QUASI-NORMAL MODES OF THE BLACK CIRCLE INTERIOR

GIOVANNI LEVRATTI

Abstract. Paper 12 established that standard Wronskian-shooting is invalid for the Black Circle quasi-normal mode (QNM) problem: the Frobenius exponents at $r = r_s$ are $\alpha_{\rm in} = \pm i\omega/2$ (de Sitter interior) and $\alpha_{\rm out} = \pm i\omega$ (Schwarzschild exterior), with exponent ratio $\alpha_{\rm out}/\alpha_{\rm in} = 2$ equal to the Frobenius algebraic ratio $\|L_a\|_F^2|_{n=4}/\|L_a\|_F^2|_{n=3} = 2$ from Paper 7 (Proposition 4.1 of Paper 12, *[Derived]*). This paper closes Open Problem 12.6 in three stages.

First (§3): a feasibility analysis of the connection matrix at r_s establishes that a naive 2×2 algebraic matrix $M(\omega)\colon (c_+, c_-) \to (b_+, b_-)$ does not exist for $\omega \neq 0$ (*[Derived]*, Proposition 3.1). The correct ingredient is a hypergeometric connection formula for the de Sitter RW ODE, deferred to OP 13.1.

Second (§4): the interior series is derived — it is purely even (Proposition 4.1) — yielding a step-2 three-term recurrence with explicit coefficients $\alpha_n, \beta_n, \gamma_n$. The key structural result *[Derived]* is that the matched QNM condition reduces to a **single purely interior equation**: $H_{\rm BC}(\omega) = D_{\rm in}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$, where $D_{\rm in}$ is the Frobenius coefficient of $(1-r)^{-i\omega/2}$ in the interior solution and the right-hand side encodes the EF-frame phase constants. The exterior Schwarzschild geometry provides the radiation channel but does not enter the eigenvalue equation.

Third (§5): the complete QNM spectrum for Levels 1, 2, 4, 5 (tensor sector, $\xi = 0$) is computed. The spectrum has two structurally distinct families. **Type I** (imaginary-axis modes, $\omega = -i\kappa$): twelve overdamped non-oscillatory modes, one per (level, l) entry, residuals $< 5 \times 10^{-15}$; longest-lived: Level 2, $l = 0$: $\tau = 12.305\, r_s/c$. **Type II** (oscillatory modes, $\mathrm{Re}(\omega) > 0$): four modes, all $l = 2$, $n = 0$, ratio $|\omega_{\rm BC}/\omega_{\rm Sch}| = 2.168 \pm 0.002$ — the $\sqrt{2}$ prediction *[Conjecture]* of Paper 10 is **not confirmed** *[Derived]*. An **instability signature** is found for Level 1, $l = 0$: $\omega_{\rm grow} = +4.139i$, e-folding time $\tau_{\rm grow} = 0.242\, r_s/c$ — first numerical determination of the BF instability exponent.

Four further results: (i) **Hawking equivalence** *[Derived]*: Black Circle Hawking spectrum equals Schwarzschild exactly. (ii) **BF cavity bound** *[Derived]*: $\lambda GM^2 < 0.003$, $\approx 225\times$ more restrictive than the de Sitter BF bound $\lambda GM^2 < 27\pi/128$ of Paper 11. (iii) **General EF matching** *[Derived]*: formula for any two-domain metric. (iv) **Perturbative mass coupling** *[Derived]* (structure): $\mathrm{Im}(\omega_{\rm BC}) \approx -0.0847 - 0.0728\, m^2$; $\Gamma_1 = \mathrm{Im}[-\partial_{m^2} D_{\rm in}/\partial_\omega H_{\rm BC}]$.

Epistemic key. *[Derived]* — follows from Papers 1–12 and established mathematics. *[Conjecture]* — precise statement; additional assumptions stated explicitly. *[Speculation]* — structurally motivated; not currently falsifiable.

1. Introduction

1.1. **What Papers 10–12 established.** Paper 10 [10] constructed the full 84-mode scalar action S_{84} on the static Black Circle interior and identified the ZD pair graph on $\mathrm{ZD}^*(\mathcal{A}_4)$ as a 4-regular graph with 168 edges.

Paper 11 [11] initiated canonical quantisation of S_{84} and established: (a) the Barrabès–Israel junction condition for $\xi = 0$ gives only $[\varphi_k] = 0$ at r_s — no Neumann condition (Prop. 3.1, *[Derived]*); (b) the complete mass spectrum with multiplicities $\{7, 14, 42, 14, 7\}$ and $\mathrm{PSL}(2,7)$ decomposition; (c) the unique $\mathrm{SO}(1,4)$-invariant condensate $C_0^4 = 3/(2048\pi G^3 M^2 \lambda)$; (d) BF stability condition $\lambda GM^2 < 27\pi/128$ for Level 1; (e) ξ not fixed by $\mathrm{PSL}(2,7)$ symmetry ($\dim = 28$).

Date: June 2, 2026.

2020 *Mathematics Subject Classification.* 17A35, 83C57, 34B24, 34E05, 81T20.

Key words and phrases. Cayley–Dickson algebras, sedenions, zero-divisors, Black Circle, quasi-normal modes, Frobenius exponents, EF matching condition, de Sitter interior cavity, overdamped modes, BF instability, Schwarzschild exterior.

Paper 12 [12] identified the Frobenius obstruction to QNM computation (Prop. 4.1, *[Derived]*): exponents $\alpha_{\text{in}} = \pm i\omega/2$ vs $\alpha_{\text{out}} = \pm i\omega$, exponent ratio $\alpha_{\text{out}}/\alpha_{\text{in}} = 2$ = Frobenius ratio from [7]. Wronskian-shooting is invalid. OP 12.6 requested a Leaver-type method.

1.2. **What this paper does and does not do. Does:** Analyses feasibility of the connection matrix (§3); derives the interior Leaver recurrence and EF matching condition (§4); computes the complete QNM spectrum for Levels 1,2,4,5 with $l = 0, 1, 2$ (§5); identifies Type I and II families; finds the BF instability exponent; refutes the $\sqrt{2}$ prediction; derives Hawking equivalence, the cavity BF bound, and the general EF matching formula (§6).

Does not: Prove $[d\chi/dr] = 0$ from BI alone (OP 13.2); construct $M(\omega)$ via hypergeometric connection formulas (OP 13.1); treat Level 3 spinorial modes (OP 12.1); compute overtones ($n \geq 1$) for oscillatory modes or $l = 0, 1$ oscillatory modes (OP 13.6).

2. Algebraic Foundation

All results cited are proved in Papers 1–12.

[P1] [1]. Sign-parity: $\pi_A \pi_B \pi_C = -1$. Every ZD pair in $\mathcal{A}_4$ is of mixed EF|EF type. $\dim \ker(L_a) = 4$. $\text{ZD}(\mathcal{A}_3) = \emptyset$ (Hurwitz). IIB physical conjectures: [2]. ZD zeta function and automorphism structure: [3]. Fano geometry and Weil zeta function: [4]. Reed–Muller CSS connection: [5]. Factorization threshold, $\Delta N = 4$ exact: [6]. Fano duality, CSS bridge: [8]. Quartic contact interaction, λ unique: [9].

[P7] [7]. Frobenius identity: $\|L_a\|_F^2 = 2^n N(a)$. Ratio at $n = 4$ vs $n = 3$ equals **2** — the algebraic origin of the Frobenius exponent mismatch [12].

[P10] [10]. Black Circle metric *[Conjecture]* (IIB-I/II): $f_{\text{in}}(r) = r^2/r_s^2 - 1$ (de Sitter interior, $r < r_s$); $f_{\text{out}}(r) = 1 - r_s/r$ (Schwarzschild exterior, $r \geq r_s$). Israel shell [15]: $\sigma = 0$, $p = 1/(64\pi G^3 M^2)$, $T_{\text{tot}} = 1/(4G)$.

[P11] [11]. RW equation (units $r_s = 1$, $\xi = 0$): $-d^2\chi/dr_*^2 + V_{\text{eff}}\,\chi = \omega^2\chi$. Tortoise (interior): $r_*^{(\text{in})} = -\text{arctanh}(r)$. $V_{\text{eff}} = f(r)[l(l+1)/r^2 + 2] + m_i^2$. BCs: $\chi \sim r^{l+1}$ at $r = 0$; $[\chi] = 0$ at r_s; outgoing at ∞.

Mass levels ($\xi = 0$):

Level	μ_i	m_i^2	Mult	Irreps
1	-4	$-32\lambda C_0^2$	7	irr_7
2	-2	0	14	$\text{irr}_6 \oplus \text{irr}_8$
3	0	$+32\lambda C_0^2$	42	(spinorial, not treated)
4	$+2$	$+64\lambda C_0^2$	14	irr_7
5	$+4$	$+96\lambda C_0^2$	7	$\text{irr}_1 \oplus \text{irr}_3 \oplus \text{irr}_3'$

[P12] [12]. Frobenius obstruction (Prop. 4.1, *[Derived]*): $\alpha_{\text{in}} = \pm i\omega/2$ (from $f'_{\text{in}}(r_s) = 2$); $\alpha_{\text{out}} = \pm i\omega$ (from $f'_{\text{out}}(r_s) = 1$). Ratio $\alpha_{\text{out}}/\alpha_{\text{in}} = 2 = \|L_a\|_F^2|_{n=4}/\|L_a\|_F^2|_{n=3}$ [7]. Wronskian-shooting invalid.

3. Connection Matrix at r_s — Feasibility Analysis

3.1. **Setup.** Let $u = r_s - r$ (interior local variable), $v = r - r_s$ (exterior). Frobenius solutions near r_s:

$$\text{Interior:} \quad \varphi_\pm^{(\text{in})} \sim u^{\pm i\omega/2},$$

$$\text{Exterior:} \quad \psi_\pm^{(\text{out})} \sim v^{\pm i\omega}.$$

Proposition 3.1 (*[Derived]*)**.** *For $\omega \neq 0$, no 2×2 matrix $M(\omega)$ over $\mathbb{C}$ satisfies*

$$c_+\, u^{i\omega/2} + c_-\, u^{-i\omega/2} = b_+\, v^{i\omega} + b_-\, v^{-i\omega}$$

as an identity for $\varepsilon = |r - r_s| \to 0$.

Proof. The four functions $\{\varepsilon^{i\omega/2}, \varepsilon^{-i\omega/2}, \varepsilon^{i\omega}, \varepsilon^{-i\omega}\}$ are linearly independent over $\mathbb{C}$ for $\omega \neq 0$. A linear identity between a combination of the first two and the last two forces all coefficients to zero. □

Remark 3.2 (*[Derived]*)**.** The mismatch ratio $2 = \alpha_{\text{out}}/\alpha_{\text{in}}$ originates in $f'_{\text{in}}(r_s)/f'_{\text{out}}(r_s) = 2 = \|L_a\|_F^2|_{n=4}/\|L_a\|_F^2|_{n=3}$ [7]. It is an algebraic invariant of the Black Circle geometry.

Remark 3.3 (Epistemic flag — Neumann condition at r_s, OP 13.2)**.** Paper 12 §4.1 states $[\chi] = 0$ and $[d\chi/dr] = 0$ at r_s, both labelled *[Derived]*. However, Paper 11 Proposition 3.1 gives only $[\varphi_k] = 0$; no Neumann condition follows from BI for $\xi = 0$. The condition $[d\chi/dr] = 0$ is not established and is carried as OP 13.2.

Working hypothesis [Conjecture] (BC assumption): both $[\chi] = 0$ and $[d\chi/dr] = 0$ hold at r_s in EF coordinates. All QNM results in §5 are conditional on this assumption until OP 13.2 closes.

3.2. **Path to $M(\omega)$ via hypergeometric connection formula [OP 13.1].** The interior de Sitter RW equation admits reduction to ${}_2F_1(a, b; c; z)$ via $z = r^2$. The Kummer connection formula at $z = 1$ relates the regular-at-0 solution to the Frobenius pair at $z = 1$ ($r = r_s$), with coefficients as Γ-function ratios of (a, b, c, ω). The Frobenius factor 2 from [7] appears in $\Gamma(c - a - b)/(\Gamma(a)\Gamma(b))$ at $z = 1$. Explicit identification of (a, b, c) as functions of (ω, l, m_i^2) is OP 13.1.

Remark 3.4 (*[Derived]*)**.** The connection matrix is not needed for the QNM computation: the EF matching condition of §4.5 bypasses $M(\omega)$ entirely. OP 13.1 is of algebraic interest but is not a prerequisite for §5.

4. EF Matching Condition and Interior Recurrence

4.1. **Interior ODE.** In units $r_s = 1$, $\xi = 0$:

$$(r^2 - 1)^2\chi'' + 2r(r^2 - 1)\chi' + \big[\omega^2 - m_i^2 - (r^2 - 1)\big(l(l + 1)/r^2 + 2\big)\big]\chi = 0. \tag{1}$$

4.2. **Interior series: parity and initial conditions.**

Proposition 4.1 (*[Derived]*)**.** *The regular solution of* (1) *at* $r = 0$ *with* $\chi \sim r^{l+1}$ *is purely even:*

$$\chi_{\text{in}}(r) = r^{l+1}\sum_{k=0}^{\infty} c_{2k}\, r^{2k}.$$

Proof. $r = 0$ is an ordinary point ($f_{\text{in}}(0) = -1 \neq 0$). Setting $\chi = c_0 r^{l+1} + c_1 r^{l+2} + \cdots$ and evaluating (1) forces $c_1 = 0$ for all $l \geq 0$. The ODE has only even powers of r in its coefficients, so even and odd index chains decouple, and $c_{2j+1} = 0$ for all j. □

Corollary 4.2 (*[Derived]*)**.** *The radius of convergence of the interior series is* 1 *(*$= r_s$*).*

4.3. **Interior three-term recurrence.**

Theorem 4.3 (*[Derived]*)**.** *For* $n = 0, 2, 4, \ldots$ *with* $c_{-2} = 0$, $c_0 = 1$*:*

$$\alpha_n\, c_{n+2} + \beta_n\, c_n + \gamma_n\, c_{n-2} = 0, \tag{2}$$

with

$$\begin{aligned}
\alpha_n &= (n + l + 3)(n + l + 2) + l(l + 1),\\
\beta_n &= \omega^2 - m_i^2 - 2(n + l + 1)^2 - l(l + 1) + 2,\\
\gamma_n &= (n + l - 1)(n + l) - 2.
\end{aligned}$$

Proof. Direct substitution of χ_{in} into (1) and collection at each power r^{n+l+1}. □

Verifications. ($n = 0, l = 0, m^2 = 0$): $\alpha_0 = 6$, $\beta_0 = \omega^2$, $c_2 = -\omega^2/6$. ✓ ($n = 0, l = 2, m^2 = 0$): $\gamma_0 = (2 - 1)(2) - 2 = 0$. ✓

4.4. The Frobenius mismatch vanishes in tortoise coordinates.

Proposition 4.4 (*[Derived]*). *In the respective tortoise coordinates, both sets of Frobenius solutions at $r \to r_s$ behave as $e^{\pm i\omega r_*}$. The exponent mismatch of Paper 12, Prop. 4.1 is a coordinate artifact.*

Proof. *Interior:* $u = 1 - r \to 0^+$, $f_{\rm in} \approx -2u$, $dr_*^{\rm (in)}/dr \approx -1/(2u)$, $r_*^{\rm (in)} \approx \frac{1}{2}\ln u + \text{const}$, so $u^{\pm i\omega/2} = e^{\pm i\omega r_*^{\rm (in)}}$. *Exterior:* $v = r - 1 \to 0^+$, $f_{\rm out} \approx v$, $dr_*^{\rm (out)}/dr \approx 1/v$, $r_*^{\rm (out)} \approx \ln v + \text{const}$, so $v^{\pm i\omega} = e^{\pm i\omega r_*^{\rm (out)}}$. The factor of 2 difference in the log-structures is absorbed into r_* (coming from $f'_{\rm in}(r_s)/f'_{\rm out}(r_s) = 2$ [7]). □

Corollary 4.5 (*[Derived]*). *The EF matching method is free of the Frobenius obstruction of Paper 12, Prop. 4.1.*

4.5. The EF matched QNM condition.

Theorem 4.6 (*[Derived]*). *The Black Circle QNM frequencies are the values $\omega \in \mathbb{C}$ with $\mathrm{Im}(\omega) < 0$ satisfying*

$$H_{\rm BC}(\omega; l, m_i^2) := D_{\rm in}(\omega; l, m_i^2) - 2^{i\omega/2}\, e^{i\omega} = 0, \tag{3}$$

where $D_{\rm in}$ is the Frobenius coefficient of $(1-r)^{-i\omega/2}$ in $\chi_{\rm in}$, extracted via the exact Wronskian formula:

$$D_{\rm in} = u^{i\omega/2}\left[\frac{\chi_{\rm in}(r)}{2} + \frac{u}{i\omega}\frac{d\chi_{\rm in}}{dr}(r)\right], \qquad u = 1 - r. \tag{4}$$

The right-hand side $2^{i\omega/2}e^{i\omega}$ encodes: $2^{i\omega/2}$ from the interior tortoise phase at $u \to 0$; $e^{i\omega}$ from the exterior tortoise phase at $v \to 0$.

Proof. In EF coordinates the field $F = e^{i\omega r_*}\chi$ is continuous at r_s. Taking the limit from both sides and imposing $[\chi] = 0$ (working hypothesis, Remark 3.3): $F_{\rm in}(r_s) = 2^{-i\omega/2} D_{\rm in}$ (interior) and $F_{\rm out}(r_s) = e^{i\omega} D_{\rm out}$ (exterior). With the S_1 normalisation $D_{\rm out} = 1$ (outgoing-at-∞ exterior solution, ingoing Frobenius component coefficient $= 1$ at r_s), matching gives $D_{\rm in} = 2^{i\omega/2}e^{i\omega}$, i.e. (3). □

Remark 4.7 (*[Derived]*). Condition (3) involves **only the interior de Sitter ODE**. The exterior Schwarzschild geometry provides the radiation channel (and determines the field profile for $r > r_s$) but does not enter the eigenvalue equation. This decoupling is exact, not perturbative.

Proposition 4.8 (*[Derived]*). *$H_{\rm BC}(-i\kappa; l, m_i^2) \in \mathbb{R}$ for all $\kappa, l, m_i^2 \in \mathbb{R}$.*

Proof. For $\omega = -i\kappa$: all recurrence coefficients $\alpha_n, \beta_n = -\kappa^2 - m_i^2 - \cdots$ are real, hence $c_{2k} \in \mathbb{R}$. The extraction gives $D_{\rm in} = u^{\kappa/2}[\chi_{\rm in}/2 + u\, d\chi_{\rm in}/dr/\kappa] \in \mathbb{R}$. The RHS $2^{\kappa/2}e^{\kappa} \in \mathbb{R}$. Hence $H_{\rm BC}(-i\kappa) \in \mathbb{R}$. □

Corollary 4.9 (*[Derived]*). *The imaginary-axis QNMs (Type I, §5.2) are solutions of a real 1D equation $\kappa \mapsto H_{\rm BC}(-i\kappa) = 0$. They lie exactly on the imaginary axis.*

4.6. General EF matching formula for two-domain metrics.

Corollary 4.10 (*[Derived]*). *Let (M, g) be any spherically symmetric spacetime with a thin shell at $r = r_s$, with surface-gravity derivatives $\kappa_{\rm in} = f'_{\rm in}(r_s)$ and $\kappa_{\rm out} = f'_{\rm out}(r_s)$. The EF matching condition for scalar QNMs is*

$$H(\omega) = D_{\rm in}(\omega) - \kappa_{\rm in}^{-i\omega/\kappa_{\rm in}} \cdot e^{i\omega/\kappa_{\rm out}} \cdot D_{\rm out}(\omega) = 0. \tag{5}$$

For the Black Circle: $\kappa_{\rm in} = 2$, $\kappa_{\rm out} = 1$, recovering (3).

The ratio $\kappa_{\rm in}/\kappa_{\rm out} = 2$ is the Frobenius factor from [7]. Formula (5) applies to de Sitter/Schwarzschild, de Sitter/Reissner–Nordström, Minkowski/Schwarzschild, and any two-domain metric with a thin shell.

Corollary 4.11 (*[Derived]*)**.** *The Hawking radiation spectrum of the Black Circle is identical to the Schwarzschild Hawking spectrum for a black hole of the same mass M.*

Proof. The Hawking spectrum depends on the exterior surface gravity $\kappa_{\text{out}} = 1/(4GM) = f'_{\text{out}}(r_s)/4$, which is unchanged by the interior geometry. The Black Circle interior enters only through $H_{\text{BC}}(\omega) = 0$ (Theorem 4.6) — a separate eigenvalue problem that does not affect the exterior ODE. □

5. QNM Spectrum — Numerical Results

5.1. **Method.** Interior series (2): $N = 500$ terms (250 even coefficients); D_{in} evaluated at $r_1 = 0.96$ and $r_2 = 0.92$ via (4), averaged. H_{BC} evaluated on a 90×70 grid, $\text{Re}(\omega) \in (0.05, 3.2)$, $\text{Im}(\omega) \in (-0.85, -0.02)$; Newton refinement, tolerance 10^{-11}. Imaginary-axis scan via `brentq` on $H_{\text{BC}}(-i\kappa)$, $\kappa \in (0.005, 10)$. Growing-mode scan on $H_{\text{BC}}(+i\kappa)$, $\kappa \in (0.01, 8)$.

Schwarzschild reference: Konoplya & Zhidenko [14], scalar $s = 0$, $2M = r_s = 1$. All results verified on two independent machines.

5.2. **Type I modes — imaginary axis.** Non-oscillatory overdamped modes: field decays as $e^{-\kappa t}$. All residuals $|H_{\text{BC}}| < 5 \times 10^{-15}$.

Level	m^2	l	κ	$\tau = 1/\kappa$ (r_s/c)	label
2	0.0	0	0.081269	**12.305**	*[Derived]*
2	0.0	1	0.589896	1.695	*[Derived]*
2	0.0	2	0.741920	1.348	*[Derived]*
1	−0.5	0	—	—	[OP 13.9]
1	−0.5	1	0.539190	1.855	*[Derived]*
1	−0.5	2	0.710620	1.407	*[Derived]*
4	+1.0	0	0.406133	2.462	*[Derived]*
4	+1.0	1	0.687723	1.454	*[Derived]*
4	+1.0	2	0.804054	1.244	*[Derived]*
5	+1.5	0	0.517384	1.933	*[Derived]*
5	+1.5	1	0.735086	1.360	*[Derived]*
5	+1.5	2	0.834882	1.198	*[Derived]*

κ increases monotonically with l (fixed m^2) and with m^2 (fixed l). The Level 1, $l = 0$ entry is replaced by a growing mode (§5.4). Note: m^2 entries $\{-0.5, 0, +1.0, +1.5\}$ are representative dimensionless values $m_i^2 r_s^2$ with $r_s = 1$.

5.3. **Type II modes — oscillatory sector.** Found only at $l = 2$, $n = 0$. No $l = 0$ or $l = 1$ oscillatory modes in $\text{Re}(\omega) \in (0.05, 3.5)$, $\text{Im}(\omega) \in (-4.0, -0.02)$ (§6.2).

Level	m^2	$\text{Re}(\omega_{\text{BC}})$	$\text{Im}(\omega_{\text{BC}})$	$\text{Re}(\omega_{\text{Sch}})$	$\text{Im}(\omega_{\text{Sch}})$	$\lvert\omega_{\text{BC}}/\omega_{\text{Sch}}\rvert$
2	0.0	+2.136719	−0.084741	+0.9677	−0.1935	**2.167**
1	−0.5	+2.139847	−0.049687	+0.9677	−0.1935	**2.169**
4	+1.0	+2.133203	−0.156896	+0.9677	−0.1935	**2.167**
5	+1.5	+2.132808	−0.193633	+0.9677	−0.1935	**2.170**

ω_{Sch}: from [14]. Residuals $|H_{\text{BC}}| < 10^{-10}$.
Structural laws *[Derived]*:

$$\text{Re}(\omega_{\text{BC}}) = 2.135 \pm 0.004 \quad \text{(independent of } m^2\text{, 0.3\% variation)},$$
$$\text{Im}(\omega_{\text{BC}}) = -0.0847 - 0.0728\, m^2 \quad \text{(linear fit, residuals } < 0.002\text{)}.$$

The coupling coefficient $\Gamma_1 = d\,\mathrm{Im}(\omega_{\mathrm{BC}})/d(m^2) = 0.0728\,r_s/c$ is derivable via the implicit function theorem on H_{BC}:

$$\Gamma_1 = \mathrm{Im}\left[\frac{-\partial_{m^2} D_{\mathrm{in}}}{\partial_\omega D_{\mathrm{in}} - i(1+\frac{\ln 2}{2})\cdot 2^{i\omega/2}e^{i\omega}}\right]_{\omega_{\mathrm{BC}}^{(0)}} \tag{6}$$

where $\partial_{m^2} D_{\mathrm{in}}$ and $\partial_\omega D_{\mathrm{in}}$ each satisfy a recurrence derivable from (2) (OP 13.8).

5.4. **Level 1, $l = 0$: BF instability growing mode.** For $m^2 = -0.5$ (representative Level 1 mass), $l = 0$: no Type I damped mode on the lower imaginary axis. A zero of H_{BC} exists on the **upper** imaginary axis:

$$\omega_{\mathrm{grow}} = +4.138696\,i, \qquad \tau_{\mathrm{grow}} = 1/4.1387 = \mathbf{0.2416}\,r_s/c. \tag{7}$$

Growing mode as function of m^2 ($l = 0$):

m^2	κ_{grow}	τ_{grow} (r_s/c)
-0.30	3.524770	0.2837
-0.50	4.138696	0.2416
-0.70	4.692635	0.2131
-1.00	5.432411	0.1841
-1.50	6.490675	0.1541

All residuals $|H_{\mathrm{BC}}(+i\kappa_{\mathrm{grow}})| < 5 \times 10^{-11}$. The transition from stable Type I to growing mode occurs at $m^2_{\mathrm{crit}} \in (-0.30, 0)$ for $l = 0$ (OP 13.9).

5.5. **The $\sqrt{2}$ prediction — negative result.**

$$\text{mean}\ |\omega_{\mathrm{BC}}(l=2, n=0)/\omega_{\mathrm{Sch}}(l=2, n=0)| = 2.168 \pm 0.002.$$

The $\sqrt{2}$ prediction of Paper 10 is not confirmed. Label change: *[Conjecture]* (Paper 10) $\to$ *[Derived]*: false for $l = 2$, $n = 0$, all four levels. Discrepancy: 53% from $\sqrt{2} = 1.4142$.

Remark 5.1 (*[Derived]*)**.** Why the $\sqrt{2}$ argument was structurally wrong: it assumed coupling between the interior potential coefficient 2 and the exterior coefficient 1 in the eigenvalue equation. But Theorem 4.6 shows the eigenvalue equation is purely interior — the exterior coefficient 1 enters only as the phase $e^{i\omega}$ in H_{BC}, not as a potential term.

6. Physical Consequences

6.1. **Two mode families: de Sitter cavity picture.** The Black Circle QNM condition (3) is a purely interior condition. Its spectrum has two structurally distinct families.

Type I (overdamped, $\mathrm{Re}(\omega) = 0$): the cavity acts as a pure attractor. The $l = 0$ Level 2 mode ($\tau = 12.3\,r_s/c$) is the longest-lived mode in the BC spectrum.

Type II (oscillatory, $\mathrm{Re}(\omega) > 0$): found only for $l = 2$. The centrifugal barrier supports ringing modes at $\mathrm{Re}(\omega) \approx 2.14$.

6.2. **Why $l = 0$, $l = 1$ have no oscillatory modes.** For $l = 0$: $V_{\mathrm{eff}} = 2(r^2 - 1) < 0$ throughout $r \in (0, 1)$ — no potential barrier, only overdamped resonances. For $l = 1$: same qualitative picture. For $l = 2$: the centrifugal term $-6/r^2$ dominates near $r = 0$, supporting oscillatory modes at $\mathrm{Re}(\omega) \approx 2.14$. Whether $l = 0, 1$ oscillatory modes exist at larger $\mathrm{Re}(\omega)$ is OP 13.6.

6.3. **BF cavity stability bound vs. the BF bound.**

Proposition 6.1 (*[Derived]*)**.** *The stability condition for the Black Circle interior cavity is strictly more restrictive than the BF bound for infinite de Sitter* [16].

Proof. From the QNM computation: a growing mode ($\mathrm{Im}(\omega) > 0$) exists for $l = 0$ when $m^2_{\text{dimless}} < m^2_{\text{crit}}$, with $m^2_{\text{crit}} \in (-0.30, 0)\, r_s^{-2}$. The BF bound gives $m^2_{\text{BF}} = -9/(4r_s^2)$, i.e. $m^2_{\text{BF,dimless}} = -2.25$. Since $m^2_{\text{crit}} \gg m^2_{\text{BF}}$ ($-0.15 \gg -2.25$), there exists a range $m^2 \in (-2.25, -0.15)\, r_s^{-2}$ where the field is BF-stable in infinite de Sitter but generates a growing mode in the Black Circle cavity. □

Quantification. $m^2_{\text{Level1,dimless}} = -2.765\sqrt{\lambda G M^2}$. For $m^2_{\text{crit}} \approx -0.15\, r_s^{-2}$:

$$\lambda G M^2 < \left(\frac{0.15}{2.765}\right)^2 \approx 0.0030 \quad \text{[cavity bound, \textit{[Derived]}]}, \tag{8}$$

against the Paper 11 condition $\lambda G M^2 < 27\pi/128 \approx 0.663$. The cavity bound is $\approx 225\times$ more restrictive.

Remark 6.2. The Paper 11 BF stability claim ($\lambda G M^2 < 27\pi/128$) refers to the infinite de Sitter background. The correct statement for the Black Circle cavity is (8) (estimated; to be refined by OP 13.9).

6.4. **The Frobenius factor 2 as structural invariant.** The chain from algebra to spectrum:

[7]: $\|L_a\|_F^2$ ratio $= 2$	(sedenion algebra)
[10]: $f'_{\text{in}}(r_s)/f'_{\text{out}}(r_s) = 2$	(metric geometry)
[12]: Frobenius exponent ratio $= 2$	(ODE singularity)
Prop. 4.4: factor 2 absorbed into r_*	(tortoise log-structure)
Thm. 4.6: $2^{i\omega/2}$ phase in H_{BC}	(EF matching)
§5: exact QNM frequencies	(numerical spectrum)

6.5. **One-loop $T_{\mu\nu}$ estimate.** *Type II contribution:* $\sum_i N_i\, \mathrm{Im}(\omega^{(II)}_{\text{BC}}(m_i^2)) \approx -5.09\, r_s^{-1}$.
Type I contribution: $\sum_i N_i\, \kappa_l(m_i^2)$:

Level (N)	$\kappa_{l=0}$	$\kappa_{l=1}$	$\kappa_{l=2}$	Total $-\sum\kappa$
2 (14)	0.0813	0.5899	0.7419	$-14 \times 1.4131 = -19.78$
1 (7)	—	0.5392	0.7106	$-7 \times 1.2498 = -8.75$
4 (14)	0.4061	0.6877	0.8041	$-14 \times 1.8979 = -26.57$
5 (7)	0.5174	0.7351	0.8349	$-7 \times 2.0874 = -14.61$

Total Type I: $-(19.78 + 8.75 + 26.57 + 14.61) = -69.71\, r_s^{-1}$. Type I dominates by a factor ≈ 14 over Type II. Total estimate: $\delta\mathrm{Im}(W_{1\text{loop}}) \approx -(69.71 + 5.09)/r_s \approx -74.8\, r_s^{-1}$ [estimate, *[Derived]*]. Exact one-loop formula requires mode-function normalisations (OP 13.11).

7. Open Problems

OP 13.1 (Connection matrix via ${}_2F_1$). Reduce (1) to Gauss hypergeometric form via $z = r^2$. Identify (a, b, c) as functions of (ω, l, m_i^2). Apply Kummer's connection formula [13] to obtain $M(\omega)$ as Γ-function ratios. The $\omega = 0$ special case (relevant for OP 13.9) is analytically simpler and should be attempted first.

OP 13.2 (Derivation of $[d\chi/dr] = 0$ from EF regularity). Reformulate the RW equation in EF coordinates. Determine whether $[\partial_r \varphi_k] = 0$ follows from ODE regularity at r_s.

OP 13.3 (Existence theory for $H_{\rm BC}=0$). Prove that $H_{\rm BC}(\omega;l,m^2)$ has a discrete set of zeros with $\mathrm{Im}(\omega)<0$. In particular, prove uniqueness of the Type I mode per (l,m^2) using the sign structure of $H_{\rm BC}(-i\kappa)$.

OP 12.1 (carried forward). Prove or disprove $\chi(r_s)=0$ for spinorial Level-3 modes.

OP 13.5 (Level 3 bosonic decomposition, Track C). The 38 bosonic Level-3 modes are not yet decomposed into PSL(2, 7) irreps. Required: GAP 4.15.1 isotypic decomposition.

OP 13.6 ($l=0$, $l=1$ oscillatory BC QNMs). Targeted searches: (a) $\mathrm{Re}(\omega)\in(3.2,10)$ for $l=1$; (b) full lower-axis scan for $l=1$ for all levels.

OP 13.7 (Analytic origin of ratio 2.168). The Type II ratio $|\omega_{\rm BC}(l=2)/\omega_{\rm Sch}(l=2)|\approx 2.168$ has no identified algebraic form. OP 13.1 should yield the exact value via the Kummer connection formula.

OP 13.8 (Analytic Γ_1 from perturbation theory). Evaluate (6) numerically from the ω- and m^2-differentiated recurrences derived from (2).

OP 13.9 (Critical mass $m^2_{\rm crit}$ and cavity stability bound, priority for Paper 14). (a) Numerically: bisection of $H_{\rm BC}(-i\kappa;l=0,m^2)$ as $m^2\to 0^-$. (b) Analytically: at $\omega=0$, the condition $H_{\rm BC}(0;l=0,m^2_{\rm crit})=0$ becomes $D^{\rm static}_{\rm in}(m^2_{\rm crit})=1$ — via OP 13.1 ($\omega=0$ hypergeometric case), this gives an analytic equation for $m^2_{\rm crit}$ in terms of Γ-functions. (c) Physical: translate $m^2_{\rm crit}$ to the tight bound $\lambda GM^2<0.003$ and compare with the Paper 11 BF bound.

OP 13.10 (Iso-lifetime coincidence). Level 2: $\tau(\text{Type I}, l=0)=12.305$, $\tau(\text{Type II}, l=2)=11.81$ r_s/c; ratio 1.042. Algebraic origin unknown.

OP 13.11 (One-loop $T_{\mu\nu}$ from the full QNM spectrum). Compute normalised mode functions $\chi_n(r)$ and inner products $\langle\chi_m|\chi_n\rangle$ on $[0,r_s]$ for the exact one-loop contribution to $T_{\mu\nu}$.

8. Conclusions

8.1. **What this paper has done.** **§3** *[Derived]*: non-existence of naive $M(\omega)$ (Proposition 3.1); epistemic flag on $[d\chi/dr]=0$ (OP 13.2); OP 13.1 identified as path to $M(\omega)$ via Kummer's formula; EF method bypasses $M(\omega)$ entirely (Remark 3.4).

§4 *[Derived]*: interior series purely even (Proposition 4.1); step-2 recurrence (2) with explicit $(\alpha_n,\beta_n,\gamma_n)$; Frobenius mismatch vanishes in tortoise coordinates (Proposition 4.4); exact EF matched QNM condition $H_{\rm BC}=D_{\rm in}-2^{i\omega/2}e^{i\omega}=0$ (Theorem 4.6) — purely interior; $H_{\rm BC}(-i\kappa)\in\mathbb{R}$ (Proposition 4.8); general EF matching formula for two-domain metrics (Corollary 4.10); Hawking equivalence (Corollary 4.11).

§5 *[Derived]*: 12 Type I modes (machine precision); 4 Type II modes, all $l=2$, $n=0$; $\sqrt{2}$ prediction refuted (ratio 2.168 ± 0.002); linear mass law $\mathrm{Im}(\omega_{\rm BC})\approx -0.0847-0.0728\,m^2$; Γ_1 derivable from recurrence via implicit function theorem; BF instability exponent $\omega_{\rm grow}=+4.139i$, $\tau_{\rm grow}=0.242\,r_s/c$; one-loop action estimate $\approx -74.8\,r_s^{-1}$.

§6 *[Derived]*: cavity BF bound $\lambda GM^2<0.003$ ($\approx 225\times$ tighter than Paper 11); Hawking equivalence confirmed; Frobenius chain from algebra to spectrum complete.

8.2. **The $\sqrt{2}$ prediction: epistemic update.** The prediction $|\omega_{\rm BC}/\omega_{\rm Sch}|\approx\sqrt{2}$ *[Conjecture]* (Paper 10) is now *[Derived]*: false for $l=2$, $n=0$, all four mass levels. The ratio 2.168 has no identified algebraic form (OP 13.7).

The structural reason: the eigenvalue equation is purely interior (Theorem 4.6); the exterior coefficient 1 enters only as the phase $e^{i\omega}$, not as a potential term. The ratio $f'_{\rm in}(r_s)/f'_{\rm out}(r_s)=2$ affects the EF phase constants (Corollary 4.10) but not the resonant frequency itself.

8.3. **What opens for Paper 14.** From Paper 13: (1) cavity stability bound via OP 13.9; (2) analytic Γ_1 via OP 13.8; (3) oscillatory modes for $l=0,1$ via OP 13.6; (4) exact one-loop $T_{\mu\nu}$ via OP 13.11; (5) general EF formula applied to Reissner–Nordström or other two-domain metrics.

Appendix A: Recurrence Coefficients — Explicit Values

A.1 Interior recurrence — selected values. $l = 0$**:** $\alpha_n = (n+3)(n+2)$, $\beta_n = \omega^2 - m_i^2 - 2(n+1)^2 + 2$, $\gamma_n = (n-1)n - 2$.

$$n = 0: \quad \alpha = 6,\ \beta = \omega^2 - m_i^2,\ \gamma = -2 \qquad (c_{-2} = 0)$$
$$n = 2: \quad \alpha = 20,\ \beta = \omega^2 - m_i^2 - 16,\ \gamma = 0$$
$$n = 4: \quad \alpha = 42,\ \beta = \omega^2 - m_i^2 - 48,\ \gamma = 10$$

$l = 1$**:** $\alpha_n = (n+4)(n+3) + 2$, $\beta_n = \omega^2 - m_i^2 - 2(n+2)^2$, $\gamma_n = n(n+1) - 2$.

$$n = 0: \quad \alpha = 14,\ \beta = \omega^2 - m_i^2 - 8,\ \gamma = -2$$
$$n = 2: \quad \alpha = 32,\ \beta = \omega^2 - m_i^2 - 32,\ \gamma = 4$$

$l = 2$**:** $\alpha_n = (n+5)(n+4) + 6$, $\beta_n = \omega^2 - m_i^2 - 2(n+3)^2 - 4$, $\gamma_n = (n+1)(n+2) - 2$.

$$n = 0: \quad \alpha = 26,\ \beta = \omega^2 - m_i^2 - 22,\ \gamma = 0$$
$$n = 2: \quad \alpha = 48,\ \beta = \omega^2 - m_i^2 - 54,\ \gamma = 10$$

A.2 Schwarzschild reference QNMs. From [14], scalar $s = 0$, $2M = r_s = 1$:

(l, n)	$\mathrm{Re}(\omega_{\mathrm{Sch}})$	$\mathrm{Im}(\omega_{\mathrm{Sch}})$
$(0, 0)$	0.2209	−0.2098
$(1, 0)$	0.5854	−0.1955
$(2, 0)$	0.9677	−0.1935

Acknowledgements

Computational assistance was provided by Claude (Anthropic) for algebraic verification, structural analysis, and proof scaffolding. The author takes full responsibility for any errors. Numerical computations: Python 3 (`numpy`, `scipy`), $N = 500$ recurrence terms, verified on two independent machines.

References

[1] G. Levratti, *Sign-Parity Classification of Zero Divisors in Cayley–Dickson Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20315751.

[2] G. Levratti, *The Informational Identity Boundary: Algebraic Foundations and Physical Conjectures*, Zenodo preprint (2026), v4, doi:10.5281/zenodo.20411845.

[3] G. Levratti, *Sign Structure, Automorphism Groups, and the ZD Zeta Function of Cayley–Dickson Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20397587.

[4] G. Levratti, *Zero-Divisors in Cayley–Dickson Algebras: Fano Geometry, Automorphisms, and the Weil Zeta Function*, Zenodo preprint (2026), doi:10.5281/zenodo.20417859.

[5] G. Levratti, *Active Stabilizers in Reed–Muller CSS Codes from Cayley–Dickson Zero-Divisors*, Zenodo preprint (2026), doi:10.5281/zenodo.20431754.

[6] G. Levratti, *The Factorization Threshold in Cayley–Dickson Integer Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20442390.

[7] G. Levratti, *A Clifford Representation of the Fano Zero-Divisors*, Zenodo preprint (2026), doi:10.5281/zenodo.20445208.

[8] G. Levratti, *Fano Duality, Projective Orthogonality, and the CSS Bridge*, Zenodo preprint (2026), doi:10.5281/zenodo.20445734.

[9] G. Levratti, *A Quartic Contact Interaction with Fano–CSS Structure*, Zenodo preprint (2026), doi:10.5281/zenodo.20506165.

[10] G. Levratti, *Loop, Threshold, and Residue: The Algebraic Structure of the $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$ Transition*, Zenodo preprint (2026), doi:10.5281/zenodo.20507204.

[11] G. Levratti, *Canonical Quantisation of the Black Circle Interior*, Zenodo preprint (2026), doi:10.5281/zenodo.20508375.

[12] G. Levratti, *Boundary Conditions and Spinorial Statistics of the Black Circle Interior*, Zenodo preprint (2026), doi:10.5281/zenodo.20509490.

[13] E. W. Leaver, An analytic representation for the quasi-normal modes of Kerr black holes, *Proc. R. Soc. London A* **402** (1985), 285–298.

[14] R. A. Konoplya and A. Zhidenko, Quasinormal modes of black holes: from astrophysics to string theory, *Rev. Mod. Phys.* **83** (2011), 793–836.

[15] C. Barrabès and W. Israel, Thin shells in general relativity and cosmology: the lightlike limit, *Phys. Rev. D* **43** (1991), 1129–1142.

[16] P. Breitenlohner and D. Z. Freedman, Stability in gauged extended supergravity, *Ann. Phys.* **144** (1982), 249–281.

Independent Researcher, Modena, Italy
Email address: giovanni.levratti.research@gmail.com