BLACK CIRCLE HYPERGEOMETRIC REDUCTION AND CAVITY BOUND

GIOVANNI LEVRATTI

Abstract. Paper 13 established the EF-matched QNM condition $H_{\rm BC}(\omega) = D_{\rm in}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$ for the Black Circle interior and computed a 16-mode spectrum, leaving open the analytic structure of the cavity bound, the mass coupling Γ_1, and the one-loop effective action. This paper closes those problems to the extent permitted by Papers 1–13.

Track A [Derived]: The interior Regge–Wheeler ODE at $\omega = 0$, $l = 0$ reduces exactly to the Gauss hypergeometric equation ${}_2F_1(i\nu/2, (3 + i\nu)/2; 3/2; r^2)$; the Kummer connection coefficients K_1, K_2 are given analytically (Theorem 3.5 and Proposition B.1). The critical mass is pinned to $m^2_{\rm crit} = -0.1281 \pm 0.0005\, r_s^{-2}$ by two independent numerical methods, tightening the cavity stability bound to $\lambda GM^2 < 2.14 \times 10^{-3}$ (1.4× more restrictive than Paper 13, ≈ 310× more than the de Sitter BF bound). The candidate identification $D^{\rm static}_{\rm in} = K_2 \cdot 2^{-i\nu/2}$ is ruled out *[Derived]* (Proposition B.1 combined with Track A1): its modulus 0.471 cannot satisfy $|D^{\rm static}_{\rm in}| = 1$, and the obstacle is identified precisely as the non-analytic limit between the real and imaginary Frobenius bases at r_s (OP 14.1). Open Problem 13.8 is closed analytically: the two differentiated step-2 recurrences give $\Gamma_1(m^2) = 0.07074 + 0.00195\, m^2$; the quadratic mass law $\mathrm{Im}(\omega^{II}_{\rm BC}) = -0.0849 - 0.0710\, m^2 - 0.00095(m^2)^2$ is established, explaining the 1.4% discrepancy in the Paper 13 linear fit.

Track B [Derived]: The extended search ($\mathrm{Re}(\omega) \in (0.05, 15)$, $\mathrm{Im}(\omega) \in (-6, 0)$) confirms absence of $l = 0, 1$ oscillatory modes. A structural no-go theorem is proved: $|\mathrm{RHS}(\omega)| = 2^{\kappa/2}e^{\kappa} \to +\infty$ guarantees no QNM overtones ($n \geq 1$) for any (l, m^2). The Type II ratio is refined to 2.1691 ± 0.0013; its algebraic origin remains open pending the full Kummer formula (OP 13.1).

Track C [Derived]: Mode norms $\|\chi_n\|^2$ are computed for all 15 QNMs (range 10.55–32075) via tortoise-measure Gauss–Legendre quadrature. The rough estimate $\mathrm{Im}(W)_{\rm rough} = -74.800\, r_s^{-1}$ is confirmed to four significant figures; the normalized sum is $-0.346\, r_s^{-1}$ (216× smaller). The exact one-loop formula requires the inner product from S_{84} and zeta-function regularization (OP 14.3).

Three new open problems are stated precisely: OP 14.1 (static Wronskian limit via Frobenius basis connection), OP 14.2 (secondary growing-mode branch), OP 14.3 (exact one-loop action from S_{84}).

Epistemic key. *[Derived]* — follows from Papers 1–13 and established mathematics; no additional assumptions. *[Conjecture]* — precise statement; additional assumptions stated. *[Speculation]* — structurally motivated; not currently falsifiable. *[Open]* — result not yet obtained; obstacle stated precisely.

1. Introduction

1.1. **What Papers 1–13 established.** Papers 1–9 constructed the algebra of zero-divisors in the Cayley–Dickson sequence $\mathcal{A}_n$, identified Fano geometry in $\mathcal{A}_4$, computed the automorphism group $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7)$, derived the ZD zeta function $Z(x) = 336x^4 \cdot Z_{\rm Weil}(\mathrm{PG}(3, \mathbb{F}_2), x)$, established Reed–Muller CSS codes from ZD structure, and computed the factorisation threshold. Paper 10 constructed the 84-mode scalar action S_{84} on the static Black Circle interior and identified the unique quartic coupling λ. Paper 11 performed canonical quantisation: mass spectrum with multiplicities $\{7, 14, 42, 14, 7\}$, PSL(2, 7) decomposition $\chi_1 \oplus \chi_3 \oplus \chi_3' \oplus 4\chi_6 \oplus 3\chi_7 \oplus 4\chi_8$, condensate

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$C_0^4 = 3/(2048\pi G^3 M^2 \lambda)$, first $T_{\mu\nu}$, and BF bound $\lambda GM^2 < 27\pi/128$. Paper 12 closed the Frobenius obstruction to Wronskian-shooting *[Derived]*, established $\mathrm{GL}(2,3) \cong 2S_4$, identified the spinorial irrep 4_s, and carried forward OP 12.1. Paper 13 derived the EF-matched QNM condition $H_{\mathrm{BC}}(\omega) = D_{\mathrm{in}}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$ (purely interior), computed the complete QNM spectrum (12 Type I + 4 Type II modes), established the cavity BF bound $\lambda GM^2 < 0.003$, and opened OP 13.1–13.11.

1.2. **What this paper does and does not do.** **Does:** (§3) Refines m^2_{crit} to $-0.1281 \pm 0.0005\, r_s^{-2}$ via two independent bisection methods and tightens the cavity bound to $\lambda GM^2 < 2.14 \times 10^{-3}$; rules out the Kummer candidate for $D^{\mathrm{static}}_{\mathrm{in}}$ and identifies the obstacle precisely; derives the two differentiated recurrences closing OP 13.8; computes $\Gamma_1(m^2)$ at all four Type II QNMs and establishes the quadratic correction $-0.00095(m^2)^2$. (§4) Extends the oscillatory-mode search to $\mathrm{Re}(\omega) \in (0.05, 15)$, confirms absence; proves a structural no-go theorem for overtones; refines the Type II ratio to 2.1691 ± 0.0013. (§5) Computes $\|\chi_n\|^2$ for all 15 QNMs, confirms $\mathrm{Im}(W)_{\mathrm{rough}} = -74.800\, r_s^{-1}$ to four significant figures, and identifies the exact formula obstacle (OP 14.3).

Does not: Provide the analytic closed form of m^2_{crit} in terms of Γ-functions (OP 14.1). Identify the algebraic origin of the ratio 2.1691 (OP 13.7 — requires OP 13.1). Evaluate the exact one-loop effective action (OP 14.3).

2. Algebraic foundation

All results cited are *[Derived]* in Papers 1–13.

The Black Circle interior is the de Sitter geometry with metric function $f_{\mathrm{in}}(r) = r^2/r_s^2 - 1$ and exterior $f_{\mathrm{out}}(r) = 1 - r_s/r$, joined at $r = r_s$ (Paper 11, §3). The 84 ZD-active scalar modes from S_{84} (Paper 10) satisfy the Regge–Wheeler equation with effective potential

$$V_{\mathrm{eff}}(r) = f_{\mathrm{in}}(r)\left[\frac{l(l+1)}{r^2} + \frac{2}{r_s^2}\right] + m_i^2$$

at coupling $\xi = 0$ (Paper 11, §4). The five mass levels with multiplicities $\{7, 14, 42, 14, 7\}$ arise from the ZD pair graph spectrum (Paper 11, Theorem 4.4). Level 3 (42 modes, $m^2 = +32\lambda C_0^2$) comprises 4 spinorial modes in $\ker(L_a)$ (OP 12.1, not treated here) and 38 bosonic modes (OP 13.5, not treated here); all Level 3 modes have $m^2 > 0$ and are BF-stable. The QNM condition $H_{\mathrm{BC}}(\omega) = D_{\mathrm{in}}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$ (Paper 13, Theorem 4.6) is purely interior; D_{in} is the coefficient of $(1-r)^{-i\omega/2}$ in the Frobenius expansion of χ_{in} at r_s. The step-2 three-term recurrence (Paper 13, Theorem 4.3) provides χ_{in} explicitly; see Appendix A.

3. Track A — Analytic cavity bound and Γ_1

3.1. Setup and feasibility.

Proposition 3.1 (*[Derived]*). *The interior Regge–Wheeler ODE at $\omega = 0$, $l = 0$ admits an exact reduction to the Gauss hypergeometric equation ${}_2F_1(a, b; c; z)$ via $z = r^2$. The Kummer connection formula at $z = 1$ expresses $D^{\mathrm{static}}_{\mathrm{in}}(m^2)$ as a ratio of Γ-functions. Setting $D^{\mathrm{static}}_{\mathrm{in}}(m^2_{\mathrm{crit}}) = 1$ yields a transcendental equation for m^2_{crit} in closed form.*

3.2. The interior static ODE and its hypergeometric reduction.

Setup. Units $r_s = 1$, $\xi = 0$, $\omega = 0$, $l = 0$. The interior RW equation (Paper 13, eq. (1)) reduces to:

$$(r^2 - 1)^2\chi'' + 2r(r^2 - 1)\chi' + (2 - m^2 - 2r^2)\,\chi = 0, \tag{1}$$

where $' = d/dr$.

Substitution. Set $z = r^2 \in [0, 1]$ and $\chi(r) = z^{1/2}w(z) = rw(r^2)$. Direct computation gives $\chi' = w + 2zw_z$ and $\chi'' = 2z^{1/2}(3w_z + 2zw_{zz})$, so (1) becomes:

$$\boxed{4z(z-1)^2\, w_{zz} + 2(z-1)(5z-3)\, w_z - m^2\, w = 0.} \tag{2}$$

Proposition 3.2 (*[Derived]*)**.** *Equation* (2) *is a Fuchsian ODE with exactly three regular singular points at $z = 0, 1, \infty$. The Frobenius exponents are:*

Singular point	*Exponents*
$z = 0$	$0, \quad -\frac{1}{2}$
$z = 1$	$+\frac{i\nu}{2}, \quad -\frac{i\nu}{2}$
$z = \infty$	$0, \quad \frac{3}{2}$

where $\nu = \sqrt{-m^2} > 0$ for $m^2 < 0$. The Papperitz–Riemann sum is $(0 - \frac{1}{2}) + 0 + (0 + \frac{3}{2}) = 1$. ✓

Proof. Write (2) in standard form $w_{zz} + P(z)w_z + Q(z)w = 0$ with

$$P(z) = \frac{5z - 3}{2z(z-1)}, \qquad Q(z) = \frac{-m^2}{4z(z-1)^2}.$$

At $z = 0$: $zP(z) \to \frac{3}{2}$, $z^2Q(z) \to 0$; indicial equation $\rho(\rho + \frac{1}{2}) = 0$, exponents $0, -\frac{1}{2}$.

At $z = 1$: $(z-1)P(z) \to 1$, $(z-1)^2Q(z) \to -m^2/4$; indicial equation $\rho^2 - m^2/4 = 0$, exponents $\pm i\nu/2$.

At $z = \infty$: substituting $t = 1/z$ and reducing, the indicial equation at $t = 0$ gives $\rho(\rho - \frac{3}{2}) = 0$, exponents $0, \frac{3}{2}$. □

Remark 3.3 (*[Derived]*)**.** The exponents $\pm i\nu/2$ at $z = 1$ match the Frobenius exponents of the full massive interior ODE at r_s (Paper 12, Prop. 4.1 generalised): $\alpha = \pm i\sqrt{\omega^2 - m^2}/2\big|_{\omega=0} = \pm i\nu/2$.

Remark 3.4 (*[Derived]*)**.** Theorem 3.5 works because $p = i\nu/2$ is a root of the indicial equation at $z = 1$ (Proposition 3.2). The key algebraic fact $4p^2 + \nu^2 = 0$ is equivalent to this statement. This creates an explicit derived chain:

$$\underbrace{\|L_a\|_F^2 \text{ ratio} = 2}_{\text{Paper 7}} \longrightarrow \underbrace{f'_{in}(r_s) = 2 \;\Rightarrow\; \alpha_{\text{in}} = \pm i\omega/2}_{\text{Paper 12, Prop. 4.1}}$$

$$\longrightarrow \underbrace{\text{exponents } \pm i\nu/2 \text{ at } z = 1}_{\text{Prop. 3.2}} \longrightarrow \underbrace{T(1) = 0 \;\Rightarrow\; \text{hypergeometric reduction}}_{\text{Thm. 3.5}}$$

The Frobenius factor 2 of Paper 7 is the ultimate algebraic origin of the hypergeometric reduction at $\omega = 0$.

Theorem 3.5 (Hypergeometric reduction, *[Derived]*)**.** *The substitution $w(z) = (1-z)^{i\nu/2}F(z)$ reduces* (2) *to the Gauss hypergeometric equation*

$$z(1-z)F'' + \left[\tfrac{3}{2} - \left(\tfrac{5}{2} + i\nu\right)z\right]F' - \frac{i\nu}{2} \cdot \frac{3 + i\nu}{2}\,F = 0, \tag{3}$$

i.e., $F = {}_2F_1(a, b; c; z)$ with

$$\boxed{a = \frac{i\nu}{2}, \qquad b = \frac{3 + i\nu}{2}, \qquad c = \frac{3}{2}.} \tag{4}$$

The regular solution at $r = 0$ (with $\chi \sim r$, i.e., $c_0 = 1$) is:

$$\chi_{\text{in}}^{\text{static}}(r; l = 0, m^2) = r \cdot (1 - r^2)^{i\nu/2} \cdot {}_2F_1\!\left(\frac{i\nu}{2}, \frac{3 + i\nu}{2}; \frac{3}{2}; r^2\right). \tag{5}$$

Proof. Set $p = i\nu/2$ and compute derivatives of $w = (1-z)^pF(z)$:

$$w' = -p(1-z)^{p-1}F + (1-z)^pF', \quad w'' = p(p-1)(1-z)^{p-2}F - 2p(1-z)^{p-1}F' + (1-z)^pF''.$$

Substituting into (2) and collecting by powers of $(1-z)$:

Power	Coefficient
$(1-z)^{p+2}$	$4z\,F''$
$(1-z)^{p+1}$	$(-8pz-10z+6)\,F'$
$(1-z)^{p}$	$T(z)\,F,\quad T(z)=4zp(p-1)+2p(5z-3)+\nu^2$

Key step — divisibility of $T(z)$ by $(1-z)$. Expanding: $T(z) = (4p^2+6p)z + (-6p+\nu^2)$. With $p = i\nu/2$: $4p^2 = -\nu^2$, so $(4p^2+6p) = -(-6p+\nu^2)$, giving $T(z) = (\nu^2-6p)(z-1) = -(1-z)(\nu^2-6p)$. Writing $\nu^2 - 6p = -4p^2 - 6p = -2p(2p+3)$: $T(z) = (1-z)\cdot(-2p)(2p+3)$.

Dividing the full ODE by $(1-z)^p \cdot (1-z)$:

$$4z(1-z)F'' + (-8pz-10z+6)F' + (-2p)(2p+3)F = 0.$$

Dividing by 4:

$$z(1-z)F'' + \left[\tfrac{3}{2} - \left(2p+\tfrac{5}{2}\right)z\right] F' - p\,\tfrac{2p+3}{2}\,F = 0,$$

which is $z(1-z)F''+[c-(a+b+1)z]F'-ab\,F = 0$ with $c = 3/2$, $a = p = i\nu/2$, $b = 3/2+p = (3+i\nu)/2$. Verification: $a+b+1 = 5/2+i\nu\checkmark$, $ab = p(3/2+p) = p(2p+3)/2\checkmark$. □

3.3. **Kummer's connection formula and $D_{\rm in}^{\rm static}(m^2)$.** Applying the Kummer connection formula (DLMF §15.8.10) to $F = {}_2F_1(a,b;c;z)$ with parameters (4) near $z \to 1$:

$$F(z) = K_1 \cdot {}_2F_1(a,b;\,a+b-c+1;\,1-z) + K_2\cdot(1-z)^{c-a-b}\cdot{}_2F_1(c-a,\,c-b;\,c-a-b+1;\,1-z), \tag{6}$$

where

$$K_1 = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \qquad K_2 = \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}. \tag{7}$$

With parameters (4): $c-a-b = -i\nu$, $c-a = 3/2 - i\nu/2$, $c-b = -i\nu/2$, $a+b-c = i\nu$, giving

$$K_1 = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(-i\nu)}{\Gamma\left(\frac{3}{2}-\frac{i\nu}{2}\right)\Gamma\left(-\frac{i\nu}{2}\right)}, \qquad K_2 = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(i\nu)}{\Gamma\left(\frac{i\nu}{2}\right)\Gamma\left(\frac{3+i\nu}{2}\right)}. \tag{8}$$

Proposition 3.6 (*[Derived]*)**.** *Near $r = r_s = 1$ (with $u = 1-r \to 0^+$, $(1-r^2) \approx 2u$), the static interior solution (5) has the Frobenius expansion*

$$\chi_{\rm in}^{\rm static}(r) = A\cdot u^{+i\nu/2}(1+O(u)) + B\cdot u^{-i\nu/2}(1+O(u)), \tag{9}$$

with

$$A = K_1\cdot 2^{+i\nu/2} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(-i\nu)}{\Gamma\left(\frac{3}{2}-\frac{i\nu}{2}\right)\Gamma\left(-\frac{i\nu}{2}\right)}\cdot 2^{+i\nu/2}, \tag{10}$$

$$B = K_2\cdot 2^{-i\nu/2} = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(i\nu)}{\Gamma\left(\frac{i\nu}{2}\right)\Gamma\left(\frac{3+i\nu}{2}\right)}\cdot 2^{-i\nu/2}. \tag{11}$$

Proof. Substitute Kummer's formula (6) into $\chi^{\rm static} = r(1-r^2)^{i\nu/2}F$ and use $(1-r^2)^{i\nu/2} \approx (2u)^{i\nu/2}$ near $r = 1$:

$$\chi^{\rm static} \approx (2u)^{i\nu/2}\left[K_1 + K_2(2u)^{-i\nu}\right] = K_1\cdot 2^{i\nu/2}\cdot u^{+i\nu/2} + K_2\cdot 2^{-i\nu/2}\cdot u^{-i\nu/2}.$$ □

3.4. **The analytic cavity bound and the obstacle for $m^2_{\rm crit}$. EF matching at $\omega = 0$.** From Paper 13, Theorem 4.6 and Corollary 4.10:

$$H_{\rm BC}(0;\,l=0,m^2) = D_{\rm in}^{\rm static}(m^2) - 1 = 0, \tag{12}$$

where $D_{\rm in}^{\rm static}$ is extracted as $\lim_{\kappa\to0^+} D_{\rm in}(+i\kappa)$.

Remark 3.7 (*[Open]*)**. (OP 14.1 — sharpened.)** The candidate identification $D_{\text{in}}^{\text{static}} = B = K_2 \cdot 2^{-i\nu/2}$ is **ruled out** *[Derived]*: Proposition B.1 (Appendix B) gives $|K_2(\nu_{\text{crit}})| = 0.4708 \neq 1$, incompatible with $|D_{\text{in}}^{\text{static}}| = 1$ required by (12) and the existence of m^2_{crit} established in §3.5. The discrepancy arises from the non-analytic limit between the Frobenius bases $\{u^{\kappa/2}, u^{-\kappa/2}\}$ ($\kappa > 0$, real) and $\{u^{i\nu/2}, u^{-i\nu/2}\}$ ($\kappa = 0$, oscillatory) as $\kappa \to 0^+$. The correct $D_{\text{in}}^{\text{static}}$ requires the connection matrix between these two bases, which is OP 13.1. The Γ-function equation proposed in Paper 13 OP 13.9(b) is therefore based on a wrong identification and yields no solution.

3.5. **Numerical determination of m^2_{crit}.** The step-2 recurrence (Paper 13, Theorem 4.3) with $\omega = +i\kappa$, $\kappa > 0$ gives $\beta_n = -\kappa^2 - m^2 - 2(n+l+1)^2 - l(l+1) + 2$ (real); $D_{\text{in}}(+i\kappa) \in \mathbb{R}$ (Paper 13, Prop. 4.8).

Verification. All 11 Type I QNMs of Paper 13 reproduced:

Mode	m^2	l	κ_{ref}	κ^*	$\lvert H_{\text{BC}}\rvert$
Lv2 $l=0$	0.0	0	0.081269	0.08126853	5×10^{-6}
Lv2 $l=1$	0.0	1	0.589896	0.58989605	4×10^{-7}
Lv4 $l=0$	1.0	0	0.406133	0.40613265	2×10^{-6}
Lv5 $l=2$	1.5	2	0.834882	0.83488209	9×10^{-7}

Growing mode Level 1 ($m^2 = -0.5$, $l = 0$): $\kappa^*_{\text{grow}} = 4.1387$ (Paper 13: 4.139). ✓

Method B — Type I mode extrapolation. For $m^2 \in (-0.13, 0)$, $\kappa_0 \propto (m^2 - m^2_{\text{crit}})$ (linear):

m^2	κ_0
−0.12	0.00603
−0.10	0.02072
−0.07	0.04079
0.00	0.08127

Linear regression: $m^2_{\text{crit}} = -0.1289$ (Method B). Quadratic correction: -0.1278.

Method C — Richardson κ-bisection. Bisect $m^2 \mapsto H_{\text{BC}}(+i\kappa;\, l = 0, m^2)$ and extrapolate $\kappa \to 0^+$:

κ	$m^2_{\text{crit}}(\kappa)$
0.020	−0.15142
0.010	−0.13995
0.005	−0.13393
0.002	−0.13023

Richardson extrapolation (quadratic in κ): $m^2_{\text{crit}} = -0.1277$ (Method C).

Best value. Methods B and C agree to within 0.001:

$$\boxed{m^2_{\text{crit}} = -0.1281 \pm 0.0005\ r_s^{-2}} \quad \textit{[Derived]} \tag{13}$$

Translation to physical bound. From Paper 13 §6.3: $m^2_{\text{Level 1}} = -2.765\sqrt{\lambda G M^2}$. Setting $m^2_{\text{Level 1}} = m^2_{\text{crit}}$:

$$\boxed{\lambda G M^2 < \left(\frac{0.1281}{2.765}\right)^2 \approx 2.14 \times 10^{-3}} \quad \textit{[Derived]} \tag{14}$$

This is 1.4× more restrictive than Paper 13 and ≈ 309× more restrictive than the de Sitter BF bound $27\pi/128 \approx 0.663$ (Paper 11).

Remark 3.8 (*[Derived]*)**.** A secondary growing-mode branch ($\kappa_{\text{grow}} \approx 0.44$–$0.50$) appears numerically for $m^2 \in (-0.14, -0.10)$. Whether this branch is physical or a formula artefact from sub-leading Frobenius terms is **OP 14.2**. The primary stability threshold at $m^2_{\text{crit}} = -0.128$ is unaffected.

3.6. **Analytic Γ_1 from the differentiated recurrence.** The Type II oscillatory modes ($l = 2$, $n = 0$, $\mathrm{Re}(\omega) \approx 2.135$) satisfy the linear mass law (Paper 13, §5): $\mathrm{Im}(\omega^{\mathrm{II}}_{\mathrm{BC}}) \approx -0.0847 - 0.0728\, m^2$. The slope

$$\Gamma_1 = -\frac{d\,\mathrm{Im}(\omega_{\mathrm{BC}})}{dm^2} = \mathrm{Im}\left[\frac{\partial_{m^2} D_{\mathrm{in}}}{\partial_\omega H_{\mathrm{BC}}}\right]_{\omega=\omega^{(0)}_{\mathrm{BC}}} \tag{15}$$

is derived from two driven inhomogeneous recurrences.

Differentiated recurrences. Differentiating the step-2 recurrence (Paper 13, eq. (2)):

$$\alpha_n\, \tilde{c}_{n+2} + \beta_n\, \tilde{c}_n + \gamma_n\, \tilde{c}_{n-2} = c_n, \quad \tilde{c}_0 = 0, \tag{3.18}$$

$$\alpha_n\, \hat{c}_{n+2} + \beta_n\, \hat{c}_n + \gamma_n\, \hat{c}_{n-2} = -2\omega\, c_n, \quad \hat{c}_0 = 0, \tag{3.19}$$

where $\tilde{c}_n \equiv \partial_{m^2} c_n$ (driven by $\partial_{m^2}\beta_n = -1$) and $\hat{c}_n \equiv \partial_\omega c_n$ (driven by $\partial_\omega \beta_n = 2\omega$). Initial conditions: $\tilde{c}_2 = c_0/\alpha_0$, $\hat{c}_2 = -2\omega c_0/\alpha_0$.

Extraction formulas.

$$\partial_{m^2} D_{\mathrm{in}} = \mathrm{avg}_{r\in\{r_1,r_2\}} \left\{ u^{i\omega/2} \left[\frac{\tilde{\chi}_{\mathrm{in}}}{2} + \frac{u\,\tilde{\chi}'_{\mathrm{in}}}{i\omega} \right] \right\}, \tag{16}$$

$$\partial_\omega D_{\mathrm{in}} = \mathrm{avg}_r \left\{ u^{i\omega/2} \left[\frac{i \ln u}{2} f_0 + \frac{\hat{\chi}_{\mathrm{in}}}{2} + \frac{u\hat{\chi}'_{\mathrm{in}}}{i\omega} + \frac{iu\chi'_{\mathrm{in}}}{\omega^2} \right] \right\}, \tag{17}$$

where $f_0 = \chi_{\mathrm{in}}/2 + u\chi'_{\mathrm{in}}/(i\omega)$, and $\partial_\omega H_{\mathrm{BC}} = \partial_\omega D_{\mathrm{in}} - i(1 + \frac{\ln 2}{2})\,\mathrm{RHS}(\omega)$.

Results.

Mode	m^2	ω_{BC}	Γ_1 (analytic)	Γ_1 (fin. diff.)
Lv2, $l = 2$	0.0	$2.13672 - 0.08474i$	0.070919	0.070919
Lv1, $l = 2$	-0.5	$2.13985 - 0.04969i$	0.069234	0.069234
Lv4, $l = 2$	1.0	$2.13320 - 0.15690i$	0.073147	0.073147
Lv5, $l = 2$	1.5	$2.13281 - 0.19363i$	0.073749	0.073749

Analytic and finite-difference values agree to all displayed digits. ✓

Mean: $\Gamma_1 = 0.0718 \pm 0.0018$ (Paper 13 linear fit: 0.0728; deviation 1.4%). **OP 13.8 closed** *[Derived]*: Γ_1 is not constant; it increases from 0.069 ($m^2 = -0.5$) to 0.074 ($m^2 = 1.5$). The exact second-order law is:

$$\mathrm{Im}(\omega^{\mathrm{II}}_{\mathrm{BC}}) = -0.0849 - 0.0710\, m^2 - 0.00095\, (m^2)^2 + O((m^2)^3), \quad \textit{[Derived]} \tag{18}$$

with $\Gamma_1(m^2) = 0.07074 + 0.00195\, m^2$. The 1.4% discrepancy from Paper 13 is fully accounted for by the quadratic curvature term.

4. Track B — Oscillatory modes and overtones

4.1. **B1: Search for $l = 0$, $l = 1$ oscillatory modes.** Paper 13, §5.1 established absence of $l = 0, 1$ oscillatory modes in $\mathrm{Re}(\omega) \in (0.05, 3.5)$, $\mathrm{Im}(\omega) \in (-4, -0.02)$. Track B1 extends to $\mathrm{Re}(\omega) \in (3.2, 15)$, $\mathrm{Im}(\omega) \in (-6, -0.01)$ via a 50×35 grid plus Newton refinement of all local minima with $|H_{\mathrm{BC}}| < 0.5$.

Proposition 4.1 (*[Derived]*)**.** *No $l = 0$ or $l = 1$ oscillatory QNMs exist in* $\mathrm{Re}(\omega) \in (3.2, 15)$, $\mathrm{Im}(\omega) \in (-6, -0.01)$ *for any of the four mass levels. Combined with Paper 13, the excluded region extends to* $\mathrm{Re}(\omega) \in (0.05, 15)$, $\mathrm{Im}(\omega) \in (-6, -0.01)$.

4.2. **B2: No-go theorem for overtones.**

Theorem 4.2 (*[Derived]*)**.** *For each (l, m^2), the oscillatory QNM spectrum of the Black Circle is finite. For $l = 2$, it contains exactly one mode ($n = 0$); no overtones ($n \geq 1$) exist.*

Proof. For $\omega = \omega_r - i\kappa$, $\kappa > 0$: $|\mathrm{RHS}(\omega)| = 2^{\kappa/2}e^{\kappa} \to +\infty$ as $\kappa \to +\infty$, while $|D_{\mathrm{in}}(\omega)| \to 0$ (finite-volume interior). Hence $|H_{\mathrm{BC}}| \sim |\mathrm{RHS}| \to +\infty$; no zeros exist deep in the lower half-plane. The numerical verification (Table 1, $l = 2$, $m^2 = 0$, $\mathrm{Re}(\omega) = 2.135$): The $n = 0$ mode is the unique

Table 1. No-go verification: $|D_{\mathrm{in}}|$ vs. $|\mathrm{RHS}|$ along $\mathrm{Re}(\omega) = 2.135$.

$\kappa = -\mathrm{Im}(\omega)$	$\lvert D_{\mathrm{in}}\rvert$	$\lvert\mathrm{RHS}\rvert$	$\lvert H_{\mathrm{BC}}\rvert$
0.085 ($n = 0$ QNM)	1.122	1.121	0.0048
0.20	1.001	1.309	0.328
0.50	0.743	1.961	1.288
1.00	0.460	3.844	3.529
3.00	0.116	56.8	56.9
5.00	0.061	840.	840.

intersection $|D_{\mathrm{in}}| \approx |\mathrm{RHS}|$. A 40×50 grid on $\mathrm{Re}(\omega) \in (1.8, 2.6)$, $\mathrm{Im}(\omega) \in (-4, -0.01)$ finds no further zeros. ∎

4.3. **B3: Refined Type II spectrum.** All four $l = 2$, $n = 0$ QNMs refined to $|H_{\mathrm{BC}}| < 10^{-15}$:

Mode	m^2	$\mathrm{Re}(\omega_{\mathrm{BC}})$	$\mathrm{Im}(\omega_{\mathrm{BC}})$	$\lvert\omega_{\mathrm{BC}}/\omega_{\mathrm{Sch}}\rvert$
Lv1	-0.5	$+2.139847$	-0.049687	2.16965
Lv2	0.0	$+2.136719$	-0.084741	2.16760
Lv4	$+1.0$	$+2.133203$	-0.156896	2.16817
Lv5	$+1.5$	$+2.132808$	-0.193633	2.17082

Using $\omega_{\mathrm{Sch}}(l = 2, n = 0, s = 0) = 0.96736 - 0.19353i$ [17]:

$$\overline{|\omega_{\mathrm{BC}}/\omega_{\mathrm{Sch}}|} = 2.1691 \pm 0.0013. \quad \textit{[Derived]}$$

The ratio is not $\sqrt{2}$ (refuted, Paper 13), not 2 (the Frobenius factor), and not any of $\{\sqrt{5}, \pi/\sqrt{2}, 7/\pi\}$. The closest rational is $13/6 = 2.1\overline{6}$ (deviation 0.11%), but no algebraic derivation is available. **OP 13.7 remains open**: the ratio cannot be identified without OP 13.1.

5. Track C — One-loop $T_{\mu\nu}$

5.1. **Mode norms.** The norm with the de Sitter tortoise measure ($dr_* = -r_s\, dr/(r_s^2 - r^2)$ at $r_s = 1$):

$$\|\chi_n\|^2 = \int_0^{r_s} \chi_n(r)^2 \, \frac{dr}{1 - r^2/r_s^2}$$

is evaluated via Gauss–Legendre quadrature on $[0, 0.98] \cup [0.98, 0.9999]$ plus the analytic tail $\chi_n(r) \sim A_n(1 - r)^{\kappa_n/2}$:

$$\int_{u_0}^{0} \frac{A_n^2 u^{\kappa_n - 1}}{2 - u} \, du \approx \frac{A_n^2 \, u_0^{\kappa_n}}{2\kappa_n}.$$

Type II norms ($l = 2$, $n = 0$):

Table 2. Type I mode norms ($r_s = 1$).

Mode	m^2	l	κ_n	$\lVert\chi_n\rVert^2$	N
Lv2 $l=0$	0.0	0	0.081269	10.55	14
Lv2 $l=1$	0.0	1	0.589896	1143.5	14
Lv2 $l=2$	0.0	2	0.741920	5421.2	14
Lv1 $l=1$	-0.5	1	0.539190	483.6	7
Lv1 $l=2$	-0.5	2	0.710620	2905.6	7
Lv4 $l=0$	1.0	0	0.406133	415.9	14
Lv4 $l=1$	1.0	1	0.687723	5729.7	14
Lv4 $l=2$	1.0	2	0.804054	18047.3	14
Lv5 $l=0$	1.5	0	0.517384	1671.9	7
Lv5 $l=1$	1.5	1	0.735086	12177.3	7
Lv5 $l=2$	1.5	2	0.834882	32075.2	7

Mode	m^2	$\lvert\mathrm{Im}(\omega)\rvert$	$\lVert\chi_n\rVert^2$
Lv1	-0.5	0.04969	13.07
Lv2	0.0	0.08474	16.69
Lv4	1.0	0.15690	30.43
Lv5	1.5	0.19363	43.90

Remark 5.1 (*[Derived]*)**.** $\lVert\chi_n\rVert^2$ grows by 1–3 orders of magnitude from $l = 0$ to $l = 2$ at fixed m^2. For $l = 0$ as $\kappa \to 0^+$: $\lVert\chi\rVert^2 \sim \kappa^{-0.9}$ (diverges), consistent with the mode becoming marginal at m^2_{crit}.

5.2. **Inner products.** For modes at different (l, m^2), the tortoise inner product $\langle\chi_m|\chi_n\rangle$ is generically non-zero: these are eigenfunctions of different operators. Representative value:

$$\langle\chi_{\mathrm{Lv2},l=0}|\chi_{\mathrm{Lv2},l=1}\rangle = 23.3, \qquad \frac{\langle\chi_{\mathrm{Lv2},l=0}|\chi_{\mathrm{Lv2},l=1}\rangle}{\lVert\chi_{\mathrm{Lv2},l=0}\rVert\cdot\lVert\chi_{\mathrm{Lv2},l=1}\rVert} = 0.213.$$

The 16-mode spectrum does not form an orthonormal basis of a single Hilbert space.

5.3. **One-loop** $\mathrm{Im}(W)$**. Rough formula** (Paper 13, §6.5): $\mathrm{Im}(W)_{\mathrm{rough}} = -\sum_n N_n|\mathrm{Im}(\omega_n)|$ (unit-norm approximation):

Sector	$-\sum N_n\lvert\mathrm{Im}(\omega_n)\rvert$
Type I	$-69.714\, r_s^{-1}$
Type II	$-5.086\, r_s^{-1}$
Total	$\mathbf{-74.800\, r_s^{-1}}$

$$\boxed{\mathrm{Im}(W)_{\mathrm{rough}} = -74.800\, r_s^{-1}} \quad \text{(Paper 13 estimate confirmed to 4 s.f.)} \quad \textit{[Derived]} \tag{19}$$

Normalized formula:

Sector	$-\sum N_n\lvert\mathrm{Im}(\omega_n)\rvert/\lVert\chi_n\rVert^2$
Type I	$-0.1453\, r_s^{-1}$
Type II	$-0.2008\, r_s^{-1}$
Total	$\mathbf{-0.3461\, r_s^{-1}}$

The normalized result is 216× smaller. The exact one-loop formula requires the inner product from S_{84} (Paper 10) and zeta-function regularization near m^2_{crit} where $\lVert\chi\rVert^2_{l=0} \to \infty$ — this is **OP 14.3** *[Open]*.

6. Physical consequences

6.1. **Tightened cavity stability bound and critical slowing down.** The bound (14) is algebraically self-consistent: $m^2_{\text{Level 1}} = -2.765 \times \sqrt{0.002146} = -0.12810 = m^2_{\text{crit}}$ to within the stated uncertainty.

Mode count. Paper 13 reported 11 stable Type I modes ($\text{Im}(\omega) < 0$) plus 1 growing mode ($\text{Im}(\omega) > 0$, Level 1 $l = 0$, OP 13.9). For $\lambda GM^2 < 2.14 \times 10^{-3}$, the growing mode becomes a stable Type I mode; the complete stable count is Level 1 ($l = 0, 1, 2$) + Level 2 ($l = 0, 1, 2$) + Level 4 ($l = 0, 1, 2$) + Level 5 ($l = 0, 1, 2$) = **12 stable Type I modes** *[Derived]*.

λGM^2	$m^2_{\text{Level 1}}$	κ_0	τ_1 (r_s/c)
5×10^{-4}	-0.0618	0.0461	~ 22
1×10^{-3}	-0.0874	0.0283	~ 35
1.5×10^{-3}	-0.1071	0.0146	~ 69
2.0×10^{-3}	-0.1237	0.0031	~ 324
2.14×10^{-3}	-0.1281	$\to 0$	$\to \infty$

Critical slowing down *[Derived]*. Two results established above: (i) $\kappa_0 \propto (m^2_{\text{Level 1}} - m^2_{\text{crit}})$ with exponent 1 (linear, §3.5); (ii) expanding $m^2_{\text{Level 1}} = -2.765\sqrt{\lambda GM^2}$ near the threshold gives $m^2_{\text{Level 1}} - m^2_{\text{crit}} \propto (\lambda GM^2_{\text{crit}} - \lambda GM^2)$. Combining:

$$\tau_1 = \frac{1}{\kappa_0} \propto \frac{1}{\lambda GM^2_{\text{crit}} - \lambda GM^2}. \tag{20}$$

The divergence has **critical exponent 1** (not 1/2). Whether this constitutes a spectral phase transition in a thermodynamic sense is *[Speculation]*: naming the universality class requires an order parameter and a free energy functional outside the scope of this paper.

6.2. **Non-linear mass coupling.** The linear mass law of Paper 13 is the first-order truncation of the quadratic law (18), with $\Gamma_1(m^2)$ now computable analytically via the differentiated recurrences (3.18)–(3.19) at any $(l = 2, m^2)$.

6.3. **Structural completeness of the QNM spectrum.** Two negative results with structural force: (i) no $l = 0, 1$ oscillatory modes in $\text{Re}(\omega) \in (0.05, 15)$ (Proposition 4.1); (ii) no overtones for any (l, m^2) (Theorem 4.2). The Black Circle has the simplest possible QNM structure compatible with its geometry: one Type I mode per (l, m^2) and one Type II mode at $l = 2$, totalling 16 modes.

6.4. **One-loop action.** $\text{Im}(W)_{\text{rough}} = -74.800\, r_s^{-1}$ is a well-defined spectral quantity (sum of imaginary QNM frequencies). The normalized sum $-0.346\, r_s^{-1}$ is 216× smaller; the physical one-loop action requires the inner product from S_{84} (OP 14.3).

7. Open problems

Label	Statement	Origin
OP 14.1	Compute $\lim_{\kappa \to 0^+}$[coeff. of $u^{\kappa/2}$ in $\chi_{\text{in}}(r; +i\kappa)$] via the connection matrix between the $\{u^{\kappa/2}, u^{-\kappa/2}\}$ and $\{u^{i\nu/2}, u^{-i\nu/2}\}$ Frobenius bases. The Kummer candidate $K_2 \cdot 2^{-i\nu/2}$ is ruled out (Prop. B.1 + §3.5). Requires OP 13.1.	§3.4
OP 14.2	Secondary growing-mode branch ($\kappa \approx 0.44$, $m^2 \in (-0.14, -0.10)$): physical or formula artefact?	§3.5

Label	Statement	Origin
OP 14.3	Exact $\mathrm{Im}(W_{\text{1-loop}})$: identify the inner product from S_{84} (Paper 10) and apply zeta-function regularization near m^2_{crit}.	§5
OP 13.2	(carried) Derive $[d\chi/dr] = 0$ at r_s from EF regularity.	Paper 12
OP 12.1	(carried) Prove/disprove $\chi(r_s) = 0$ for spinorial Level-3 modes.	Paper 12
OP 13.3	(carried) Existence/uniqueness of Type I modes: numerical evidence (exactly 1 sign change of $H_{\text{BC}}(-i\kappa)$ per $(l = 0, m^2)$ for $m^2 > m^2_{\text{crit}}$) supports uniqueness; analytic proof open.	Paper 13
OP 13.5	(carried) Level-3 bosonic $\mathrm{PSL}(2,7)$ decomposition (GAP).	Paper 13
OP 13.10	(carried) Iso-lifetime coincidence Level 2: $\tau_I/\tau_{II} = \lvert\mathrm{Im}(\omega_{II})\rvert/\kappa_I = 1.04274$ (this paper); algebraic origin unknown.	Paper 13
OP 12.7	(carried) PCT involution for $\mathrm{GL}(2,3)$ on 4_s; Osterwalder–Schrader axioms.	Paper 12

8. Conclusions

8.1. **What this paper has done. Mode count cascade *[Derived]*.** For $\lambda GM^2 < 2.14 \times 10^{-3}$, all 12 imaginary-axis QNM modes are stable. Paper 13 had 11 stable Type I modes ($\mathrm{Im}(\omega) < 0$) plus 1 growing mode ($\mathrm{Im}(\omega) > 0$) at Level 1, $l = 0$; the 12th stable mode is recovered within the cavity bound (§6.1).

§3 *[Derived]*. The interior RW ODE at $\omega = 0$, $l = 0$ reduces to ${}_2F_1(i\nu/2, (3+i\nu)/2; 3/2; r^2)$ (Theorem 3.5); the static connection coefficients K_1, K_2 are given in terms of Γ-functions (Appendix B). The Kummer candidate $D^{\text{static}}_{\text{in}} = K_2 \cdot 2^{-i\nu/2}$ is ruled out *[Derived]*: its modulus 0.471 cannot satisfy $|D^{\text{static}}_{\text{in}}| = 1$ (Proposition B.1 combined with §3.5); the correct identification requires OP 13.1 (OP 14.1, sharpened). The critical mass is determined numerically to $m^2_{\text{crit}} = -0.1281 \pm 0.0005\, r_s^{-2}$ by two independent methods, tightening the cavity bound to $\lambda GM^2 < 2.14 \times 10^{-3}$. OP 13.8 is closed analytically: the differentiated step-2 recurrences give $\Gamma_1(m^2) = 0.07074 + 0.00195\, m^2$; the quadratic mass law (18) is established; the 1.4% discrepancy from Paper 13 is fully explained by the curvature term.

§4 *[Derived]*. No $l = 0$ or $l = 1$ oscillatory modes in $\mathrm{Re}(\omega) \in (0.05, 15)$ (combined Paper 13 + this paper). Structural no-go theorem for overtones proved: $|\mathrm{RHS}| \to +\infty$ guarantees no zeros of H_{BC} for $\kappa \gg 1$. The Type II ratio is refined to 2.1691 ± 0.0013; OP 13.7 remains open.

§5 *[Derived]*. Mode norms $\|\chi_n\|^2$ computed for all 15 QNMs (range 10.55–32075). The rough estimate $-74.800\, r_s^{-1}$ is confirmed to 4 significant figures; the normalized sum is $-0.346\, r_s^{-1}$, $216\times$ smaller. The obstacle to the exact one-loop formula is stated as OP 14.3.

8.2. **What this paper has not done.** The analytic closed form of m^2_{crit} as a Γ-function equation (OP 14.1) is not achieved. The candidate $D^{\text{static}}_{\text{in}} = K_2 \cdot 2^{-i\nu/2}$ is now ruled out *[Derived]*; the correct $D^{\text{static}}_{\text{in}}$ requires the connection matrix between the $\{u^{\kappa/2}, u^{-\kappa/2}\}$ and $\{u^{i\nu/2}, u^{-i\nu/2}\}$ Frobenius bases (OP 13.1). The algebraic origin of 2.1691 (OP 13.7) requires OP 13.1. The exact one-loop effective action (OP 14.3) requires additional QFT input not present in Papers 1–13.

8.3. **What opens for Paper 15.** The most pressing open problem is **OP 13.1/14.1**: the full Kummer reduction for $\omega \neq 0$. This would simultaneously close OP 13.7 (ratio 2.1691), OP 14.1 (m^2_{crit} analytic), and provide the connection matrix $M(\omega)$ that Paper 12 identified as the key missing ingredient. A secondary priority is **OP 14.3**: identifying the inner product from S_{84} to convert the normalized sum $-0.346\, r_s^{-1}$ into the exact one-loop action. The spinorial sector (OP 12.7, OP 12.1) and the Level-3 bosonic decomposition (OP 13.5) remain independent open threads.

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Appendix A. Step-2 recurrence (from Paper 13)

$$\alpha_n c_{n+2} + \beta_n c_n + \gamma_n c_{n-2} = 0, \quad n = 0, 2, 4, \ldots \tag{A.1}$$

$$\alpha_n = (n + l + 3)(n + l + 2) + l(l + 1), \tag{A.2a}$$

$$\beta_n = \omega^2 - m^2 - 2(n + l + 1)^2 - l(l + 1) + 2, \tag{A.2b}$$

$$\gamma_n = (n + l - 1)(n + l) - 2. \tag{A.2c}$$

Boundary conditions: $c_{-2} = 0$, $c_0 = 1$.

Appendix B. Hypergeometric parameters and Proposition B.1

For $l = 0$, $\omega = 0$, $m^2 = -\nu^2 < 0$:

Quantity	Value
Substitution	$z = r^2,\ \chi = rw(r^2),\ w = (1 - z)^{i\nu/2} F(z)$
a	$i\nu/2$
b	$(3 + i\nu)/2$
c	$3/2$
$c - a - b$	$-i\nu$
$a + b - c$	$i\nu$
K_1	$\Gamma(3/2)\Gamma(-i\nu)/\big[\Gamma(3/2 - i\nu/2)\Gamma(-i\nu/2)\big]$
K_2	$\Gamma(3/2)\Gamma(i\nu)/\big[\Gamma(i\nu/2)\Gamma((3 + i\nu)/2)\big]$
$\lvert K_2\rvert$	$1/(2\sqrt{1 + \nu^2})$
Papperitz sum	$-1/2 + 0 + 3/2 = 1$✓

Proposition B.1 (*[Derived]*)**.** $|K_2| = \dfrac{1}{2\sqrt{1+\nu^2}}$ *for all* $\nu > 0$.

Proof. Using DLMF 5.4.3: $|\Gamma(iy)|^2 = \pi/(y \sinh \pi y)$; and $\Gamma(3/2 + iy) = (1/2 + iy)\Gamma(1/2 + iy)$ with $|\Gamma(1/2 + iy)|^2 = \pi/\cosh \pi y$ (DLMF 5.4.4), giving $|\Gamma(3/2 + iy)|^2 = (1/4 + y^2)\pi/\cosh \pi y$. Setting $y = \nu/2$:

$$|K_2|^2 = \frac{|\Gamma(3/2)|^2|\Gamma(i\nu)|^2}{|\Gamma(i\nu/2)|^2|\Gamma(3/2 + i\nu/2)|^2} = \frac{\frac{\pi}{4} \cdot \frac{\pi}{\nu \sinh \pi\nu}}{\frac{2\pi}{\nu \sinh(\pi\nu/2)} \cdot \frac{\pi(1+\nu^2)}{4\cosh(\pi\nu/2)}}.$$

The denominator simplifies via $\sinh(\pi\nu/2)\cosh(\pi\nu/2) = \sinh(\pi\nu)/2$ to $\pi^2(1+\nu^2)/(\nu \sinh \pi\nu)$, giving $|K_2|^2 = \frac{1}{4(1+\nu^2)}$, hence $|K_2| = 1/(2\sqrt{1 + \nu^2})$. □

Numerical verification: $\nu = 1$: formula $= 1/(2\sqrt{2}) = 0.35355339$, SciPy: 0.35355339, error 2×10^{-15}.✓ $\nu = 2$: formula $= 1/(2\sqrt{5}) = 0.22360680$, SciPy: 0.22360680, error 5×10^{-15}.✓

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Independent Researcher, Modena, Italy
Email address: giovanni.levratti.research@gmail.com