BOUNDARY CONDITIONS AND SPINORIAL STATISTICS OF THE BLACK CIRCLE INTERIOR

GIOVANNI LEVRATTI

ABSTRACT. Paper 11 established the complete mass spectrum for the 84 ZD-active modes on the de Sitter Black Circle interior, the unique $\mathrm{SO}(1,4)$-invariant condensate $C_0^4 = 3/(2048\pi G^3 M^2 \lambda)$, the first $T_{\mu\nu}$ for $\mathcal{A}_4$ matter content, and two negative results on spinorial boundary conditions.

This paper pursues three objectives.

Objective (i) (§3): option (iii) of Remark 3.6, Paper 11 is closed as a *[Derived]* negative result (Proposition 3.1): the sign-parity constraint $\pi_A\pi_B\pi_C = -1$ does not force the overlap of $\ker(L_a)$ modes with outgoing $\mathcal{A}_3$ plane waves to vanish; $\mathcal{A}_3$ elements are of pure ε-type and lie outside the ZD graph. Neither the $\mathcal{A}_4$ inner product $\langle b|g\rangle$ nor the product $b\cdot g$ is annihilated by sign-parity. OP 11.1 remains open. OP 11.6 sub-question (a) is closed *[Derived]* (Proposition 3.2): standard CCR are valid for S_{84}. A weaker result is derived (Proposition 3.5): Fano support orthogonality.

Objective (ii) (§4): a fundamental obstacle is identified. The Frobenius exponents at r_s are $\alpha_{\mathrm{in}} = \pm i\omega/2$ and $\alpha_{\mathrm{out}} = \pm i\omega$, differing because $f'_{\mathrm{in}}(r_s) = 2 \neq 1 = f'_{\mathrm{out}}(r_s)$ (Proposition 4.1, *[Derived]*). Standard Wronskian-shooting is invalid for this geometry. Preliminary candidates under the Frobenius-continuation ansatz are reported as *[Conjecture]*; the prediction $|\omega_{\mathrm{BC}}/\omega_{\mathrm{Sch}}| \approx \sqrt{2}$ is not confirmed. Rigorous QNM computation via Leaver-type continued fractions is deferred to OP 12.6.

Objective (iii) (§5): $\mathrm{GL}(2,3) \cong 2S_4$ *[Derived]* (Proposition 5.1); the unique 4-dimensional irrep $\mathrm{irr}_8 = 4_s$ is spinorial *[Derived]* (Proposition 5.3); under the discrete spin-statistics theorem (*[Conjecture]*, Assumption 5.5), the 4 modes are fermionic with CAR algebra and Grassmann–Fock space of dimension $2^4 = 16$ (Theorem 5.6).

Epistemic key. *[Derived]* — follows from Papers 1–11 and established mathematics. *[Conjecture]* — precise statement; additional assumptions stated explicitly. *[Speculation]* — structurally motivated; not currently falsifiable.

1. Introduction

1.1. **What Paper 11 established.** Paper 11 [11] initiated canonical quantisation of S_{84} on the static de Sitter interior and established:

- *[Derived]*: BI analysis at $\xi = 0$: only $[\varphi_k] = 0$ at r_s; no Neumann condition (Prop. 3.1).
- *[Derived]*: Born Rule alone does not force $\chi(r_s) = 0$ for spinorial modes; gap $\pi(\ker(L_a)) = V_{\bar{\ell}} \neq 0$ (Prop. 3.5).
- *[Derived]*: Complete mass spectrum, multiplicities $\{7, 14, 42, 14, 7\}$; $\mathrm{PSL}(2,7)$ decomposition $\chi_1 \oplus \chi_3 \oplus \chi_3' \oplus 4\chi_6 \oplus 3\chi_7 \oplus 4\chi_8$.
- *[Derived]*: Condensate $C_0^4 = 3/(2048\pi G^3 M^2 \lambda)$; first $T_{\mu\nu}$ for $\mathcal{A}_4$ matter.
- *[Derived]*: BF stability $\lambda G M^2 < 27\pi/128$ for Level 1; ξ not fixed by $\mathrm{PSL}(2,7)$ (dim 28).

1.2. **Three resolution paths for OP 11.1.** Paper 11, Remark 3.6 identified three paths toward closing the spinorial BC gap:

Date: June 2, 2026.

2020 *Mathematics Subject Classification.* 17A35, 83C57, 81T20, 81T75, 20C15.

Key words and phrases. Cayley–Dickson algebras, sedenions, zero-divisors, Black Circle, boundary conditions, quasi-normal modes, Frobenius exponents, spinorial statistics, GL(2,3), binary octahedral group, Grassmann–Fock space.

(i) QFT over $\mathcal{A}_4$ (OP 11.6 deep form).
(ii) IIB-I/II [2]: confinement implies $T = 0$.
(iii) Sign-parity projection theorem: despite $V_{\bar{\ell}} \neq 0$, the overlap vanishes via $\pi_A\pi_B\pi_C = -1$ [1].

This paper delivers the verdict on (iii) and the relevant sub-question of (i).

1.3. **What this paper does and does not do. Does:** Closes option (iii) as *[Derived]* negative result (§3). Closes OP 11.6 sub-question (a): standard CCR valid for S_{84}. Identifies Frobenius obstruction to Wronskian-shooting (§4). Reports preliminary QNM candidates under Frobenius ansatz. Determines spinorial statistics as fermionic, conditionally (§5).

Does not: Prove $\chi(r_s) = 0$ for spinorial modes from algebra alone. Compute rigorous QNM frequencies (deferred to OP 12.6). Construct full QFT over $\mathcal{A}_4$ (OP 11.6 deep form).

2. Algebraic Foundation

All results in this section are proved in Papers 1–11.

[P1] [1]. Sign-parity: $\pi_A\pi_B\pi_C = -1$ for every 2-flat (Thm 7.18). Every ZD pair in $\mathcal{A}_4$ is of mixed EF|EF type (Thm 7.7). $\dim\ker(L_a) = 4$ for every ZD unit (Thm 5.17). $\mathrm{ZD}(\mathcal{A}_3) = \emptyset$ (Hurwitz). Born Rule: unique for $n \leq 3$, undefined for $n \geq 4$.

[P2] [2]. IIB-I/II *[Conjecture]*: Born-inaccessible modes cannot propagate as free asymptotic states; under IIB-I/II, $\chi(r_s) = 0$ for spinorial modes. The only currently available path (ii) to close OP 12.1.

[P3] [3]. ZD zeta function $\zeta_{\mathrm{ZD}}(s)$; sign structure of ZD pairs classified; automorphism group verified via GAP.

[P4] [4]. Fano geometry: ZD pairs correspond to geometry of $\mathrm{PG}(2, \mathbb{F}_2)$. Weil zeta function for the ZD variety. $\mathrm{Aut}(\mathcal{A}_4) \cong C_2 \times ((C_2^3) \,.\, \mathrm{PSL}(3,2))$, order 2688 (GAP-verified, [9]). The irr_7 thread: $\mathrm{PSL}(2,7)$ acts irreducibly on the 7 imaginary octonionic directions.

[P5] [5]. Reed–Muller CSS codes; active stabilisers; $\pi_A\pi_B\pi_C = -1$ maps to the global phase -1 of the stabiliser product; CSS bridge ZD graph $\leftrightarrow$ $[[15, 7, 3]]$ code.

[P6] [6]. Factorization threshold; $\Delta N(a, b) = 4$ for every ZD-active pair (Thm 3.8, exact).

[P7] [7]. Clifford map ρ: $\mathrm{ZD}^*(\mathcal{A}_4) \to \mathrm{Cliff}(0, 7)$. Frobenius identity: $\|L_a\|_F^2 = 2^n N(a)$; ratio at $n = 4/n = 3$ equals 2 — the algebraic origin of the Frobenius exponent mismatch in Proposition 4.1.

[P8] [8]. Fano duality; CSS bridge; outgoing $\mathcal{A}_3$ plane waves at r_s identified with the CSS-dual sector (used in Proposition 3.5).

[P9] [9]. $\mathcal{L}_{\mathrm{int}} = \lambda\Delta N(\varphi, \psi)$, λ unique (Cor. 3.20). ZD-active modes algebraically massless. Fano conservation: $m(a) = m(b)$ for all ZD pairs.

[P10] [10]. Black Circle ansatz; S_{84}; Israel shell [12] $\sigma = 0$, $p = 1/(64\pi G^3 M^2)$, $T_{\mathrm{tot}} = 1/(4G)$. Fano complement structure of $\ker(L_a)$ (Rem. 7.10): for $a = e_p + \sigma e_{8+k}$, Fano line $\ell = \{p, k, p \oplus k\}$: $\ker(L_a) = V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)}$ where $\bar{\ell} = \{1, \ldots, 7\} \setminus \ell$, $V_{\bar{\ell}} = \mathrm{span}\{e_i : \mathrm{Fano}(i) \in \bar{\ell}\} \subset \mathcal{A}_3$, $V_{\bar{\ell}}^{(8)} \subset \mathcal{A}_4 \setminus \mathcal{A}_3$. $\mathrm{GL}(2,3)$ acts on $\ker(L_a)$ as irr_8 (GAP, Thm 7.11). Symmetry chain $S_4 \subset \mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2 \subset SO(7) \leftarrow \mathrm{Cliff}(0,7)$ (OP 10.7c, GAP-verified).

[P11] [11]. Black Circle metric (*[Conjecture]* under IIB-I/II): $f_{\mathrm{in}} = r^2/r_s^2 - 1$, $f_{\mathrm{out}} = 1 - r_s/r$. BI result ($\xi = 0$): $[\varphi_k] = 0$ at r_s; RW equation, tortoise coordinate, mass spectrum $m_i^2 = 16\lambda C_0^2(2 + \mu_i)$, condensate $C_0^4 = 3/(2048\pi G^3 M^2 \lambda)$, BF condition $\lambda G M^2 < 27\pi/128$, $T_{\mu\nu}^{\mathrm{bulk}}|_{C_0} = -g_{\mu\nu} \cdot 64\lambda C_0^4$.

3. Sign-Parity Overlap Argument — Objective (i)

Proposition 3.1 (*[Derived]* — negative result on option (iii)). *The sign-parity constraint* $\pi_A\pi_B\pi_C = -1$ [1, Thm 7.18] *does not imply*

$$\langle b \mid g \rangle = 0 \qquad \forall\, b \in \ker(L_a),\ g \in \mathcal{A}_3$$

for any natural algebraic definition of the inner product.

Proof. Step 1 — Domain of the constraint. Theorem 7.18 of [1] applies to *active ZD splits*, i.e., to pairs $(a,b) \in \mathrm{ZD}(\mathcal{A}_4)$. Its domain is the ZD graph of $\mathcal{A}_4$.

Step 2 — $\mathcal{A}_3$ elements are outside the ZD graph. By Theorem 7.7 of [1], every ZD pair in $\mathcal{A}_4$ consists of two elements each of mixed EF-type. Elements of $\mathcal{A}_3$, embedded in $\mathcal{A}_4$ as ε-type basis vectors e_m ($m \in \{0,\dots,7\}$), are of pure ε-type and therefore not mixed EF. No element $g \in \mathcal{A}_3 \hookrightarrow \mathcal{A}_4$ belongs to any ZD pair in $\mathcal{A}_4$; the constraint $\pi_A\pi_B\pi_C = -1$ does not apply to any triple (a,b,g) with $g \in \mathcal{A}_3$.

Step 3 — Algebraic product does not vanish. Let $b \in \ker(L_a)$: $b = e_k + \sigma e_{8+l}$ ($k \in \bar{\ell}$). Let $g = e_m \in \mathcal{A}_3$ ($m \in \{1,\dots,7\}$). Cayley–Dickson multiplication:

$$b \cdot g = (e_k + \sigma e_{8+l}) \cdot e_m = \varepsilon_{k,m}\, e_{k\oplus m} - \sigma\varepsilon_{l,m}\, e_{8+(l\oplus m)}.$$

Since ε-signs are always ± 1, $b \cdot g \neq 0$.

Step 4 — $\mathcal{A}_4$ inner product does not vanish. $\langle b|g\rangle_{\mathcal{A}_4} = \sum_i b_i g_i$. For $b = e_k + \sigma e_{8+l}$ and $g = \sum_{m\le 7} g_m e_m$: $\langle b|g\rangle = g_k$. Since $k \in \bar{\ell} \subset \{1,\dots,7\}$, the component g_k of a generic $\mathcal{A}_3$ wave is non-zero.

The sign-parity argument acts in the *wrong direction*: the fact that $(b,g) \notin \mathrm{ZD}(\mathcal{A}_4)$ means $b \cdot g \neq 0$, establishing algebraic coupling rather than its absence. □

Proposition 3.2 (*[Derived]* — partial closure of OP 11.6(a))**.** *The canonical commutation relations*

$$[\varphi_k(\mathbf{x},t),\, \pi_l(\mathbf{x}',t)] = i\hbar\,\delta_{kl}\,\delta^3(\mathbf{x}-\mathbf{x}')$$

are valid without modification for the 84 real scalar fields ϕ_k of S_{84}. The non-associativity of $\mathcal{A}_4$ does not obstruct canonical quantisation of S_{84}.

Proof. The fields $\varphi_k = \phi_k(x) \cdot A_k$ factor as a real scalar $\phi_k(x) \in \mathbb{R}$ times an algebraic label $A_k \in \mathrm{ZD}^*(\mathcal{A}_4)$. Every term in S_{84} is a product of *real numbers*: $\phi_k(x) \in \mathbb{R}$, $\phi_A^2\phi_B^2 \in \mathbb{R}$. The non-associativity of $\mathcal{A}_4$ determines *which pairs* (A,B) belong to ZD — the topology of the interaction graph — but is not invoked in any product of field values. The canonical momenta $\pi_k = \partial\mathcal{L}/\partial(\partial_t\phi_k)$ are real scalars. Standard CCR for 84 coupled real scalars on a curved background apply without modification. □

Remark 3.3 (Residual open question of OP 11.6)**.** The deeper form of OP 11.6 asks whether a full $\mathcal{A}_4$-valued quantum field $\hat{\Phi}(x) = \sum_k \hat{\phi}_k(x) \cdot e_k$ can be defined. This requires an associative enveloping algebra of $\mathcal{A}_4$ — no canonical construction exists; $\mathcal{A}_4$ is neither Lie nor alternative. Explicit non-alternativity: $(ab)b - a(b^2) = -2e_{15} \neq 0$ (Paper 1, Thm 4.6). This construction remains open as the deep form of OP 11.6 (Stratum III, Paper 10).

Remark 3.4 (No cubic vertex in S_{84})**.** The interaction in S_{84} is purely quartic: $4\lambda \sum \phi_A^2\phi_B^2$. There is no cubic vertex. A 3-point amplitude $\langle 0|T[\phi_a\phi_b\phi_g]|0\rangle$ vanishes by Wick's theorem in free theory and does not arise from a fundamental vertex. The "overlap via 3-point amplitude" interpretation of option (iii) has no natural home in S_{84}.

Proposition 3.5 (*[Derived]* — Fano support orthogonality)**.** *Let $a = e_p + \sigma e_{8+k}$ with Fano line $\ell = \{p, k, p\oplus k\}$. Then*

$$\langle V_{\bar{\ell}} \mid \mathrm{span}\{e_p, e_k, e_{p\oplus k}\}\rangle_{\mathcal{A}_4} = 0.$$

Proof. Elements of $V_{\bar{\ell}} = \mathrm{span}\{e_i : i \in \bar{\ell}\}$ have ε-indices in $\bar{\ell} = \{1,\dots,7\}\setminus\ell$, disjoint from $\{p,k,p\oplus k\}$. Basis elements with different indices are orthogonal in the $\mathcal{A}_4$ inner product. □

Remark 3.6. This is *index support orthogonality*, not sign-parity cancellation. It states that the $\mathcal{A}_3$-projection of $\ker(L_a)$ is orthogonal to the specific $\mathcal{A}_3$ elements *on the Fano line ℓ* — not to all outgoing $\mathcal{A}_3$ waves. An arbitrary outgoing wave has components on $\bar{\ell}$, which overlap with $V_{\bar{\ell}}$.

Under IIB-I/II [2] *[Conjecture]*: if the outgoing Hawking quanta are the Born-accessible modes associated with ℓ, then $\langle \ker(L_a)\ \text{mode}|\text{outgoing wave on}\ \ell\rangle = 0$ *[Derived]*. This uses IIB-I/II — precisely what option (iii) was meant to avoid.

Resolution path	Status	Source
(i) QFT over $\mathcal{A}_4$ (deep form)	Open — Stratum III	[10]
(ii) IIB-I/II confinement	Available *[Conjecture]*	[2]
(iii) Sign-parity overlap	*[Derived]* **negative**	Prop. 3.1
CCR for S_{84} (OP 11.6a)	*[Derived]* closed	Prop. 3.2
Fano support orthogonality	*[Derived]*, not sufficient	Prop. 3.5

The only currently available path to $\chi(r_s) = 0$ for spinorial modes is **(ii) IIB-I/II** [2] (*[Conjecture]*). OP 11.1 is carried forward.

4. Tensoriale QNM Frequencies — Objective (ii)

4.1. **Setup.** The Regge–Wheeler equation for tensoriale Level 2 modes ($m^2 = 0$, $\xi = 0$) on the static Black Circle, in units $r_s = 1$:

Interior (de Sitter, $r < 1$): $f_{\text{in}}(r) = r^2 - 1$, $r_*^{(\text{in})} = -\text{arctanh}(r)$,

$$V_{\text{in}}(r, l) = f_{\text{in}}(r)\left[\frac{l(l+1)}{r^2} + 2\right].$$

Exterior (Schwarzschild, $r > 1$): $f_{\text{out}}(r) = 1 - 1/r$, $r_*^{(\text{out})} = r + \ln(r - 1)$,

$$V_{\text{out}}(r, l) = f_{\text{out}}(r)\left[\frac{l(l+1)}{r^2} + \frac{1}{r^3}\right].$$

Boundary conditions. At $r \to 0$: regularity, $\chi \sim r^{l+1}$. At $r \to \infty$: purely outgoing, $\chi \sim e^{+i\omega r_*^{(\text{out})}}$. At $r = 1$: $[\chi] = 0$, $[d\chi/dr] = 0$ (Prop. 3.1, Paper 11, *[Derived]*).

4.2. **The Frobenius obstruction.**

Proposition 4.1 (*[Derived]*). *The regular singular points of the interior and exterior ODE at $r = r_s$ carry different Frobenius exponents:*

$$\alpha_{\text{in}} = \pm\tfrac{i\omega}{2} \quad (\textit{from } f'_{\text{in}}(r_s) = 2), \qquad \alpha_{\text{out}} = \pm i\omega \quad (\textit{from } f'_{\text{out}}(r_s) = 1).$$

Any Wronskian-shooting method continuing the interior solution through r_s via the Frobenius-continuation ansatz imposes initial conditions on the exterior ODE with exponent $i\omega/2 \neq i\omega$. The zeros of $W_{\text{BC}}(\omega)$ do not coincide in general with the physical QNM eigenvalues.

Proof. $f'_{\text{in}}(1) = 2$; indicial equation near $u = 1 - r$: $\alpha^2 + \omega^2/4 = 0$, giving $\alpha = \pm i\omega/2$. $f'_{\text{out}}(1) = 1$; indicial near $v = r - 1$: $\alpha^2 + \omega^2 = 0$, giving $\alpha = \pm i\omega$. Equal iff $\omega = 0$.

The factor $1/2$ in $\alpha_{\text{in}}/\alpha_{\text{out}}$ is the reciprocal of the Frobenius algebraic ratio

$$\|L_a\|_F^2|_{n=4}/\|L_a\|_F^2|_{n=3} = 16N(a)/8N(a) = 2,$$

[7], which propagates into QNM physics as the surface-gravity jump $\kappa_{\text{in}}/\kappa_{\text{out}} = 2$ at r_s [10]. □

Corollary 4.2 (*[Derived]*). *Rigorous Black Circle QNM computation requires either a matched Frobenius expansion at r_s (Leaver-type method adapted to the two-domain geometry) or an explicit connection formula. This is OP 12.6.*

4.3. **Numerical candidates [*[Conjecture]* — ansatz Frobenius].** The following computation is performed with the caveat of Proposition 4.1: results are *candidates* under the Frobenius-continuation ansatz, not rigorously established eigenvalues.

Method. Wronskian shooting with truncated exterior domain $[1+\delta, R_M]$, $R_M = 4.0$ ($l = 0$) or 6.0 ($l \geq 1$), $\delta = 0.02$, matching point $r_m = 3.0$ ($l = 0$) or 4.0 ($l \geq 1$). Newton's method in $\mathbb{C}$; convergence threshold $|W| < 10^{-10}$.

Schwarzschild reference (same method, $r_s = 1$):

(l,n)	$\omega_R^{\rm Sch}$	$\omega_I^{\rm Sch}$	Lit. ω_R	Lit. ω_I	$\lvert W\rvert$
$(0,0)$	0.2527	−0.2746	0.2209	−0.2098	3×10^{-14}
$(1,0)$	0.5505	−0.1468	0.5854	−0.1955	2×10^{-14}
$(2,0)$	1.0020	−0.1469	0.9677	−0.1935	7×10^{-14}

Deviations from literature [16]: $\Delta\omega_R \approx$ 4–14%, $\Delta\omega_I \approx$ 24–31%, consistent with the V/ω^2 truncation estimate at R_M.

Black Circle candidates (Level 2, $n=0$):

l	$\omega_R^{\rm BC}$	$\omega_I^{\rm BC}$	$\lvert W_{\rm BC}\rvert$	$\lvert\omega_{\rm BC}/\omega_{\rm Sch}\rvert$	Note
0	0.2919	−0.1073	2×10^{-15}	0.833	candidate
1	0.5505	−0.1468	5×10^{-11}	1.000	**spurious**
2	1.2754	−0.1870	1×10^{-17}	1.273	candidate

The $l=1$ result is spurious — Newton converged to $\omega_{\rm Sch}$, an artefact of the exponent mismatch.

Prediction test. $|\omega_{\rm BC}/\omega_{\rm Sch}| \approx \sqrt{2}$ predicts ratio ≈ 1.414. Numerical candidates give 0.833 ($l=0$) and 1.273 ($l=2$).

Remark 4.3 (*[Derived]*)**.** The Wronskian method cannot validate or invalidate the $\sqrt{2}$ prediction, because Proposition 4.1 establishes that the method is unreliable for this geometry. The prediction remains *[Conjecture]* and is referred to OP 12.6.

4.4. **Open Problem OP 12.6.** Leaver's method [15] for Schwarzschild uses the expansion $\chi_{\rm out}(r) = e^{i\omega r}(r-1)^{-i\omega}\sum_{n=0}^{\infty} a_n\left(\frac{r-1}{r}\right)^n$ and derives a 3-term recurrence for a_n. For the Black Circle, both sides of r_s admit such expansions:

- **Interior:** $\chi_{\rm in}(r) = r^{l+1}\sum_n c_n r^n$ (regular at $r=0$).
- **Exterior:** $\chi_{\rm out}(r) = (r-1)^{-i\omega}\sum_n b_n (r-1)^n$ (ingoing at r_s).

The QNM condition is simultaneous convergence of both series at $r=r_s$, expressed as a determinantal equation involving the c_n and b_n recurrences. This is tractable and free of the exponent-mismatch problem. Test of $|\omega_{\rm BC}/\omega_{\rm Sch}| \approx \sqrt{2}$ follows upon completion.

5. Spinorial Statistics — Objective (iii)

5.1. $\mathrm{GL}(2,3) \cong 2S_4$.

Proposition 5.1 (*[Derived]*)**.** $\mathrm{GL}(2,\mathbb{F}_3)$ *is isomorphic to the binary octahedral group* $2S_4$*, i.e., the unique non-trivial central extension of* S_4 *by* $\mathbb{Z}_2$*.*

Proof. $Z(\mathrm{GL}(2,3)) = \{\lambda I : \lambda\in\mathbb{F}_3^*\} = \{I, 2I\}$, order 2 ($(2I)^2 = 4I = I$ in $\mathbb{F}_3$). The quotient $\mathrm{PGL}(2,3) = \mathrm{GL}(2,3)/Z$ has order $48/2 = 24 = |S_4|$. Since $\mathrm{PGL}(2,3)\cong S_4$ and $H^2(S_4, U(1)) = \mathbb{Z}_2$ (Schur multiplier of S_4), the extension $\mathbb{Z}_2 \to \mathrm{GL}(2,3) \to S_4$ is the unique non-trivial one. □

Corollary 5.2 (*[Derived]*)**.** $\mathrm{GL}(2,3)$ *has exactly* 8 *conjugacy classes, hence* 8 *irreducible complex representations with dimensions:*

$$\{\underbrace{1,1,2,3,3}_{\textit{bosonic (from } S_4)},\ \underbrace{2,2,4}_{\textit{spinorial}}\}.$$

$\sum d_i^2 = 1+1+4+9+9+4+4+16 = 48 = |\,\mathrm{GL}(2,3)|$ ✓. *The* 5 *bosonic irreps pull back from* S_4*; the* 3 *spinorial irreps have* $\rho(2I) = -\mathrm{Id}_{d_i}$.

5.2. The unique 4-dimensional spinorial irrep.

Proposition 5.3 (*[Derived]*)**.** *The notation* irr_8 *refers to the unique* 4*-dimensional irreducible representation* 4_s *of* $\mathrm{GL}(2,3)$*, which is spinorial: the central element* $2I \in Z(\mathrm{GL}(2,3))$ *acts as* $-\mathrm{Id}_4$*.*

Proof. By Corollary 5.2, there is exactly one irrep of dimension 4, and dimension 4 does not appear among the bosonic irreps $(1,1,2,3,3)$. Therefore the 4-dimensional irrep is spinorial. In GAP canonical ordering, it is the 8th irrep, hence the label irr_8. □

Remark 5.4 (*[Derived]*)**.** The 4 spinorial modes of the Black Circle ([10, Rem. 7.10]) span a 4-dimensional representation space of $\mathrm{GL}(2,3)$. By Proposition 5.3, this is 4_s, which does not factor through $S_4 = \mathrm{GL}(2,3)/Z_2$. No consistent integer-spin quantum number assignment exists for these modes within the symmetry group.

5.3. Fermionic statistics.

Assumption 5.5 (*[Conjecture]* — discrete spin-statistics theorem)**.** In the static de Sitter–Schwarzschild geometry of the Black Circle, the CPT structure of the Bunch–Davies vacuum [14] is preserved and the Osterwalder–Schrader reconstruction extends to the compact sector. Under these conditions, the spin-statistics theorem of Streater–Wightman applies and associates spinorial representations with Fermi statistics.

This assumption is *[Conjecture]*. It becomes *[Derived]* iff the axiomatic QFT setup is constructed in full (OP 12.7).

Theorem 5.6 (*[Conjecture]* | Assumption 5.5)**.** *Under Assumption 5.5, the* 4 *spinorial modes are fermionic, obeying the canonical anti-commutation relations (CAR):*

$$\{\hat{a}_k, \hat{a}_l^\dagger\} = \delta_{kl}\hat{1}, \quad \{\hat{a}_k, \hat{a}_l\} = 0, \quad k, l \in \{1,2,3,4\}.$$

Proof sketch. The spinorial irrep 4_s cannot be extended to a bosonic representation: any bosonic field on the Black Circle transforms in a representation of $S_4 = \mathrm{GL}(2,3)/\mathbb{Z}_2$, but 4_s is not a pull-back from S_4. Under Assumption 5.5, a field in a representation that genuinely uses the double cover obeys Fermi statistics. □

5.4. CAR algebra and Grassmann–Fock space.

Under Theorem 5.6, the spinorial sector admits the following structure.

The CAR algebra is isomorphic to the Clifford algebra on $2n = 8$ generators:

$$\mathrm{CAR}(\mathbb{C}^4) \cong \mathrm{Cliff}(\mathbb{R}^8)_{\mathbb{C}}, \qquad \dim = 2^8 = 256.$$

Mode counting. S_{84} has 84 ZD-active modes total. Level 3 ($\mu = 0$) has multiplicity 42. The 4 modes in $4_s \subset$ Level 3 are identified by their $\mathrm{GL}(2,3)$ transformation law; the remaining 38 Level-3 modes transform in bosonic irreps and obey CCR. Accordingly, the bosonic sector contains $84 - 4 = 80$ modes.

The full Hilbert space:

$$\mathcal{H}_{\mathrm{BC}} = \mathcal{F}_{\mathrm{sym}}(\mathbb{C}^{80}) \otimes \bigwedge^{\bullet}(\mathbb{C}^4),$$

where $\mathcal{F}_{\mathrm{sym}}(\mathbb{C}^{80})$ is the symmetric Fock space of the 80 bosonic modes and $\bigwedge^\bullet(\mathbb{C}^4)$ is the Grassmann Fock space of the 4 spinorial modes.

Grassmann–Fock space. $|\Omega\rangle$ satisfies $\hat{a}_k|\Omega\rangle = 0$. $\mathcal{F} = \bigoplus_{n=0}^4 \mathcal{F}_n \cong \bigwedge^\bullet(\mathbb{C}^4)$, $\dim \mathcal{F}_n = \binom{4}{n}$:

n	Sector	Basis	Dim
0	vacuum	$\|\Omega\rangle$	1
1	1-particle	$\hat{a}_k^\dagger\|\Omega\rangle$	4
2	2-particle	$\hat{a}_k^\dagger\hat{a}_l^\dagger\|\Omega\rangle$	6
3	3-particle	$\hat{a}_k^\dagger\hat{a}_l^\dagger\hat{a}_m^\dagger\|\Omega\rangle$	4
4	4-particle	$\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_3^\dagger\hat{a}_4^\dagger\|\Omega\rangle$	1

Total: $\dim\mathcal{F} = 1+4+6+4+1 = 16 = 2^4$. Pauli exclusion: $(\hat{a}_k^\dagger)^2 = 0$.

6. Physical Consequences

6.1. **Mixed boson-fermion structure.** The quantisation of S_{84} yields a mixed boson-fermion Hilbert space:

- **80 bosonic modes** (Levels 1,2,4,5 plus 38 Level-3 modes in bosonic irreps of $\mathrm{GL}(2,3)$) — CCR, *[Derived]*.
- **4 fermonic modes** (Level 3, 4_s of $\mathrm{GL}(2,3)$) — CAR, *[Conjecture]* | Assumption 5.5.

6.2. **BF bound and Level 1 instability.** BF condition for Level 1 from [11, 13]: $\lambda GM^2 < 27\pi/128$. For $M > M_{\mathrm{BF}} \equiv \sqrt{27\pi/(128\lambda G)}$, Level 1 modes are BF-unstable. Independent of the spinorial sector.

6.3. $T_{\mu\nu}$ **corrections from the spinorial sector.** If the 4 fermionic modes contribute to $T_{\mu\nu}$ at one loop, the fermionic contribution carries a relative sign compared to the bosonic contribution. The precise form requires the heat-kernel expansion for spinorial modes on the de Sitter interior. *[Speculation]* — deferred to a future paper.

7. Open Problems

OP 12.1 (OP 11.1 carried forward). Prove or disprove $\chi(r_s) = 0$ for spinorial modes without invoking IIB-I/II. Option (iii) of Paper 11 is **closed as negative** (Proposition 3.1). Available paths: QFT over $\mathcal{A}_4$ (OP 12.2); new algebraic construction not in Papers 1–11.

OP 12.2 (OP 11.6 deep form). Construct the full $\mathcal{A}_4$-valued quantum field $\hat{\Phi}(x) = \sum_k \hat{\phi}_k(x)\cdot e_k$. Requires an associative enveloping algebra of $\mathcal{A}_4$, or a weaker structure sufficient for QFT.

OP 12.3 (from Proposition 3.5). Does the Fano support orthogonality admit a physical interpretation without IIB-I/II? Is there a natural restriction of "outgoing $\mathcal{A}_3$ plane waves at r_s" to Fano-line modes from first principles?

OP 12.4 (OP 11.5 — partial closure by §5). The 4 spinorial modes are fermionic under Assumption 5.5 (Theorem 5.6); the full CAR algebra and Grassmann–Fock space of dimension 16 are constructed.

OP 12.5 (OP 11.6(b)). Precisely characterise the non-associativity obstruction for the full $\mathcal{A}_4$-valued QFT. Does the power-associativity of $\mathcal{A}_4$ (x^n well-defined for all x) provide sufficient structure for a restricted class of observables?

OP 12.6 (Leaver-type QNM, see §4.4). Compute Black Circle QNM frequencies for Levels 1,2,4,5 via matched continued fractions, free of the Frobenius exponent mismatch. Test $|\omega_{\mathrm{BC}}/\omega_{\mathrm{Sch}}| \approx \sqrt{2}$.

OP 12.7 (from Assumption 5.5). Derive the discrete spin-statistics theorem for compact modes on the Black Circle: construct the PCT involution compatible with the $\mathrm{GL}(2,3)$ action on 4_s; verify Osterwalder–Schrader axioms; confirm positivity of energy in the spinorial sector. Subsumes the deep form of OP 12.2.

8. Conclusions

8.1. **What this paper has done. Objective (i)** (§3). Option (iii) of Paper 11 is *[Derived]* negative (Proposition 3.1): sign-parity constraint $\pi_A\pi_B\pi_C = -1$ does not annihilate $\langle \ker(L_a)\ \mathrm{mode}|g\rangle$ for $g \in \mathcal{A}_3$; $\mathcal{A}_3$ elements lie outside the ZD graph and the overlap $\langle b|g\rangle = g_k \neq 0$ generically. OP 11.1 remains open.

OP 11.6 sub-question (a) closed *[Derived]* (Proposition 3.2): standard CCR valid for S_{84}. Non-associativity governs interaction graph topology, not field-value products.

Fano support orthogonality derived (Proposition 3.5): $V_{\bar{\ell}} \perp \mathrm{span}\{e_p, e_k, e_{p\oplus k}\}$ by index support. Weaker than option (iii); under IIB-I/II gives Dirichlet BC.

Objective (ii) (§4). *[Derived]*: Proposition 4.1: Frobenius exponents at r_s are $\alpha_{\mathrm{in}} = \pm i\omega/2$ vs $\alpha_{\mathrm{out}} = \pm i\omega$ (from $f'_{\mathrm{in}}(r_s) = 2 \neq 1 = f'_{\mathrm{out}}(r_s)$). Standard Wronskian-shooting is invalid. This is the algebraic signature of the Frobenius ratio of Paper 7 propagating into QNM physics.

[Conjecture] | ansatz Frobenius: $l = 0$ and $l = 2$ QNM candidates reported; $l = 1$ result spurious. Prediction $|\omega_{\mathrm{BC}}/\omega_{\mathrm{Sch}}| \approx \sqrt{2}$ not confirmed. OP 12.6 (Leaver method) is the required next step.

Objective (iii) (§5). *[Derived]*: $\mathrm{GL}(2,3) \cong 2S_4$ (Proposition 5.1); $\mathrm{irr}_8 = 4_s$ is the unique 4-dimensional spinorial irrep (Proposition 5.3). *[Conjecture]* | Assumption 5.5: the 4 modes are fermionic (Theorem 5.6); CAR algebra, Grassmann–Fock space $\dim 16 = 2^4$. Full Hilbert space $\mathcal{H}_{\mathrm{BC}} = \mathcal{F}_{\mathrm{sym}}(\mathbb{C}^{80}) \otimes \bigwedge^{\bullet}(\mathbb{C}^4)$.

8.2. **What this paper has not done.** $\chi(r_s) = 0$ for spinorial modes not proved from algebra alone. Rigorous QNM frequencies not computed (OP 12.6). Assumption 5.5 not derived (OP 12.7). Full $\mathcal{A}_4$-valued QFT not constructed (OP 12.2).

Acknowledgements

The author thanks the mathematical physics community and the foundational work on octonions and Cayley–Dickson algebras by J. C. Baez [17] and G. M. Dixon. Computational assistance was provided by Claude (Anthropic) for algebraic verification, structural analysis, and proof scaffolding. The author takes full responsibility for any errors.

References

[1] G. Levratti, *Sign-Parity Classification of Zero Divisors in Cayley–Dickson Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20315751.
[2] G. Levratti, *The Informational Identity Boundary: Algebraic Foundations and Physical Conjectures*, Zenodo preprint (2026), v4, doi:10.5281/zenodo.20411845.
[3] G. Levratti, *Sign Structure, Automorphism Groups, and the ZD Zeta Function of Cayley–Dickson Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20397587.
[4] G. Levratti, *Zero-Divisors in Cayley–Dickson Algebras: Fano Geometry, Automorphisms, and the Weil Zeta Function*, Zenodo preprint (2026), doi:10.5281/zenodo.20417859.
[5] G. Levratti, *Active Stabilizers in Reed–Muller CSS Codes from Cayley–Dickson Zero-Divisors*, Zenodo preprint (2026), doi:10.5281/zenodo.20431754.
[6] G. Levratti, *The Factorization Threshold in Cayley–Dickson Integer Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20442390.
[7] G. Levratti, *A Clifford Representation of the Fano Zero-Divisors*, Zenodo preprint (2026), doi:10.5281/zenodo.20445208.
[8] G. Levratti, *Fano Duality, Projective Orthogonality, and the CSS Bridge*, Zenodo preprint (2026), doi:10.5281/zenodo.20445734.
[9] G. Levratti, *A Quartic Contact Interaction with Fano–CSS Structure*, Zenodo preprint (2026), doi:10.5281/zenodo.20506165.
[10] G. Levratti, *Loop, Threshold, and Residue: The Algebraic Structure of the $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$ Transition*, Zenodo preprint (2026), doi:10.5281/zenodo.20507204.
[11] G. Levratti, *Canonical Quantisation of the Black Circle Interior*, Zenodo preprint (2026), doi:10.5281/zenodo.20508375.

[12] C. Barrabès and W. Israel, Thin shells in general relativity and cosmology: the lightlike limit, *Phys. Rev. D* **43** (1991), 1129–1142.
[13] P. Breitenlohner and D. Z. Freedman, Stability in gauged extended supergravity, *Ann. Phys.* **144** (1982), 249–281.
[14] T. S. Bunch and P. C. W. Davies, Quantum field theory in de Sitter space: renormalization by point-splitting, *Proc. R. Soc. London A* **360** (1978), 117–134.
[15] E. W. Leaver, An analytic representation for the quasi-normal modes of Kerr black holes, *Proc. R. Soc. London A* **402** (1985), 285–298.
[16] R. A. Konoplya and A. Zhidenko, Quasinormal modes of black holes: from astrophysics to string theory, *Rev. Mod. Phys.* **83** (2011), 793–836.
[17] J. C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002), no. 2, 145–205.

Independent Researcher, Modena, Italy
Email address: `giovanni.levratti.research@gmail.com`