraic transition locus. $r_{\mathrm{AH}}(v) = r_s(v) = 2GM(v)$ *coincides with the* $\ker = 0 \to \ker = 4$ *jump.*

(iii) Algebraic discreteness forces thin null shell. *The ker jump is discrete; a smooth* $M(v)$ *would give continuous horizon growth, incompatible with the jump. Therefore* $M(v) = M_f \cdot \Theta(v - v^*)$ *— only impulsive formation is algebraically consistent.*

(iv) Shell pressure. *For the null shell* ($\dot{r}_s = 0$)*:* $[K_{\theta\theta}] = 0$ *for* $\dot{r}_s \neq 0$ (SymPy)*; the Barrabès–Israel null formalism gives* $\sigma = 0$*,* $p = 1/(64\pi G^3 M^2)$*,* $T_{\mathrm{total}} = 1/(4G)$ *(OP 2).*

Two falsifiable predictions. **P1:** The quasi-normal mode spectrum near r_s differs from Schwarzschild by the Frobenius factor 2 [4]. **P2:** No Black Circle forms via smooth collapse — only impulsive (thin null shell) formation is algebraically consistent.

(11) **First consistency test for QFT over $\mathcal{A}_4$ — PARTIALLY CLOSED.** *[Derived]* (algebra) + *[Conjecture]* (physical identification); [SymPy + Python]

Theorem 7.8 (Direzione 2)**.** *Let* $ZD^*(4)$ *denote the set of* 84 *ZD-active elements of* $\mathcal{A}_4$.

(i) Single-mode vanishing. [Derived] *[Python, 84 elements]. For any single ZD-active mode* $\varphi = \phi A$*:* $A^2 = -2e_0$ *(since* $\{e_p, e_{8+k}\} = 0$*,* [4] *Thm 5.1),* $N(A^2) = 4$*, and* $\Delta N(\varphi,\varphi) = 0$ *identically. The* $\lambda\Delta N$ *term vanishes on a single mode; the action reduces to a free scalar.*

(ii) ZD pair graph is 4-regular. [Derived] *[Python]. The ZD pair graph on $ZD^*(4)$ has* 168 *unordered edges and every vertex has degree* 4. *The* 84 *elements split as:*

Sector	*Elements*	*Internal pairs*	*Cross-pairs*
Spinorial $(\ker(L_a))$	4	0 *(independent set)*	16
Tensoriale	80	152	16

Geometric reason. The 4 *spinorial elements have* ε*-indices in the Fano complement* $\bar{\ell} = \{1, \ldots, 7\} \setminus \ell_a$. *For any two spinorial elements with indices* $p, r \in \bar{\ell}$: $p \oplus r \in \ell_a$; *the sign-parity constraints of* [1] *forbid ZD pairing within* $\bar{\ell}$.

(iii) Spinorial sector is free. [Derived]. $V\big|_{\ker(L_a)} = 4\lambda \sum_{(A,B)\in ZD_{\mathrm{int}}} \phi_A^2\phi_B^2 = 0$ *identically (zero internal pairs). The* 4 *fermionic modes carry no* ΔN *self-interaction; all* 16 *cross-couplings go to the tensoriale sector.*

(iv) Mixed condensate and bulk matching. [Derived] *[given ansatz].* Λ_{eff} *cannot be sourced by the spinorial sector alone. For a cross-pair (A spinorial, B tensoriale), the interaction potential per pair is* $4\lambda\phi_S^2\phi_T^2$. *There are exactly* 16 *cross-pairs, so the total potential in the symmetric VEV* $\phi_S = \phi_T = \phi_0$ *is:*

$$V_{\mathrm{total}} = 4\lambda \cdot 16 \cdot \phi_0^4 = 64\lambda\phi_0^4.$$

Note on the factor 16*: this counts all cross-pairs (*4 *spinorial modes* $\times$ 4 *cross-pairs each* $= 16$*), not the number of fields; the VEV symmetry* $\phi_S = \phi_T = \phi_0$ *then reduces the sum to a single amplitude. This is a consequence of the* 4*-regular graph structure, not an additional assumption.*

Bulk matching to de Sitter ($\Lambda_{\mathrm{eff}} = 3/r_s^2$, $r_s = 2GM$)*:*

$$\Lambda_{\mathrm{eff}} = 8\pi G \cdot 64\lambda\phi_0^4 \;\Rightarrow\; \phi_0^4 = \frac{3}{2048\pi G^3 M^2 \lambda} \;\propto\; M^{-2}.$$

Larger Black Circles have weaker condensate, consistent with $T_H \propto 1/M$.

(v) Surface: GHY term is mandatory. *The field VEV contributes zero to surface stress at* r_s *(null surface). The Israel values come entirely from the Gibbons–Hawking–York term, which must be added to the action.*

The full 84-mode action.

$$S = \int \sqrt{-g}\Big[\sum_{k=1}^{84} \tfrac{1}{2}(\nabla\varphi_k)^2 + 4\lambda \sum_{(A,B)\in ZD} \phi_A^2\phi_B^2 + \tfrac{\xi}{2}R\sum_k N(\varphi_k)\Big]\, d^4x + \frac{1}{8\pi G}\int_{r=r_s} \sqrt{|h|}\,[K]\, d^3x. \tag{2}$$

The curvature coupling $\xi R/2$ stabilises the potential in de Sitter ($R = 12/r_s^2 > 0$, $m_{\mathrm{eff}}^2 = 2\xi R > 0$ for $\xi > 0$). Via OP 9: the 7D-irrep path gives $m(\mathbf{1}) = 1$ — one invariant 3-form, consistent with two couplings (λ, ξ) of (2).

Remark 7.9 (Full algebraic chain, *[Derived]*)**.** The computations of OP 7–9 and classical Lie theory establish:

$$S_4 \subset \mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2 \subset SO(7) \supset \mathrm{Spin}(7) \leftarrow \mathrm{Cliff}(0,7).$$

$\mathrm{PSL}(2,7)$ is already present in $\mathcal{A}_3$ as a finite subgroup of $G_2 = \mathrm{Aut}(\mathcal{A}_3)$ via two non-conjugate embeddings (OP 9). The $\mathcal{A}_3 \to \mathcal{A}_4$ transition does not create $\mathrm{PSL}(2,7)$: it *transforms* it. The continuous G_2 symmetry is broken at the threshold; the discrete subgroup $\mathrm{PSL}(2,7)$ survives and governs the ZD structure of $\mathcal{A}_4$.

The natural geometric home of $\mathrm{PSL}(2,7)$ is $SO(6)$ (compact, 6-dimensional): it does not embed in $SO(4,1)$ (de Sitter) nor in $SO(4,2)$ (conformal 4D). $SO(6)$ is one dimension below the Clifford representation space $SO(7)$ of Paper 7 [4].

Remark 7.10 (Fano complement structure of $\ker(L_a)$, *[Derived]*)**.** For any ZD-active $a = e_p + \sigma e_{8+k} \in ZD^*(4)$, with Fano line $\ell = \{p, k, p \oplus k\}$:

$$\ker(L_a) = V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)},$$

where $\bar{\ell} = \{1, \ldots, 7\} \setminus \ell$ is the Fano complement (4 points), $V_{\bar{\ell}} = \mathrm{span}\{e_i : \mathrm{Fano}(i) \in \bar{\ell}\} \subset \mathcal{A}_3$, and $V_{\bar{\ell}}^{(8)} = \mathrm{span}\{e_{8+i} : \mathrm{Fano}(i) \in \bar{\ell}\} \subset \mathcal{A}_4 \setminus \mathcal{A}_3$. Verified: $\mathrm{rank}(L_a) = 12$, $\dim\ker = 4$, max residual 2.22×10^{-16} [Python].

The Born-inaccessible content of a Black Circle formed from a lives on the 4 Fano points *not* in ℓ — the geometrically complementary structure of the Fano plane. The action of $S_4 = \mathrm{Stab}(a)$ on $\ker(L_a)$ is a **projective representation** with Schur multiplier $\mathbb{Z}_2$; the underlying linear decomposition is consistent with $\mathbf{1} \oplus \mathbf{3}$, the restriction of the SU(3) representation via the chain $\mathrm{PSL}(2,7) \subset \mathrm{SU}(3)$ (Remark 7.9). *[Conjecture]* pending projective character theory (Schur cover of S_4).

8. Conclusions

8.1. **What this paper has done. Objective (i).** The tower map $T\colon \varrho(n) \mapsto \varrho(n{+}1)$ is an exactly defined discrete dynamical system with no fixed point (Theorem 3.3), unique asymptote $\varrho^* = 0$ (Theorem 3.4), and exact aeon-to-aeon rate converging to 1/2 (Theorem 3.5). Theorem 3.10 proves $\overline{\Delta N}(n) = 4$ for all $n \geq 4$, establishing that the physical map G is exactly isomorphic to T for all levels, with the rate formula Q.3 universal. The 5%/95% ratio is shown not derivable — a precise negative result (Proposition 3.13). Conjecture 3.14 identifies the algebraic condition any future QFT over $\mathcal{A}_4$ must satisfy for this ratio to be a fixed point.

Objective (ii). A two-region metric ansatz: exact Schwarzschild for $r \geq r_s$, de Sitter interior for $r < r_s$ with $L = r_s$. No free parameters beyond r_s. $K(0) = 6\Delta N/r_s^4 = 24/r_s^4$, finite and algebraically fixed by $\Delta N = 4$ (Theorem 4.8). The Frobenius factor 2 manifests identically in the operator norm ratio, the metric derivative ratio, and the Kretschmann ratio (Remark 4.12).

The Barrabès–Israel thin shell at r_s has $\sigma = 0$, $p = 1/(64\pi G^3 M^2)$, and $T_{\text{total}} = 1/(4G)$ universal (OP 2, SymPy-verified). The Vaidya dynamic extension gives $\sigma = 0$ in all three regimes, $r_{\text{AH}} = r_s$ exact, and algebraic discreteness forces thin null shell formation (OP 10). Two falsifiable predictions: **P1** quasi-normal modes near r_s differ from Schwarzschild by the Frobenius factor 2; **P2** no smooth-collapse Black Circle exists.

The symmetry chain (OP 7–9, `GAP`): $\mathrm{Aut}(\mathcal{A}_4)$ does not embed in $SO(4,1)$; de Sitter breaks $\mathrm{PSL}(2,7)$ to S_4 (1/7); $\mathrm{PSL}(2,7)$ has two conjugacy classes in $G_2 = \mathrm{Aut}(\mathcal{A}_3)$; the full chain $S_4 \subset \mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2 \subset SO(7) \leftarrow \mathrm{Cliff}(0,7)$ is verified.

Objective (iii). $n_{\text{res}}(k) = 3{+}k$ algebraically defined. The ZD content formula $|ZD_{\text{dark}}(k)| = |ZD(3{+}k)|$ proved from the Fano filtration theorem and verified numerically for $k = 1, \ldots, 7$ (Theorem 5.4). The sign-parity constraint $\pi_A \pi_B \pi_C = -1$ is carried by the residue. The first-aeon problem of Penrose CCC does not arise (Proposition 5.8). The Fano complement theorem (Remark 7.10) proves that the Born-inaccessible content of a Black Circle lives on the 4 Fano points complementary to the source line — a direct geometric result. The S_4-action on $\ker(L_a)$ is a projective (spinorial) representation, irr8 of $\mathrm{GL}(2,3)$ (`GAP`-verified): the dark-circle content is algebraically fermionic relative to the $\mathcal{A}_3$ observer.

QFT scaffold (OP 11, Direzione 2). The ZD pair graph on the 84 active elements is 4-regular (168 edges, Python-verified). The spinorial sector $\ker(L_a)$ is a free sector (zero internal ZD pairs — independent set, Fano complement geometry). The de Sitter vacuum requires a mixed spinorial-tensoriale condensate: $\phi_0^4 = 3/(2048\pi G^3 M^2 \lambda)$ from 16 cross-pairs with symmetric VEV. The GHY term is mandatory for the surface pressure. The full 84-mode action (2) is the first concrete QFT scaffold for $\mathcal{A}_4$.

Unified result. All three objectives are governed by $T\colon n \mapsto n{+}1$. N is the unique structure satisfying universal persistence, gravitational coupling, and coincidence with the Born Rule at $n = 3$

(Proposition 6.1). At $n = 3$ the gravitational field and the quantum probability measure are the same object; at $n = 4$ they split. $\mathcal{A}_3$ is the unique level where the tower can measure itself (Remark 6.3). The intra-aeon enrichment loop (Remark 6.5) is T operating continuously within each aeon: every Black Circle event injects N-active content, writing the gravitational map that guides subsequent $\mathcal{A}_3$ structure formation.

8.2. **What this paper has not done.** The energy-algebra map E has not been constructed; QFT over $\mathcal{A}_4$ has not been built; the 5%/95% ratio has not been derived; the Black Circle ansatz does not solve the Einstein equations in the interior; $T_{\mu\nu}$ for $\mathcal{A}_4$ matter has not been derived; the conformal crossing mechanism of IIB-IV has not been modelled; the kernel dimension theorem has not been connected to any observable fraction. OP 11 (QFT scaffold) is partially closed: the 4-regular graph structure and spinorial-free sector are derived, but the complete QFT over $\mathcal{A}_4$ remains open. The three identifications of OP 5 required to make the 1/4 irrecoverable fraction a derived result are not established.

8.3. **The minimal standard of evidence.** Steps to convert the IIB-IV programme from conjecture to result:

(1) *Closed by Theorem 3.10.* $\overline{\Delta N}(n) = 4$ for all $n \geq 4$; rate formula Q.3 is universal.
(2) *Closed by OP 2.* $\sigma = 0$, $p = 1/(64\pi G^3 M^2)$, $T_{\text{total}} = 1/(4G)$. SymPy-verified; Frobenius factor is the unique algebraic origin.
(3) *Closed by OP 10.* $\sigma = 0$ universal; $r_{\text{AH}} = r_s$ exact; thin null shell forced by algebraic discreteness; shell pressure applies to null shell only.
3b. *Partially closed by OP 11 (Direzione 2).* ZD pair graph 4-regular (168 edges); spinorial sector free; mixed condensate $\phi_0^4 = 3/(2048\pi G^3 M^2 \lambda)$; GHY mandatory. Full 84-mode action (2) identified.
(4) Construct QFT over $\mathcal{A}_4$ (Open Problem 8).
(5) Test Conjecture 3.14(ii).

Steps 1–3b are accessible with current tools. Until step 4 is completed, this paper establishes the algebraic skeleton of the three objectives — a necessary prerequisite for any future physical theory, but not sufficient alone.

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References

[1] G. Levratti, *Sign-Parity Classification of Zero Divisors in Cayley–Dickson Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20315751. Under review at *Communications in Algebra*.
[2] G. Levratti, *The Informational Identity Boundary: Algebraic Foundations and Physical Conjectures*, Zenodo preprint (2026), doi:10.5281/zenodo.20411845.
[3] G. Levratti, *The Factorization Threshold in Cayley–Dickson Integer Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20442390.
[4] G. Levratti, *A Clifford Representation of the Fano Zero-Divisors*, Zenodo preprint (2026), doi:10.5281/zenodo.20445208.
[5] G. Levratti, *Fano Duality, Projective Orthogonality, and the CSS Bridge*, Zenodo preprint (2026), doi:10.5281/zenodo.20445734.
[6] G. Levratti, *A Quartic Contact Interaction with Fano–CSS Structure*, Zenodo preprint (2026), doi:10.5281/zenodo.20506165.
[7] R. Penrose, *Cycles of Time: An Extraordinary New View of the Universe*, Bodley Head, 2010.

[8] S. A. Hayward, Formation and evaporation of regular black holes, *Phys. Rev. Lett.* **96** (2006), 031103.
[9] W. Israel, Singular hypersurfaces and thin shells in general relativity, *Nuovo Cimento B* **44** (1966), 1–14.
[10] J. C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002), no. 2, 145–205.
[11] I. Juodžbalis, C. Marconcini, F. D'Eugenio, R. Maiolino, et al., A direct black-hole mass measurement in a little red dot at high redshift, *Nature* (May 2026), doi:10.1038/s41586-026-10579-4.

Independent Researcher, Modena, Italy
Email address: giovanni.levratti.research@gmail.com