# CANONICAL QUANTISATION OF THE BLACK CIRCLE INTERIOR

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Abstract. Paper 10 [7] constructed the full 84-mode scalar action $S_{84}$ on the static de Sitter interior of the Black Circle and identified the ZD pair graph on $\mathrm{ZD}^*(\mathcal{A}_4)$ as a 4-regular graph with 168 edges. The present paper initiates canonical quantisation and establishes five main results.

*First* (§3): the Klein–Gordon equation for ZD-active fields reduces to a Regge–Wheeler radial equation. The BC at $r = 0$ is regularity (complete). The Barrabès–Israel analysis at $r_s$ gives two derived results: (a) for $\xi = 0$, BI imposes only $[\varphi_k] = 0$ — no condition on $\partial_r\varphi_k$ (Proposition 3.1, negative result); (b) the Born Rule theorem [1] alone does not force $\chi(r_s) = 0$ for spinorial modes (Proposition 3.5, negative result): the gap is $\pi(\ker L_a) = V_{\bar{\ell}} \neq 0$. For $\xi \neq 0$, a curvature jump $[R]\delta$ generates a shell-field coupling (Remark 3.3).

*Second* (§4): the complete mass spectrum is established via Python and GAP 4.15.1. The ZD pair graph decomposes into 7 isomorphic connected bipartite components (one per Fano class, 12 vertices each), characteristic polynomial $(\lambda+4)(\lambda+2)^2\lambda^6(\lambda-2)^2(\lambda-4)$, full spectrum $\{-4, -2, 0, +2, +4\}$ with multiplicities $\{7, 14, 42, 14, 7\}$, all integers. The $\mathrm{PSL}(2, 7)$ permutation character on 84 elements decomposes as $\chi_1 \oplus \chi_3 \oplus \chi_3' \oplus 4\chi_6 \oplus 3\chi_7 \oplus 4\chi_8$. The **7 tachyonic modes** ($\mu = -4$) are precisely the $\mathrm{irr}_7$ representation — the same 7-dimensional irrep governing Papers 7 and 10.

*Third* (§5): the de Sitter-invariant condensate is derived from first principles. The unique classical scalar field on $\mathrm{dS}_4$ invariant under $\mathrm{SO}(1, 4)$ and self-consistent with the Einstein equation is $\varphi_0 = C_0 = \text{const}$ with $C_0^4 = 3/(2048\pi G^3 M^2\lambda)$ (Theorem 5.4, *[Derived]*, OP 11.3 closed). As an immediate corollary (Proposition 5.11), the stress-energy tensor $T^{\text{bulk}}_{\mu\nu}|_{C_0} = -g_{\mu\nu}\cdot 64\lambda C_0^4$ reproduces $\Lambda_{\text{eff}} = 3/r_s^2$ exactly — the **first explicit $T_{\mu\nu}$ for $\mathcal{A}_4$ matter content**.

*Fourth* (§5): the BF stability bound for Level 1 modes ($\mu = -4$, $m^2 < 0$) holds if and only if $\lambda G M^2 < 27\pi/128$ (Proposition 5.7, *[Derived]* analytically). This is a conditional result, not universal: for $M > M_{\text{BF}} \equiv \sqrt{27\pi/(128\lambda G)}$ the $\mathrm{irr}_7$ sector is BF-unstable. Levels 2–5 ($m^2 \geq 0$) satisfy the BF bound unconditionally.

*Fifth* (§5): a *[Derived]* negative result (OP 11.4 closed): the space of $\mathrm{PSL}(2, 7)$-invariant symmetric bilinear forms on $\mathbb{R}^{84}$ has dimension **28** (GAP 4.15.1). Therefore $\xi$ cannot be fixed by $\mathrm{PSL}(2, 7)$ symmetry. Conjecture 6.5 ($\xi = 0$ from algebraic minimality) is correctly labelled as a conjecture.

**Epistemic key.** *[Derived]* — follows from Papers 1–10 and established mathematics. *[Conjecture]* — precise statement; additional assumptions stated explicitly. *[Speculation]* — structurally motivated; not currently falsifiable.

## 1. Introduction

1.1. **What Paper 10 established.** Paper 10 [7] established three foundations for this paper.

**(a) The Black Circle metric ansatz.** *[Conjecture]* under IIB-I/II.

$$f_{\text{in}}(r) = \frac{r^2}{r_s^2} - 1, \quad ds^2 = -f_{\text{in}}\, c^2\, dt^2 + f_{\text{in}}^{-1}\, dr^2 + r^2\, d\Omega^2. \tag{1}$$

Regular at $r = 0$ ($K(0) = 6\Delta N/r_s^4$ finite); $C^0$ at $r_s$; $C^1$ jump = Frobenius factor 2 [4].

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**(b) The full 84-mode scalar action.** *[Derived]* [7].

$$S_{84} = \int \sqrt{-g} \left[ \sum_{k=1}^{84} \tfrac{1}{2}(\nabla\varphi_k)^2 + 4\lambda \sum_{(A,B)\in \mathrm{ZD}} \phi_A^2\phi_B^2 + \tfrac{\xi}{2} R \sum_k N(\varphi_k) \right] d^4x + \frac{1}{8\pi G}\int_{r_s} \sqrt{|h|}\,[K]\,d^3x \tag{2}$$

with $\lambda$ unique [6] and $\xi$ a free gravitational parameter (§6).

**(c) The ZD pair graph and condensate.** *[Derived]* [7]. The ZD pair graph on $\mathrm{ZD}^*(\mathcal{A}_4)$ is 4-regular (168 edges). The spinorial sector $\ker(L_a)$ is an independent set (zero internal ZD pairs). De Sitter vacuum: mixed condensate

$$\phi_0^4 = \frac{3}{2048\pi G^3 M^2\,\lambda}. \tag{C0}$$

1.2. **What this paper does.**

- Applies Barrabès–Israel formalism to $S_{84}$ at $r_s$: derived negative result for $\xi = 0$ — no Neumann condition (§3; OP 11.1 partially closed)
- Computes the complete mass spectrum with full $\mathrm{PSL}(2,7)$ decomposition (§4)
- Identifies the bipartite graph structure and $\mathrm{irr}_7$/tachyonic-mode connection (§4)
- Derives the de Sitter-invariant condensate $C_0$ from $\mathrm{SO}(1,4)$ (§5; OP 11.3 closed)
- Derives the BF stability condition analytically: Levels 2–5 unconditionally, Level 1 conditionally on $\lambda G M^2 < 27\pi/128$ (§5)
- Establishes that $\xi$ is not fixed by $\mathrm{PSL}(2,7)$ symmetry (§6; OP 11.4 closed, negative result)
- Derives $T_{\mu\nu}$ at the condensate (§5)

1.3. **What this paper does not do.** This paper does not construct a Hilbert space over $\mathcal{A}_4$ (OP 11.6); does not compute QNM frequencies (requires OP 11.1); does not quantise the fermionic $\ker(L_a)$ sector (OP 11.5); does not fix $\xi$ algebraically (OP 11.4 closed as negative result).

## 2. Algebraic Foundation

All results in this section are proved in Papers 1–10.

**[P1]** [1]**.** Born Rule: unique solution for $n \le 3$, undefined for $n \ge 4$. $|ZD(4)| = 336$ ordered pairs. Sign-parity: $\pi_A \cdot \pi_B \cdot \pi_C = -1$ for all ZD triples.

**[P6]** [3]**.** $\Delta N(a,b) = 4$ for every ZD-active pair in $\mathcal{A}_4$. Exact.

**[P7]** [4]**.** $\|L_a\|_F^2 = 2^n N(a)$. Frobenius ratio $n = 4/n = 3 = 2$. Clifford map $\rho$: $\mathrm{ZD}^*(4) \to \mathrm{Cliff}(0,7)$ uses the $\mathrm{irr}_7$ action of $\mathrm{PSL}(2,7)$ on $\mathbb{R}^7$; 7D-irrep embedding with $m(\mathbf{1}) = 1$ [7].

**[P9]** [6]**.** $\mathcal{L}_{\text{int}} = \lambda \Delta N(\varphi,\psi)$, $\lambda$ unique. ZD-active modes algebraically massless. Fano conservation: $m(a) = m(b)$ for all ZD pairs.

**[P10]** [7]**.** Black Circle ansatz; $S_{84}$; mixed condensate (C0); Israel shell $\sigma = 0$, $p = 1/(64\pi G^3 M^2)$, $T_{\text{total}} = 1/(4G)$; symmetry chain $S_4 \subset \mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2 \subset SO(7) \leftarrow \mathrm{Cliff}(0,7)$, GAP-verified; $\mathrm{irr}_8 = GL(2,3)$ on $\ker(L_a)$, GAP-verified; CSS bridge [5].

## 3. Canonical Quantisation on the Static de Sitter Background

3.1. **Klein–Gordon equation.** A ZD-active scalar field $\varphi_k = \phi_k(x)\cdot A_k$ is algebraically massless [6]. Its equation of motion in the static de Sitter interior is *[Derived]* from (2):

$$\Box_{\mathrm{dS}}\,\varphi_k - \xi R\,\varphi_k = (\text{interaction terms}), \qquad R = \frac{12}{r_s^2}. \tag{3}$$

The curvature mass is $m^2_{\text{curv}} = \xi R = 12\xi/r_s^2$.

3.2. **Radial equation.** Mode decomposition $\varphi_k = e^{-i\omega t} Y_{lm}\chi(r)/r$ gives *[Derived]*:

$$-\frac{d^2\chi}{dr_*^2} + V_{\text{eff}}(r)\,\chi = \omega^2\chi, \tag{4}$$

with tortoise coordinate

$$r_* = -\frac{r_s}{2}\ln\frac{r_s+r}{r_s-r} = -r_s\,\text{arctanh}(r/r_s), \quad r_* \in (-\infty, 0]: \quad r_* \to 0 \text{ as } r \to 0;\; r_* \to -\infty \text{ as } r \to r_s^-,$$

and effective potential

$$V_{\text{eff}}(r) = f(r)\left[\frac{l(l+1)}{r^2} + \frac{2}{r_s^2} + \frac{12\xi}{r_s^2}\right]. \tag{5}$$

The term $2/r_s^2$ arises from $f'(r)/r = 2/r_s^2$ and is present even for $\xi = 0$.

3.3. **Boundary conditions. At $r = 0$.** *[Derived]* from [7]: $K(0)$ finite implies $\chi \sim r^{l+1}$. Complete.
**At $r = r_s$ — Barrabès–Israel analysis.** *[Derived]* — partial closure of OP 11.1.

The null shell at $r_s$ separates Schwarzschild (outside) from de Sitter (inside). In Eddington–Finkelstein coordinates $(v, r)$, $ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2$, the metric is $C^0$ at $r_s$ ($f(r_s) = 0$ from both sides).

**Proposition 3.1** (*[Derived]*, BI junction conditions, $\xi = 0$, [8])**.** *The Barrabès–Israel formalism applied to $S_{84}$ at the null shell $r = r_s$ for $\xi = 0$ gives:*

$$[\varphi_k] = 0 \quad \textit{(continuity)}, \tag{6}$$

$$[\partial_v\varphi_k] = 0 \quad \textit{(null-derivative continuity)}. \tag{7}$$

*For modes $e^{-i\omega v}F(r)$, (7) follows from (6). No condition on $\partial_r\varphi_k$ follows from BI for $\xi = 0$.*

*Proof.* The critical term $\partial_r(r^2 f\,\partial_r\varphi_k)$: writing $A = r^2 f$ (continuous, since $f(r_s) = 0$) and $B = \partial_r\varphi_k$ (potentially with a jump $[B]$): $\partial_r(AB) = A'B + A(B'_{\text{smooth}} + [B]\delta)$. The delta term is $A(r_s)\cdot[B]\cdot\delta = r_s^2\cdot 0\cdot[B]\cdot\delta = 0$. The factor $r^2 f|_{r_s} = 0$ eliminates any distributional contribution from a jump in $\partial_r\varphi_k$. The remaining EF terms produce no delta for $[\varphi_k] = 0$. Conditions (6)–(7) follow. □

**Remark 3.2** (*[Derived]*)**.** A calculation in static $(t, r)$ coordinates using the jump $[f'] = -1/r_s$ [7] would suggest a Neumann condition. This is incorrect: the static coordinates are singular at $r_s$ ($1/f$ diverges), and $r^2 f\,\partial_r\varphi_k$ vanishes before the jump in $f'$ can contribute. The EF calculation in Proposition 3.1 is definitive.

**Remark 3.3** (*[Derived]*)**.** For $\xi \neq 0$, the term $\xi R\varphi_k$ contributes at the shell because $R$ has a distributional part $[R]_{\text{null}}\delta$ from the curvature jump $[f'']$. The BI source becomes $\mathcal{S}_k = \xi[R]_{\text{null}}\varphi_k(r_s) \neq 0$, modifying (7): the field couples to the shell geometry. This is an independent argument for $\xi = 0$: only at $\xi = 0$ does the scalar field decouple from the shell at the level of field equations (Conjecture 6.5).

**Remark 3.4** (*[Derived]*, partial; *[Conjecture]* sector-specific)**.** The BI result leaves $\partial_r\varphi_k|_{r_s}$ free. The complete BC depends on the sector. *Tensoriale modes* (Levels 1,2,4,5): QNM frequencies require matching the interior de Sitter solution to the exterior Schwarzschild QNM at $r_s$ via continuity and the Wronskian condition — a coupled eigenvalue problem. (OP 11.1, refined.) *Spinorial modes* $\ker(L_a)$ (Level 3): under IIB-I/II [2], Born-inaccessible modes cannot exit, giving Dirichlet BC $\chi(r_s) = 0$.

**Proposition 3.5** (*[Derived]*)**.** *The Born Rule theorem [1] alone does not imply the Dirichlet boundary condition $\chi(r_s) = 0$ for spinorial modes* $\ker(L_a)$*.*

*Proof.* The Born Rule theorem establishes that no canonical probability assignment exists for $\ker(L_a)$ modes in $\mathcal{A}_3$ language: for $a \in \ker(L_a)$ and $b \in \mathcal{A}_4$, $N(ab) = 0$ forces $P(a|b) = 0$ even for $a, b \neq 0$, contradicting $P(a|a) = 1$ [1]. This states that any non-zero transmission amplitude $T \neq 0$ cannot be interpreted as a Born probability by an $\mathcal{A}_3$ observer. It does not establish $T = 0$.

The gap is algebraically precise: $\ker(L_a) = V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)}$ where $V_{\bar{\ell}} \subset \mathcal{A}_3$ [7, Rem. 7.10]. The projection $\pi\colon \mathcal{A}_4 \to \mathcal{A}_3$ satisfies $\pi(\ker L_a) = V_{\bar{\ell}} \neq 0$: the spinorial modes have a non-zero $\mathcal{A}_3$ component; no algebraic statement annihilates this projection. □

**Remark 3.6.** Three ingredients that would close the gap: (i) QFT over $\mathcal{A}_4$ (OP 11.6): an explicit $S$-matrix would determine $T$ directly. (ii) IIB-I/II [2]: the confinement principle directly implies $T = 0$. (iii) A projection theorem: show that despite $V_{\bar{\ell}} \neq 0$, the overlap of $\ker(L_a)$ modes with outgoing $\mathcal{A}_3$ plane waves vanishes via the sign-parity constraint $\pi_A \pi_B \pi_C = -1$ [1]. Option (iii) is the most accessible and is proposed as a target for the next paper in the series.

## 4. Complete Mass Spectrum and PSL(2, 7) Structure

4.1. **Mass matrix.** At uniform condensate $\varphi_k = C_0 + \delta\varphi_k$, the Hessian of $V = 4\lambda \sum_{(A,B)\in \mathrm{ZD}} \phi_A^2 \phi_B^2$ is *[Derived]*:

$$H = 16\lambda C_0^2\,(2I_{84} + A), \tag{8}$$

where $A$ is the $84 \times 84$ adjacency matrix of the ZD pair graph. Mass eigenvalues:

$$m_i^2 = 16\lambda C_0^2\,(2 + \mu_i) + \frac{12\xi}{r_s^2}. \tag{9}$$

**Proposition 4.1** (*[Derived]*)**.** *Each of the* 4 *spinorial modes in* $\ker(L_a)$ *has* $m_s^2 = 32\lambda C_0^2 + 12\xi/r_s^2$, *arising entirely from the* 16 *cross-pair couplings (*$A_{SS} = 0$ [7]*).*

4.2. **Graph structure.**

**Theorem 4.2** (*[Derived]*, Python)**.** *The ZD pair graph on* $\mathrm{ZD}^*(\mathcal{A}_4)$ *has* 7 *isomorphic connected components, one per Fano class (algebraically necessary by Fano conservation* [6, Prop. 1.9]*). Each component is a connected* 4*-regular bipartite graph on* 12 *vertices (*$|A| = |B| = 6$*) with characteristic polynomial:*

$$p(\lambda) = (\lambda + 4)(\lambda + 2)^2 \lambda^6 (\lambda - 2)^2 (\lambda - 4). \tag{10}$$

*Full adjacency spectrum:*

| $\mu_i$ | *Multiplicity* | *Per Fano class* |
|---|---|---|
| −4 | 7 | 1 |
| −2 | 14 | 2 |
| 0 | 42 | 6 |
| +2 | 14 | 2 |
| +4 | 7 | 1 |

*All eigenvalues integers. Graph Laplacian eigenvalues* $\{0, 2, 4, 6, 8\}$*; algebraic connectivity (Fiedler)* $= 2$ *per component.*

**Corollary 4.3** (*[Derived]*)**.** *The ZD pair graph is bipartite. Symmetric spectrum confirmed by BFS* 2*-colouring of all* 7 *components.*

### 4.3. PSL(2,7) irreducible decomposition.

**Theorem 4.4** (*[Derived]*, GAP 4.15.1)**.** *The permutation character of* $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ *on the* 84 *ZD-active elements (stabiliser* $H$ *of order* 2*) decomposes as:*

$$\chi_{\mathrm{perm}} = \chi_1 \oplus \chi_3 \oplus \chi_3' \oplus 4\chi_6 \oplus 3\chi_7 \oplus 4\chi_8. \tag{11}$$

*Dimension check:* $1+3+3+24+21+32 = \mathbf{84}$ ✓*. Permutation character on the* 6 *conjugacy classes:* $[84, 0, 0, 0, 4, 0]$.

**Theorem 4.5** (*[Derived]*, Python + GAP 4.15.1)**.** *The eigenvalue–irrep correspondence is:*

| $\mu_i$ | *Mult* | *Isotypic component* | *Notes* |
|---|---|---|---|
| $+4$ | 7 | $\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_3'$ | *condensate direction* |
| $+2$ | 14 | $\mathrm{irr}_7$ *(2 eigvec of* $M_7$*)* | *stable Fano sector* |
| 0 | 42 | $3 \cdot \mathrm{irr}_6 \oplus 3 \cdot \mathrm{irr}_8$ | *spinorial + bulk* |
| $-2$ | 14 | $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ *(1 each)* | *geometric-mass sector* |
| $-4$ | 7 | $\mathrm{irr}_7$ *(1 eigvec of* $M_7$*)* | *complementary series (cond. on BFcond)* |

*Derivation.* $M_7$ *is the* $3\times 3$ *matrix of* $A$ *on the* 3 *copies of* $\mathrm{irr}_7$*. From* $\mathrm{trace}(M_7) = 0$ *and multiplicities:* $\det(\lambda I - M_7) = \lambda^3 - 12\lambda + 16 = (\lambda+4)(\lambda-2)^2$*, roots* $\{-4, +2, +2\}$*. Similarly* $M_6$ *and* $M_8$ *(*$4\times 4$ *each) have characteristic polynomial* $\lambda^3(\lambda+2)$*.*

**Remark 4.6** (*[Derived]*)**.** $\mathrm{irr}_7$ governs three distinct structures in the series: Paper 7 [4] (Clifford representation $\rho$: $\mathrm{ZD}^*(4) \to \mathrm{Cliff}(0,7)$, irreducible action on $\mathbb{R}^7$); Paper 10 [7] (7D-irrep path with $m(\mathbf{1}) = 1$; $S_4 \subset \mathrm{PSL}(2,7)$ breaking at $r_s$); this paper (tachyonic sector $\mu = -4$ and stable Fano sector $\mu = +2$). The same 7-dimensional representation connects the Clifford structure, the symmetry-breaking geometry, and the mass spectrum. This is a coherent algebraic thread, not a numerical coincidence.

**Theorem 4.7** (*[Derived]*)**.** *The* 7 *modes with* $\mu = -4$ *are in bijection with the* 7 *points of* $\mathrm{PG}(2,\mathbb{F}_2)$*, one per Fano class. They are the anti-uniform eigenvectors of each Fano-class component (*$\pm 1$ *on the bipartition) — the directions in field space along which the PSL*$(2,7)$*-invariant condensate* $C_0$ *is statically non-constant.*

### 4.4. Physical mass levels.

Complete table ($m_i^2 = 16\lambda C_0^2(2+\mu_i) + 12\xi/r_s^2$, $\xi = 0$ column):

| Level | $\mu_i$ | Irreducibles | $2{+}\mu_i$ | Mult | $m^2$ at $\xi{=}0$ |
|---|---|---|---|---|---|
| 1 | $-4$ | $\mathrm{irr}_7$ | $-2$ | 7 | $-32\lambda C_0^2$ |
| 2 | $-2$ | $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ | 0 | 14 | $\mathbf{0}$ |
| 3 | 0 | $3\cdot\mathrm{irr}_6 \oplus 3\cdot\mathrm{irr}_8$ | $+2$ | 42 | $+32\lambda C_0^2$ |
| 4 | $+2$ | $\mathrm{irr}_7$ | $+4$ | 14 | $+64\lambda C_0^2$ |
| 5 | $+4$ | $\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_3'$ | $+6$ | 7 | $+96\lambda C_0^2$ |

Level 2 is **exactly massless** at $\xi = 0$. Level 5 is the condensate direction.

## 5. The Condensate, BF Stability, and $T_{\mu\nu}$

### 5.1. Static ODE analysis.

**Proposition 5.1** (*[Derived]*, Python)**.** *Near* $r = r_s$*, setting* $s = 1 - r/r_s$*: the condensate ODE has two independent regular Frobenius solutions* $u_1 = 1$ *(constant) and* $u_2 = s^2$ *(quadratic); no singular solution exists. The ODE with only* $\varphi'(0) = 0$ *has a 1-parameter family of regular solutions parameterised by* $a = \varphi_0(0)$*. The slope at* $r_s$ *is fixed:* $\varphi_0'(r_s) = 16\lambda\varphi_0(r_s)^3/r_s$*.*

**Proposition 5.2** (*[Derived]*)**.** *For any $\xi \geq 0$, the uniform field $\varphi_k \equiv \phi_0 \neq 0$ does not satisfy the EOM from $S_{84}$.*

*Proof.* EOM at $\phi_0 = \text{const}$: $32\lambda\phi_0^3 + (12\xi/r_s^2)\phi_0 = 0$. For $\phi_0 \neq 0$, $\lambda > 0$, $\xi \geq 0$: the left side satisfies $32\lambda\phi_0^3 + (12\xi/r_s^2)\phi_0 \geq 32\lambda\phi_0^3 > 0$, contradicting the requirement that it equals zero. □

**Theorem 5.3** (*[Derived]*, Python, Nodal Tension)**.** *For $\xi = 0$, the regular Lorentzian solutions with $\varphi'(0) = 0$ satisfy:*

- (i) *Nodelessness bound: $a < a_{\text{node}} \approx 0.2345$ (units $r_s = \lambda = 1$); $a_{\text{node}} \approx 0.949\, C_0$.*
- (ii) *Energy-matching: $\langle\phi_0^4\rangle_V = C_0^4$ requires $a_* \approx 0.3905 \approx 1.58\, C_0$.*
- (iii) *$a_* > a_{\text{node}}$: the energy-matching solution has a node. Nodeless maximum $\approx 30.5\%$ of the $\Lambda_{\text{eff}}$ target.*

*No Lorentzian static solution simultaneously satisfies nodelessness and energy-matching. This is not a deficiency but the correct result: the physical condensate is not a static-patch solution (Theorem 5.4).*

### 5.2. **The de Sitter-invariant condensate.**

**Theorem 5.4** (*[Derived]*)**.** *The unique classical scalar field configuration on $\mathrm{dS}_4$ satisfying:*

- (i) *invariance under the full de Sitter isometry group $\mathrm{SO}(1,4)$, and*
- (ii) *self-consistency with the Einstein equation $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$,*

*is $\varphi_0 = C_0 = \text{const}$, where:*

$$C_0^4 = \frac{3}{2048\pi G^3 M^2\, \lambda}. \tag{12}$$

*Proof. (i) Uniqueness of constant.* $\mathrm{SO}(1,4)$ acts transitively on $\mathrm{dS}_4$: any scalar $f\colon \mathrm{dS}_4 \to \mathbb{R}$ satisfying $f(g\cdot x) = f(x)$ for all $g \in \mathrm{SO}(1,4)$ must be constant.

*(ii) Value.* For $\varphi_0 = \text{const}$: $\partial_\mu\varphi_0 = 0$ and $T_{\mu\nu} = -g_{\mu\nu}\cdot V(\varphi_0) = -g_{\mu\nu}\cdot 64\lambda\varphi_0^4$. The de Sitter geometry gives $G_{\mu\nu} = -3/r_s^2 \cdot g_{\mu\nu}$. Einstein's equation:

$$\frac{3}{r_s^2} = 8\pi G\cdot 64\lambda\varphi_0^4 = 512\pi G\lambda\varphi_0^4.$$

With $r_s = 2GM$: $\varphi_0^4 = 3/(2048\pi G^3 M^2\lambda) = C_0^4$. □

**Remark 5.5** (*[Derived]*)**.** The static patch has isometry group $\mathrm{SO}(2)\times\mathrm{SO}(3) \subset \mathrm{SO}(1,4)$, which does not force constancy — correctly explaining why the static ODE admits non-constant solutions. The Nodal Tension (Theorem 5.3) signals that these are not the physical vacuum. The static ODE is the correct equation for *fluctuations* around $C_0$, not for $C_0$ itself.

**Remark 5.6** (*[Derived]*)**.** $C_0 = \text{const}$ is the unique $\mathrm{PSL}(2,7)$-invariant field configuration, corresponding to the $\mathrm{irr}_1$ isotypic component (Theorem 4.4). The de Sitter isometry $\mathrm{SO}(1,4)$ and the Fano symmetry $\mathrm{PSL}(2,7)$ independently select the same configuration.

### 5.3. **Breitenlohner–Freedman stability.**

**Proposition 5.7** (*[Derived]*, analytic)**.** *For $\xi = 0$, the Breitenlohner–Freedman stability bound for $\mathrm{dS}_4$ [11],*

$$m_i^2 \geq m_{\text{BF}}^2 \equiv -\frac{9}{4r_s^2},$$

*holds unconditionally for Levels 2–5 ($m_i^2 \geq 0$). For Level 1 ($m_1^2 = -32\lambda C_0^2 < 0$), it holds if and only if:*

$$\lambda G M^2 < \frac{27\pi}{128}. \tag{BFcond}$$

*Proof.* The condition $m_1^2 > m_{\rm BF}^2$ is $32\lambda C_0^2 < 9/(4r_s^2)$. Squaring:

$$[32\lambda]^2 \cdot \frac{3}{2048\pi G^3 M^2 \lambda} < \frac{81}{256 G^4 M^4}.$$

Simplifying the left side: $\frac{1024\lambda^2 \cdot 3}{2048\pi G^3 M^2 \lambda} = \frac{3\lambda}{2\pi G^3 M^2}$. The condition becomes $\frac{3\lambda}{2\pi G^3 M^2} < \frac{81}{256 G^4 M^4}$, i.e. $\lambda G M^2 < 81 \cdot 2\pi/(256 \cdot 3) = 27\pi/128$. □

**Remark 5.8** (*[Derived]*)**.** The condition (BFcond) constrains the product $\lambda G M^2$. For fixed $\lambda$, it fails for sufficiently massive Black Circles $M > M_{\rm BF} \equiv \sqrt{27\pi/(128\lambda G)}$. For $M > M_{\rm BF}$, Level 1 modes ($\mathrm{irr}_7$, $\mu = -4$) are BF-unstable: their effective mass satisfies $m_1^2 < m_{\rm BF}^2$, implying exponential growth in the de Sitter interior; these modes are not consistently quantisable with the standard Bunch-Davies vacuum [10] for such Black Circles. The existence of $M_{\rm BF}$ is a **falsifiable prediction**: sufficiently massive Black Circles have an unstable $\mathrm{irr}_7$ sector. The BF stability of Level 1 is a conditional derived result, not a universal one.

**Corollary 5.9** (*[Derived]*, conditional on (BFcond))**.** *For $\lambda G M^2 < 27\pi/128$, the 7 Level-1 modes ($\mu = -4$, $\mathrm{irr}_7$) with $m_{\rm BF}^2 < m_1^2 < 0$ belong to the **complementary series** representations of* $\mathrm{SO}(1,4)$ [10]*: consistently quantisable in de Sitter with a well-defined Bunch-Davies vacuum; not physically unstable. Levels 2–5 ($m_i^2 \geq 0$) belong to the **massive/principal series** unconditionally. Level 2 ($m_2^2 = 0$, $\xi = 0$) is exactly massless — the boundary between the two series.*

**Remark 5.10** (*[Derived]*)**.** The complementary-series assignment of Level 1 (when (BFcond) holds) is algebraically coherent: $\mathrm{irr}_7$ is the unique $\mathrm{PSL}(2,7)$ irrep appearing in $M_7$ at both $\mu = -4$ (complementary) and $\mu = +2$ (massive). The 7-dimensional Fano irrep straddles the massless threshold, connecting the two de Sitter series.

### 5.4. $T_{\mu\nu}$ at the condensate.

**Proposition 5.11** (*[Derived]* from $S_{84}$, $\xi = 0$)**.** *For $\xi = 0$ and $\varphi_k = C_0 = \mathrm{const}$:*

$$T_{\mu\nu}^{\rm bulk}\big|_{C_0} = -g_{\mu\nu} \cdot 64\lambda C_0^4. \tag{13}$$

*Einstein's equation gives:*

$$\Lambda_{\rm eff} = 512\pi G\lambda C_0^4 = 512\pi G\lambda \cdot \frac{3}{2048\pi G^3 M^2 \lambda} = \frac{3}{r_s^2} \qquad \checkmark \tag{14}$$

***This is the first explicit derivation of $T_{\mu\nu}$ for $\mathcal{A}_4$ matter content.***

**Remark 5.12** (*[Derived]*, [7])**.** The Israel shell contributes $\sigma = 0$, $p = 1/(64\pi G^3 M^2)$, $T_{\rm total} = 1/(4G)$ from the GHY term [9] — independent of $\varphi_k$.

## 6. The Curvature Coupling $\xi$ — A Derived Negative Result

**Theorem 6.1** (*[Derived]*, GAP 4.15.1)**.** *The space of $PSL(2,7)$-invariant symmetric bilinear forms on $\mathbb{R}^{84}$ has dimension:*

$$n_{\rm sym} = \left\langle \mathrm{Sym}^2(\chi_{\rm perm}),\, \chi_1 \right\rangle = \mathbf{28}.$$

*Proof.* GAP 4.15.1: `SymmetricParts(ctG,[chi],2)` and `ScalarProduct(ctG,chi_sym2,irr[1])`. Frobenius-Schur indicators: $[1,0,0,1,1,1]$. Check: $n_{\rm sym} + n_{\rm alt} = 28 + 16 = 44 = \langle \chi_{\rm perm}, \chi_{\rm perm} \rangle$ ✓.

The 28 forms distribute (indices follow GAP's internal ordering; $\mathrm{irr}_4^{\rm GAP} = \mathrm{irr}_6$ of the text, dim 6; $\mathrm{irr}_5^{\rm GAP} = \mathrm{irr}_7$ of the text, dim 7; $\mathrm{irr}_6^{\rm GAP} = \mathrm{irr}_8$ of the text, dim 8):

| Block | FS | Copies $n_\rho$ | Forms $n_\rho(n_\rho{+}1)/2$ |
|---|---|---|---|
| $\mathrm{irr}_1$ (dim 1) | +1 | 1 | 1 |
| $\mathrm{irr}_3 \oplus \mathrm{irr}_3'$ (complex pair, dim 3) | 0 | 1+1 | 1 |
| $\mathrm{irr}_6$ (dim 6) | +1 | 4 | 10 |
| $\mathrm{irr}_7$ (dim 7) | +1 | 3 | 6 |
| $\mathrm{irr}_8$ (dim 8) | +1 | 4 | 10 |
| **Total** | | | **28** |

□

**Remark 6.2** (*[Derived]*)**.** The specific form $Q = \sum_k N(\varphi_k) = 2\sum_k \phi_k^2$ used in $S_{84}$ corresponds to $I_{84}$ — one of 28 independent PSL$(2,7)$-invariant choices. It is algebraically motivated (it uses the $\mathcal{A}_4$ norm $N$) but is not singled out by PSL$(2,7)$ symmetry alone.

**Proposition 6.3** (*[Derived]*)**.** *The coupling $\xi$ is a free gravitational parameter. PSL$(2,7)$ symmetry constrains the quadratic invariant to one of 28 independent choices but does not select a preferred value of $\xi$, including $\xi = 0$.*

**Remark 6.4** (*[Derived]*, GAP)**.** There are $n_{\mathrm{alt}} = 16$ independent PSL$(2,7)$-invariant antisymmetric bilinear forms on $\mathbb{R}^{84}$, which could appear in topological extensions of $S_{84}$. They do not enter the current action.

**Conjecture 6.5** (*[Conjecture]*, $\xi = 0$ from algebraic minimality)**.** *Although $\xi$ cannot be fixed by PSL$(2,7)$ symmetry (Proposition 6.3), the choice $\xi = 0$ is algebraically motivated by two independent arguments.*

(i) Spectral purity. *At $\xi = 0$: mass ratios $m_i^2/m_j^2 = (2{+}\mu_i)/(2{+}\mu_j)$ are dimensionless algebraic numbers, independent of $M$, $G$, $r_s$.*

(ii) Exact masslessness. *Level 2 ($\mu = -2$, $\mathrm{irr}_6 \oplus \mathrm{irr}_8$) is exactly massless: $m_2^2 = 0$ independent of all gravitational parameters.*

*Note on the BF argument. A third argument — BF stability at $\xi = 0$ imposes no condition on $\xi$ — is incorrect. The BF condition for Level 1 at $\xi = 0$ is* (BFcond)*, a gravitational constraint independent of $\xi$. Moreover, for $\lambda GM^2 > 27\pi/128$, a value $\xi > 0$ would* restore *BF stability, providing an argument for $\xi > 0$ rather than $\xi = 0$. The BF structure does not favour $\xi = 0$.*

*OP 11.4 is **closed as a derived negative result**: symmetry does not fix $\xi$.*

## 7. Physical Consequences

7.1. **Derived predictions. (i) Five algebraically fixed mass levels.** *[Derived]*, Theorem 4.5. Multiplicities $\{7, 14, 42, 14, 7\}$ determined entirely by the PSL$(2,7)$ permutation representation. At $\xi = 0$: all masses determined by $\lambda$, $G$, $M$ alone with algebraic ratios from the ZD graph.

**(ii)** $\mathrm{irr}_7$ **governs both sectors simultaneously.** *[Derived]*, Theorem 4.5. Characteristic polynomial of $M_7$ is $(\lambda+4)(\lambda-2)^2$: the same 7-dimensional irrep contains the complementary-series Level 1 ($\mu = -4$) and the massive Level 4 ($\mu = +2$).

**(iii) QNM prediction.** *[Conjecture]* under IIB-I + Theorem 4.2. The Frobenius factor 2 at $r_s$ predicts:

$$\omega_{\mathrm{QNM}}^{(\mathrm{int})} \approx \sqrt{2}\,\omega_{\mathrm{QNM}}^{(\mathrm{ext})}, \tag{15}$$

up to corrections from $l$ and mass level. Becomes *[Derived]* once OP 11.1 closes.

**(iv) Super-selection by** $\mathrm{irr}_1$**.** *[Derived]* from Theorem 5.4 + [1]. The physical condensate occupies $\mathrm{irr}_1$ (trivial, PSL$(2,7)$-invariant). The $\ker(L_a)$ modes (Level 3, $\mu = 0$) form a super-selection sector: no Born-accessible probability assignment exists [1], making them indistinguishable to $\mathcal{A}_3$ observers. (Connection to CCC: [12].)

**(v) Two de Sitter series.** *[Derived]*, Corollary 5.9 — conditional. Level 1: complementary series ($m^2 < 0$) when (BFcond) holds. Levels 2–5: massive/principal series ($m^2 \geq 0$), unconditional. Level 2 (exactly massless, $\xi = 0$): boundary of the two series. The mass stratification is governed algebraically by the ZD graph spectrum; for Level 1, additionally by the gravitational condition (BFcond).

## 8. Open Problems

**OP 11.2** — RESOLVED by Theorem 4.2 [Python].

**OP 11.3** — RESOLVED by Theorem 5.4 [SO$(1,4)$ + GR, numerically verified].

**OP 11.4** — RESOLVED as a **derived negative result** by Theorem 6.1 [GAP 4.15.1]: $n_{\text{sym}} = 28$; $\xi$ is not fixed by PSL$(2,7)$ symmetry. Conjecture 6.5 ($\xi = 0$) remains open as a conjecture from algebraic minimality.

**OP 11.7** — RESOLVED by Theorem 4.4 [GAP] and Corollary 4.3 [Python].

**OP 11.1** (QNM boundary conditions at $r_s$ — partially closed by Propositions 3.1 and 3.5). For $\xi = 0$, BI gives only $[\varphi_k] = 0$ — no condition on $\partial_r \varphi_k$. *Tensoriale sector:* solve the coupled Regge–Wheeler eigenvalue problem with matching at $r_s$; yields complex QNM frequencies. *Spinorial sector:* Proposition 3.5 establishes that the Born Rule theorem [1] alone does not force $\chi(r_s) = 0$ — a derived negative result. The gap $\pi(\ker L_a) = V_{\bar{\ell}} \neq 0$ is precisely characterised with three resolution paths; option (iii) (sign-parity overlap argument) is the target for the next paper in the series.

**OP 11.5** (Fermionic quantisation of $\ker(L_a)$). Construct the Fock space for the 4 spinorial modes ($\text{irr}_8$ of $GL(2,3)$, Schur multiplier $\mathbb{Z}_2$ of $S_4$, Level 3, $\mu = 0$, $m^2 = 32\lambda C_0^2$ at $\xi = 0$).

**OP 11.6** (Non-associativity obstruction = OP 10.8 of [7]). State precisely whether the CCR $[\varphi_k(x), \pi_k(x')] = i\hbar\delta^3(x - x')$ is valid for $\mathcal{A}_4$-valued fields or requires modification.

## 9. Conclusions

9.1. **What this paper has done. §3 (BI analysis, OP 11.1 partially closed).** Klein-Gordon → Regge-Wheeler (4), potential (5), tortoise coordinate explicit. BC at $r = 0$ complete. Proposition 3.1 *[Derived]*: BI for $\xi = 0$ gives only $[\varphi_k] = 0$, no condition on $\partial_r \varphi_k$. Proposition 3.5 *[Derived]*: the Born Rule theorem [1] does not force $\chi(r_s) = 0$ for spinorial modes — gap $\pi(\ker L_a) = V_{\bar{\ell}} \neq 0$, three resolution paths identified. For $\xi \neq 0$: shell-field coupling appears (Remark 3.3).

**§4 (OP 11.2 and OP 11.7 resolved).** ZD pair graph: 7 isomorphic connected bipartite 4-regular components, 12 vertices each, characteristic polynomial (10). Eigenvalues $\{-4, -2, 0, +2, +4\}$, multiplicities $\{7, 14, 42, 14, 7\}$, all integers. PSL$(2,7)$ decomposition: $[1, 1, 1, 4, 3, 4]$ (GAP 4.15.1). The $\text{irr}_7$ thread (Papers 7→10→11 [4, 7]) is the central algebraic coherence of the series.

**§5 (OP 11.3 resolved; OP 11.4 resolved — negative result).** De Sitter-invariant condensate: SO$(1,4)$ transitivity + Einstein equation → $C_0$ uniquely (Theorem 5.4). BF stability: Levels 2–5 unconditional; Level 1 conditional on $\lambda GM^2 < 27\pi/128$ (Proposition 5.7, analytic derivation); Level 1 in complementary series when BFcond holds, BF-unstable otherwise — falsifiable prediction (Remark 5.8). First explicit $T_{\mu\nu}$ for $\mathcal{A}_4$ matter (Proposition 5.11). $n_{\text{sym}} = 28$: $\xi$ not fixed by symmetry — derived negative result (Theorem 6.1).

9.2. **What this paper has not done.** QNM frequencies not computed (tensoriale sector needs exterior matching; spinorial BC not derivable from [1] alone). $\xi$ not fixed algebraically (OP 11.4 closed as negative: $n_{\text{sym}} = 28$). Fermionic Fock space not constructed (OP 11.5). Non-associativity obstruction not resolved (OP 11.6). No scattering amplitudes, no direct experimental predictions beyond (15).

9.3. **Priority for the Next Paper.** **Track A** (tensoriale QNM, numerical): solve the coupled Regge-Wheeler eigenvalue problem (interior de Sitter + exterior Schwarzschild) with matching at $r_s$. Yields complex frequencies $\omega_{n,l,i}$ for Levels 1,2,4,5. Accessible with SCIPY.

**Track B** (spinorial BC, algebraic): compute the overlap integral of $\ker(L_a)$ modes with outgoing $\mathcal{A}_3$ plane waves using the sign-parity constraint $\pi_A\pi_B\pi_C = -1$ [1]. If the overlap vanishes by sign-parity cancellation, the Dirichlet BC is derived without IIB-I/II.

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