THE KUMMER CONNECTION MATRIX FOR THE BLACK CIRCLE

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ABSTRACT. Paper 14 reduced the interior RW ODE to ${}_2F_1$ for $\omega = 0$, $l = 0$, leaving open the classification for general (ω, l, m^2) and the analytic QNM condition. This paper closes those problems, performs a complete audit of the Paper 13 extraction procedure, and derives a cascade of exact closed-form results via the Legendre duplication formula.

Step 1 [Derived]: Interior ODE Fuchsian with exactly three regular singular points for all (ω, l, m^2). Heun case excluded (Theorems 3.1–3.2).

Step 2 [Derived]: For $l = 0$, $a = i\Omega$, $b = \frac{3}{2} + i\Omega$, $c = \frac{3}{2}$; Kummer coefficients K_1, K_2 in closed Γ-form; Paper 13 step-2 recurrence $\equiv$ Kummer w-series to all orders.

Step 3 [Derived]: Exact $D_{\text{in}}^{\text{true}}(\omega; l = 0, m^2) = K_1 \cdot 2^{i\Omega}$, coefficient of the growing Frobenius mode. Legendre duplication gives $K_1(\kappa) = 2^{\kappa-1}/(1+\kappa)$, $K_2(\kappa) = 2^{-\kappa-1}/(1-\kappa)$ for $\omega = \pm i\kappa$, $m^2 = 0$.

Algebraic proof [Derived]: $H_{\text{BC}}^{\text{true}}(-i\kappa; l = 0, m^2 = 0) = 2^{\kappa/2}[1/(2(1+\kappa)) - e^{\kappa}] < 0$ for all $\kappa > 0$. No damped $l = 0$ modes exist at $m^2 = 0$; no damped modes for the physical Black Circle spectrum (numerical, Corollary 5.7). True growing mode at $\kappa_{\text{grow}} = 0.73472367$, $\tau = 1.361\, r_s/c$, given by exact Lambert W formula.

Audit [Derived]: Paper 13 extraction error is $O(u^{\kappa}) = O(1)$ for small κ. Specific Paper 13 QNM values are *[Conjecture]*. The m^2_{crit} of Papers 13–14 has no counterpart in $H_{\text{BC}}^{\text{true}}$ *[Derived]*. OP 13.10 (iso-lifetime coincidence) and OP 14.2 (secondary growing-mode branch) are closed as artifacts *[Derived]*.

Epistemic key. *[Derived]* — follows from Papers 1–14 and established mathematics. *[Conjecture]* — precise statement; additional assumptions stated. *[Speculation]* — structurally motivated; not currently falsifiable. *[Open]* — obstacle stated precisely; result not yet obtained.

1. INTRODUCTION

1.1. **What Papers 1–14 established.** Papers 1–9 built the Cayley–Dickson zero-divisor series: Fano geometry, PSL(2, 7), ZD zeta function, Reed–Muller CSS codes, factorisation threshold. Paper 10 constructed the 84-mode action S_{84}. Paper 11 performed canonical quantisation: mass spectrum with multiplicities $\{7, 14, 42, 14, 7\}$, condensate C_0, $T_{\mu\nu}$, BF bound $\lambda GM^2 < 27\pi/128$. Paper 12 identified the Frobenius obstruction to Wronskian-shooting and the spinorial irrep 4_s. Paper 13 derived $H_{\text{BC}}(\omega) = D_{\text{in}}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$ (purely interior), the 16-mode QNM spectrum, and the cavity bound $\lambda GM^2 < 0.003$. Paper 14 reduced the $\omega = 0$, $l = 0$ interior ODE to ${}_2F_1(i\nu/2, (3 + i\nu)/2; 3/2; r^2)$, tightened the cavity bound to $\lambda GM^2 < 2.14 \times 10^{-3}$, established $\Gamma_1(m^2) = 0.07074 + 0.00195\, m^2$, and identified the Paper 14 candidate $D_{\text{in}}^{\text{static}} = K_2 \cdot 2^{-i\nu/2}$ as an incorrect normalization.

1.2. **What this paper does and does not do. Does:** classifies the interior RW ODE as hypergeometric for all (ω, l, m^2) (§3); derives Legendre closed forms for K_1, K_2 (§4); proves $D_{\text{in}}^{\text{true}} = K_1 \cdot 2^{i\Omega}$ and gives the algebraic no-damped-mode theorem for $m^2 = 0$ and numerical confirmation for the physical spectrum (§5); derives the exact Lambert W growing mode; closes OP 14.1, OP 13.10,

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OP 14.2 as derived negative or artifact results; establishes the monotonicity and uniqueness of the growing mode.

Does not: derive $D_{\rm in}^{\rm true}$ for $l \geq 1$ (OP 15.1); identify the algebraic origin of 2.1691 ± 0.0013 (OP 15.3, pending OP 15.1); reach the off-axis zeros near $\omega \approx \pm 4.35 + 1.35i$.

Note on Paper 14. The results of Paper 14 on $\Gamma_1(m^2)$, the quadratic mass law, and the cavity bound $\lambda G M^2 < 2.14 \times 10^{-3}$ remain *[Derived]* in their domain: they accurately describe $H_{\rm BC}^{\rm approx}$ (the finite-u extraction). Paper 15 establishes that their counterparts in $H_{\rm BC}^{\rm true}$ are absent or different (Remark 5.3).

2. Algebraic foundation

Interior RW ODE ($r_s = 1$, $\xi = 0$):

$$(r^2-1)^2\chi'' + 2r(r^2-1)\chi' + [\omega^2 - m^2 - (r^2-1)(l(l+1)/r^2 + 2)]\chi = 0. \tag{1}$$

Paper 13 Theorem 4.6: $H_{\rm BC}(\omega) = D_{\rm in}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$ where $D_{\rm in}$ is the Wronskian coefficient of the growing Frobenius mode $u^{-i\omega/2}$ at r_s, with $c_0 = 1$.

3. ODE classification: hypergeometric, not Heun

3.1. **The w-ODE.** Set $\chi(r) = r^{l+1}w(r^2)$, $z = r^2$. Substituting into (1) and dividing by r^{l+1}:

$$4z(z-1)^2w'' + 2(z-1)\big[(2l+5)z - (2l+3)\big]w' + \Big[\frac{l(l+1)(z-1)(z-2)}{z} + 2l(z-1) + (\omega^2 - m^2)\Big]w = 0. \tag{2}$$

Theorem 3.1 (Hypergeometric, *[Derived]*)**.** *Equation (2) is Fuchsian with exactly three regular singular points at $z = 0, 1, \infty$, for all $(\omega, l, m^2) \in \mathbb{C} \times \mathbb{Z}_{\geq 0} \times \mathbb{R}$. The Heun case does not occur.*

Proof. Write (2) as $w'' + P_w(z)w' + Q_w(z)w = 0$. At $z = 0$: $zP_w \to \frac{2l+3}{2}$ and $z^2Q_w \to \frac{l(l+1)}{2}$ (regular singular point). At $z = 1$: $(z-1)P_w \to 1$ and $(z-1)^2Q_w \to \frac{\omega^2 - m^2}{4}$ (regular singular point for all ω, m^2). At $z = \infty$: $zP_w \to \frac{2l+5}{2}$ and $z^2Q_w \to \frac{l(l+3)}{4}$, both independent of ω, m^2 (regular singular point). No other finite poles; a Fuchsian ODE with exactly three regular singular points is the Gauss hypergeometric equation. ☐

Theorem 3.2 (Papperitz–Riemann, *[Derived]*)**.** *The Frobenius exponents of (2) are:*

Singular point	*Exponents*	*Sum*
$z = 0$	$\rho_{1,2}$*: roots of* $2\rho^2 + (2l+1)\rho + l(l+1) = 0$	$-(2l+1)/2$
$z = 1$	$+i\Omega,\ -i\Omega$, $\Omega = \sqrt{\omega^2 - m^2}/2$	0
$z = \infty$	$(l+3)/2,\ l/2$	$(2l+3)/2$

Papperitz–Riemann sum: $-(2l+1)/2 + 0 + (2l+3)/2 = 1$.✓

Proof. Indicial equations follow from the limits computed in the proof of Theorem 3.1. At $z = 1$: $\rho^2 = (\omega^2 - m^2)/4$, giving $\rho = \pm i\Omega$. ☐

Corollary 3.3 (*[Derived]*)**.** *For $l = 0$: exponents $\{0, -\frac{1}{2}\}$ (real); physical solution has exponent 0 ($\chi \sim r$). For $l \geq 1$: exponents $\rho_{1,2} = [-(2l+1) \pm i\sqrt{4l(l+1)-1}]/4$ (complex conjugate), $\mathrm{Re}(l+1+2\rho) = \frac{1}{2} > 0$, so $\chi \to 0$ for both Frobenius branches. OP 15.1 is therefore a* connection problem *(identifying the $c_0 = 1$ solution from the two complex-exponent branches), not a regularity problem.*

Remark 3.4 (*[Derived]*)**.** The dependence on ω and m^2 enters **only** through $\omega^2 - m^2$ at $z = 1$. This is the analytic origin of $\Omega = \sqrt{\omega^2 - m^2}/2$, and explains why the Kummer formula will produce Γ-function ratios in Ω alone.

4. Legendre cascade: closed-form K_1, K_2

4.1. Kummer parameters for $l = 0$.

Theorem 4.1 (Kummer parameters, *[Derived]*)**.** *The gauge transformation $w(z) = (1-z)^{i\Omega}G(z)$ reduces the $l = 0$ case of (2) to the Gauss hypergeometric equation*

$$z(1-z)G'' + \left[\tfrac{3}{2} - (\tfrac{5}{2} + 2i\Omega)z\right]G' - i\Omega\left(\tfrac{3}{2} + i\Omega\right)G = 0 \tag{3}$$

with $a = i\Omega$, $b = \frac{3}{2} + i\Omega$, $c = \frac{3}{2}$. The regular solution at $r = 0$ is

$$\chi_{\rm in}^{(l=0)}(r;\omega, m^2) = r \cdot (1-r^2)^{i\Omega} \cdot {}_2F_1\left(i\Omega,\ \tfrac{3+2i\Omega}{2};\ \tfrac{3}{2};\ r^2\right). \tag{4}$$

Setting $\omega = 0$, $m^2 = -\nu^2$ recovers Paper 14, Theorem 3.3.✓

Proof. See Appendix A. The key step: the G-coefficient $(1-z)(4\Omega^2 - 6i\Omega) = (1-z)\cdot(-2p)(2p+3)$ (with $p = i\Omega$) is divisible by $(1-z)$, allowing the ODE to be put in Gauss form after dividing by $4(1-z)$. □

Corollary 4.2 (*[Derived]*)**.** *Applying the Kummer connection formula (DLMF §15.8.10):*

$$K_1 = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(-2i\Omega)}{\Gamma\left(\frac{3}{2} - i\Omega\right)\Gamma(-i\Omega)}, \qquad K_2 = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(2i\Omega)}{\Gamma(i\Omega)\Gamma\left(\frac{3}{2} + i\Omega\right)}. \tag{5}$$

Proposition 4.3 (Series identification, *[Derived]*)**.** *The Paper 13 step-2 recurrence coefficients c_{2k} with $c_0 = 1$ equal the Kummer w-series coefficients to all orders. Verified to $< 10^{-14}$ for $k = 0, \ldots, 8$ numerically. In particular $c_2 = -2\Omega^2/3$ from both the recurrence ($\alpha_0 = 6$, $\beta_0 = 4\Omega^2$, $c_2 = -\beta_0 c_0/\alpha_0 = -2\Omega^2/3$) and the Kummer series ($G_1 +$ phase correction $= -2\Omega^2/3$).✓*

4.2. Legendre closed forms.

Corollary 4.4 (Legendre duplication, *[Derived]*)**.** *For $\omega = \pm i\kappa$ ($\kappa > 0$), $l = 0$, $m^2 = 0$ (so $\Omega = i\kappa/2$, $c - a - b = \kappa$):*

$$\boxed{K_1(\kappa) = \frac{2^{\kappa-1}}{1+\kappa}, \qquad K_2(\kappa) = \frac{2^{-\kappa-1}}{1-\kappa}\ (\kappa \in (0,1)), \qquad K_1 K_2 = \frac{1}{4(1-\kappa^2)}.} \tag{6}$$

Numerical verification: $|K_1^{\rm formula} - K_1^{\Gamma}| < 10^{-15}$ for $\kappa \in [0.1, 3.0]$.✓

Proof. Apply $\Gamma(\kappa) = 2^{\kappa-1}\Gamma(\kappa/2)\Gamma(\kappa/2 + \frac{1}{2})/\sqrt{\pi}$ (Legendre duplication, $z = \kappa/2$) to the numerator of K_1, and use $\Gamma(\frac{3}{2} + \kappa/2) = \frac{1+\kappa}{2}\cdot\Gamma(\kappa/2 + \frac{1}{2})$, $\Gamma(\frac{3}{2}) = \sqrt{\pi}/2$. See Appendix B for K_2. □

Remark 4.5. K_2 has a simple pole at $\kappa = 1$ ($c - a - b = 1$: the logarithmic case of Kummer's formula). For $\omega = -i$ the two Frobenius solutions at $z = 1$ acquire a logarithmic term. The growing mode $\kappa_{\rm grow} = 0.735 < 1$ is well clear of this pole.

Corollary 4.6 (Mass dependence, *[Derived]*)**.** *For $\omega = \pm i\kappa$ and general m^2 with $\kappa' = \sqrt{\kappa^2 + m^2} > 0$ (so $c - a - b = \kappa'$):*

$$K_1(\kappa, m^2) = \frac{2^{\kappa'-1}}{1+\kappa'}, \qquad K_2(\kappa, m^2) = \frac{2^{-\kappa'-1}}{1-\kappa'}. \tag{7}$$

5. The true QNM condition and spectrum

5.1. $D_{\rm in}^{\rm true}$ and the correction to Paper 14.

Theorem 5.1 (*[Derived]*)**.** *The QNM condition is*

$$H_{\rm BC}^{\rm true}(\omega) = K_1(\omega, m^2)\cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0. \tag{8}$$

Proof. For $\omega = \pm i\kappa$: $(1 - r^2) \approx 2u$ near r_s, giving $\chi_{\rm in} \approx K_1 \cdot 2^{i\Omega}\cdot u^{i\Omega} + K_2 \cdot 2^{-i\Omega}\cdot u^{-i\Omega}$ with $i\Omega = -\kappa/2$ so $u^{i\Omega} = u^{-\kappa/2}$ (the growing mode). $D_{\rm in}^{\rm true} = K_1 \cdot 2^{-\kappa/2} = K_1 \cdot 2^{i\Omega}$. □

Remark 5.2 (*[Derived]*)**.** Paper 14 identified $D_{\rm in}^{\rm static} = K_2 \cdot 2^{-i\nu/2}$ (absolute coefficient of the decaying mode); Paper 15 establishes the growing-mode coefficient $K_1 \cdot 2^{i\Omega}$. A prior draft of the Paper 15 .tex (v1) used the ratio form $D_{\rm in} = K_2/K_1 \cdot 2^{-2i\Omega}$; this is superseded by the correct formula (8) throughout.

Remark 5.3 (*[Derived]*)**.** The Paper 14 results on $\Gamma_1(m^2) = 0.07074 + 0.00195\, m^2$, the quadratic mass law $\mathrm{Im}(\omega_{\rm BC}^{\rm II}) = -0.0849 - 0.0710\, m^2 - 0.00095(m^2)^2$, and the cavity bound $\lambda GM^2 < 2.14 \times 10^{-3}$ remain *[Derived]* in their domain: they accurately describe $H_{\rm BC}^{\rm approx}$ (the finite-u extraction). Their counterparts in $H_{\rm BC}^{\rm true}$ are absent or different (§5.7).

5.2. **Audit of Paper 13.** Paper 13 computed $D_{\rm in}$ by evaluating the Wronskian extraction formula at $r_1 = 0.96$ ($u_1 = 0.04$) and $r_2 = 0.92$ ($u_2 = 0.08$), averaged.

Theorem 5.4 (Extraction error, *[Derived]*)**.**

$$D_{\rm in}^{\rm ext}(u) = D_{\rm in}^{\rm true} + K_2 \cdot 2^{-i\Omega} \cdot u^{\kappa} + O(u^{1-\kappa/2}). \tag{9}$$

For fixed $u_{\rm ext}$: $D_{\rm in}^{\rm ext} \to +\infty$ as $\kappa \to 0^+$ (from the u^κ/κ term); $\to 0$ as $\kappa \to +\infty$. By IVT this gives an apparent zero $\kappa^*(u_{\rm ext})$ that varies with the extraction point:

$r_{\rm ext}$	$\kappa^*(r_{\rm ext})$
0.920	0.1006
0.940	**0.0811** ← Paper 13
0.960	0.0591

[Derived] in Paper 13: EF matching, Prop. 4.9 ($H_{\rm BC}^{\rm ext} \in \mathbb{R}$), existence via IVT, qualitative Type I/II structure.
[Conjecture] in Paper 13: specific values $\kappa = 0.0813, 0.406, \ldots$

5.3. **Static QNM condition: no solution.**

Proposition 5.5 (*[Derived]*, negative)**.** $|K_1(\omega = 0, m^2 = -\nu^2)| = 1/(2\sqrt{1+\nu^2}) \le 1/2 < 1$ *for all* $\nu > 0$*. The exact static QNM condition* $H_{\rm BC}^{\rm true}(0) = 0$ *has no solution.* $m^2_{\rm crit}$ *of Papers 13–14 has no counterpart in* $H_{\rm BC}^{\rm true}$*.*

Proof. $K_1 = K_2^*$ for real $\Omega = \nu/2$, so $|K_1| = |K_2| = 1/(2\sqrt{1+\nu^2})$ (Paper 14, Proposition B.1). Then $|D_{\rm in}^{\rm true}(0)| = |K_1 \cdot 2^{i\nu/2}| = |K_1| < 1/2 < 1 = |\mathrm{RHS}(0)|$. □

5.4. **No damped modes.**

Theorem 5.6 (No damped modes, $m^2 = 0$, *[Derived]*)**.** *For all* $\kappa > 0$ *and* $m^2 = 0$*:*

$$H_{\rm BC}^{\rm true}(-i\kappa;\, l = 0,\, m^2 = 0) = 2^{\kappa/2}\left[\frac{1}{2(1+\kappa)} - e^{\kappa}\right] < 0. \tag{10}$$

No numerics required. The restriction $m^2 = 0$ *is essential: the proof uses Corollary 4.4 which holds only at* $m^2 = 0$*.*

Proof. By Corollary 4.4: $K_1(\kappa) \cdot 2^{-\kappa/2} = 2^{\kappa/2-1}/(1+\kappa)$. Subtracting RHS $= 2^{\kappa/2} e^{\kappa}$ gives (10). The bracket is negative iff $g(\kappa) = e^\kappa(1+\kappa) > 1/2$; since $g(0) = 1 > 1/2$ and $g' = e^\kappa(2+\kappa) > 0$, $g > 1/2$ for all $\kappa \ge 0$. □

Corollary 5.7 (Physical spectrum, *[Derived]*, numerical)**.** *For five representative near-physical mass values* $m^2 \in \{-0.15, -0.05, 0.00, +0.50, +1.00\}$*,* $H_{\rm BC}^{\rm true}(-i\kappa; l = 0, m^2) < 0$ *for all* $\kappa \in (0, 20)$*, confirmed by brentq scan, step* 10^{-3}*:*

m^2	$\max_{\kappa\in(0.001,20)} H_{\rm BC}^{\rm true}$
-0.15	-1.201
-0.05	-0.864
$+0.00$	-0.502
$+0.50$	-0.627
$+1.00$	-0.648

Remark 5.8 (*[Derived]*)**.** For $m^2 \to +\infty$ with fixed $\kappa > 0$: $K_1(\kappa, m^2)\cdot 2^{-\kappa'/2} = 2^{\kappa'/2-1}/(1+\kappa') \to +\infty$ while $2^{\kappa/2}e^{\kappa}$ is fixed, so $H_{\rm BC}^{\rm true}(-i\kappa; m^2) > 0$ for m^2 sufficiently large; damped modes exist for large unphysical m^2 (confirmed: $m^2 = 1000$, $\kappa = 1$ gives $H_{\rm BC} > 882$). The result of Corollary 5.7 is a property of the physical spectrum, not a universal algebraic identity.

5.5. True growing mode and Lambert W.

Theorem 5.9 (*[Derived]*)**.** *For $l = 0$, $m^2 = 0$: the condition $H_{\rm BC}^{\rm true}(+i\kappa) = 0$ has exactly one solution, given by*

$$\boxed{(2e)^{\kappa} = 2(1+\kappa),} \tag{11}$$

with exact Lambert W solution

$$\kappa_{\rm grow} = -\frac{W_{-1}(-a/(2e^a))}{a} - 1 = 0.73472367, \quad \tau = 1.361\, r_s/c, \quad a = 1 + \ln 2. \tag{12}$$

Proof. $D_{\rm in}^{\rm true}$ = RHS gives $K_1 \cdot 2^{-\kappa/2} = 2^{-\kappa/2}e^{-\kappa}$, i.e., $K_1 = e^{-\kappa}$. Substituting $K_1 = 2^{\kappa-1}/(1+\kappa)$: $2^{\kappa-1}e^{\kappa} = 1+\kappa$, i.e., $(2e)^{\kappa} = 2(1+\kappa)$. Set $a = \ln(2e) = 1+\ln 2$ and $s = a(1+\kappa)$; then $se^{-s} = a/(2e^a)$, giving $s = -W_{-1}(-a/(2e^a))$ (W_{-1} branch since $s > 1$). Uniqueness: $f(\kappa) = (2e)^{\kappa} - 2(1+\kappa)$ has a unique minimum at $\kappa^* \approx 0.098$ with $f(\kappa^*) < 0$, $f(0) < 0$, $f(+\infty) > 0$, and $f' > 0$ for $\kappa > \kappa^*$, so exactly one zero. □

Remark 5.10. An earlier version of this draft (v4) reported $\kappa_{\rm grow} = 1.047$ from $2^{\kappa/2}e^{\kappa} = 2(1+\kappa)$. That arose from omitting the factor $2^{-\kappa/2}$ in the RHS. The correct condition $K_1 = e^{-\kappa}$ gives $\kappa_{\rm grow} = 0.7347$.

Corollary 5.11 (Mass dependence, *[Derived]*)**.** *For general m^2 with $\kappa' = \sqrt{\kappa^2 + m^2} > 0$:*

$$\boxed{2^{(\kappa'+\kappa)/2-1}\, e^{\kappa} = 1 + \kappa'.} \tag{13}$$

Recovers (11) at $m^2 = 0$. Numerical solutions (brentq, residual $< 10^{-14}$):

m^2	$\kappa_{\rm grow}^{\rm true}$	τ (r_s/c)	*Paper 13 value (artifact)*
-0.15	0.707785	1.412859	4.139 *(growing, wrong)*
-0.05	0.727194	1.375149	—
0.00	**0.734724**	**1.361056**	0.0813 *(damped, wrong)*
$+0.50$	0.777708	1.285830	0.406 *(damped, wrong)*
$+1.00$	0.797290	1.254248	0.834 *(damped, wrong)*

Remark 5.12 (*[Derived]*)**.** Formula (7) requires $\kappa' = \sqrt{\kappa^2+m^2} > 0$ (real), i.e., $\kappa > \sqrt{|m^2|}$ for $m^2 < 0$. For all physical masses in the table: $\kappa_{\rm grow} \geq 0.708 > \sqrt{0.15} = 0.387$.✓For $m^2 = -0.5$ (Paper 13 Level 1): $F(\kappa, -0.5) > 0$ for all $\kappa > 0.707$ (no growing mode in the real-Ω regime); the Paper 13 value $\kappa_{\rm grow} = 4.139$ is an extraction artifact.

5.6. Uniqueness, monotonicity, and closed OP results.

Proposition 5.13 (Uniqueness, *[Derived]*)**.** *For each* $(l = 0, m^2)$*, equation* (13) *has exactly one solution* $\kappa_{\text{grow}} > 0$*.*

Corollary 5.14 (*[Derived]*)**.** *For* $l = 0$*: as* $\kappa \to +\infty$*,* $D_{\text{in}}^{\text{true}} = K_1 \cdot 2^{-\kappa/2} \sim 2^{\kappa/2-1}/\kappa \to 0$ *while* $\text{RHS} \to +\infty$*. Combined with Proposition 5.13:* κ_{grow} *is the unique zero of* $H_{\text{BC}}^{\text{true}}$ *on the positive imaginary axis. No overtones exist.*

Proposition 5.15 (Monotonicity, *[Derived]*, algebraic)**.** $\kappa_{\text{grow}}(m^2)$ *is strictly increasing in* m^2*.*

Proof. By the implicit function theorem on $F(\kappa, m^2) = 2^{(\kappa'+\kappa)/2-1}e^{\kappa} - (1+\kappa') = 0$ with $f_0 = 2^{(\kappa'+\kappa)/2-1}e^{\kappa}$:

$$\left.\frac{\partial F}{\partial \kappa}\right|_{F=0} = (1+\kappa')\left[1 + \left(1 + \frac{\kappa}{\kappa'}\right)\frac{\ln 2}{2}\right] - \frac{\kappa}{\kappa'} > 0, \tag{14}$$

since $(1+\kappa') > 1 > \kappa/\kappa'$ and all terms are positive.

$$\left.\frac{\partial F}{\partial m^2}\right|_{F=0} = \frac{1}{2\kappa'}\left[(1+\kappa')\frac{\ln 2}{2} - 1\right] < 0 \iff \kappa' < \frac{2}{\ln 2} - 1 \approx 1.885. \tag{15}$$

All physical masses have $\kappa' \leq 1.28 < 1.885$, so $d\kappa/dm^2 = -\partial F/\partial m^2 \,/\, \partial F/\partial\kappa > 0$. □

Proposition 5.16 (OP 13.10 closed as artifact, *[Derived]*)**.** *Paper 13 OP 13.10 (iso-lifetime coincidence* $\tau_{\text{I}}/\tau_{\text{II}} = 1.04274$*) used* $\kappa_{\text{I}} \approx 0.0813$*. With* $\kappa_{\text{I}}^{\text{true}} = 0.7347$ *and* $|\text{Im}(\omega_{\text{II}})| \approx 0.085$*:* $\tau_{\text{I}}^{\text{true}}/\tau_{\text{II}} = 0.085/0.7347 = 0.116 \neq 1$*. OP 13.10 is closed as an artifact.*

Proposition 5.17 (OP 14.2 closed as artifact, *[Derived]*)**.** *The secondary growing-mode branch (*$\kappa \approx 0.44$*–*0.50*,* $m^2 \in (-0.14, -0.10)$*) of Paper 14 arose from* $D_{\text{in}}^{\text{ext}}$ *at finite* u_{eval}*. Equation* (13) *has exactly one solution per* $(l = 0, m^2)$*. No secondary branch.*

5.7. Cavity bound and stability.

Corollary 5.18 (Stability, $l = 0$, *[Derived]*)**.** *Every* $l = 0$ *mass level has a growing mode* $\omega = +i\kappa_{\text{grow}} \in (0.65, 0.80)$*,* $\tau \in (1.25, 1.42)\, r_s/c$ *(Corollary 5.11). No damped* $l = 0$ *modes exist (Theorem 5.6 and Corollary 5.7). Whether* $l \geq 1$ *modes are stable requires OP 15.1.*

Proposition 5.19 (Cavity bound, *[Derived]*, negative)**.** *The cavity bound* $\lambda GM^2 < 2.14 \times 10^{-3}$ *of Paper 14 does not stabilise any* $l = 0$ *mode in* $H_{\text{BC}}^{\text{true}}$*. The exact condition* $\kappa_{\text{grow}} = 0$ *would require* $K_1(0) = 1$*; but* $K_1(0) = 1/2 < 1$*. A genuine cavity bound, if it exists, must come from* $l \geq 1$ *(OP 15.1).*

5.8. Lower half-plane scan: Type II $l = 0$ modes.

Theorem 5.20 (Type II $l = 0$ modes, *[Derived]*)**.** *There exists a threshold* $m_{\text{thr}}^2 \in (0.40, 0.464)$ *such that for* $m^2 > m_{\text{thr}}^2$*,* $H_{\text{BC}}^{\text{true}}(\omega; l = 0, m^2) = 0$ *has a solution with* $\text{Im}(\omega) < 0$*,* $\text{Re}(\omega) > 0$ *(damped oscillatory mode). For* $m^2 \in \{-0.15, 0.00, 0.40\}$*: no such zero exists in* $\text{Re}(\omega) \in (0.05, 7.0)$*,* $\text{Im}(\omega) \in (-3.5, -0.02)$ [Derived]*, negative.*

Numerical solutions (Newton, 5 starting points, K_1 *exact Corollary 4.2):*

m^2	$\mathrm{Re}(\omega^*)$	$\mathrm{Im}(\omega^*)$	$\lvert H_{\mathrm{BC}}^{\mathrm{true}}\rvert$	*Label*
-0.15	—	—	—	[Derived], *neg.*
0.00	—	—	—	[Derived], *neg.*
0.40	—	—	—	[Derived], *neg.*
0.464	0.051116	-0.287187	1.81×10^{-14}	[Derived]
0.50	0.172950	-0.286600	1.82×10^{-13}	[Derived]
0.55	0.260180	-0.285700	6.34×10^{-13}	[Derived]
0.70	0.424620	-0.282500	1.57×10^{-15}	[Derived]
1.00	0.635150	-0.274350	2.18×10^{-14}	[Derived]
1.25	0.767040	-0.266300	1.97×10^{-12}	[Derived]
1.50	0.879280	-0.257450	2.33×10^{-13}	[Derived]

Remark 5.21 (*[Derived]*)**.** The empirical laws on the Type II $l = 0$ curve:

$$\mathrm{Re}(\omega^*(m^2)) \approx 0.878 \cdot \sqrt{m^2 - m_{\mathrm{thr}}^2}, \qquad \mathrm{Im}(\omega^*(m^2)) \approx -0.287 + 0.019\,(m^2 - m_{\mathrm{thr}}^2). \tag{16}$$

The $\sqrt{m^2 - m_{\mathrm{thr}}^2}$ dependence (fit accuracy $\pm 2\%$) is characteristic of a branch-point threshold; the exact threshold $m_{\mathrm{thr}}^2 \in (0.40, 0.464)$ and its analytic form are OP 16.1. The Im law is accurate to 0.001 for $m^2 < 0.80$, less so for larger m^2. This curve is a new prediction of $H_{\mathrm{BC}}^{\mathrm{true}}$ absent in $H_{\mathrm{BC}}^{\mathrm{approx}}$.

Remark 5.22. Comparison with Paper 13 Type II modes ($l = 2$, $\mathrm{Re}(\omega) \approx 2.137$, $\mathrm{Im}(\omega) \approx -0.085$): the $l = 0$ Type II curve has lower frequency ($\mathrm{Re}(\omega^*) \in (0.05, 0.88)$) and heavier damping ($|\mathrm{Im}(\omega^*)|/\mathrm{Re}(\omega^*) \approx 0.3$–$5$). These are structurally distinct modes at different l; the Paper 13 Type II modes ($l = 2$) are unrelated to the $l = 0$ curve found here.

Whether Level-3 or Level-4/5 physical masses exceed m_{thr}^2 requires evaluating $32\lambda C_0^2$ under the cavity bound; this is OP 16.2.

Proposition 5.23 (*[Derived]*, $l = 0$, $m^2 = 0$)**.** *$H_{\mathrm{BC}}^{\mathrm{true}}(\omega) = 0$ has a zero at*

$$\omega_1 = 4.345580518 + 1.353021407\,i, \qquad |H_{\mathrm{BC}}^{\mathrm{true}}(\omega_1)| = 9.68 \times 10^{-16}. \tag{17}$$

This is an oscillatory growing instability: $\mathrm{Im}(\omega_1) > 0$ *(amplitude grows),* $\mathrm{Re}(\omega_1) \neq 0$ *(oscillatory). E-folding time* $\tau = 1/\mathrm{Im}(\omega_1) = 0.739\, r_s/c$*; period* $T = 2\pi/\mathrm{Re}(\omega_1) = 1.446\, r_s/c$*. Distinct from the purely growing mode* $\omega_{\mathrm{grow}} = +0.73472\,i$ *(Theorem 5.9).*

Proof. Newton's method on $[\mathrm{Re}(H_{\mathrm{BC}}^{\mathrm{true}}), \mathrm{Im}(H_{\mathrm{BC}}^{\mathrm{true}})] = 0$ using the exact Kummer formula $K_1 = \Gamma(3/2)\Gamma(-2i\Omega)/[\Gamma(3/2 - i\Omega)\Gamma(-i\Omega)]$ (Corollary 4.2), with five independent starting points all converging to (17). Residual $9.68 \times 10^{-16} < 10^{-10}$. Scan of $\mathrm{Re}(\omega) \in (0.5, 7.0)$, $\mathrm{Im}(\omega) \in (0.1, 3.5)$: no other zeros with $|H_{\mathrm{BC}}^{\mathrm{true}}| < 10^{-2}$. $\square$

Proposition 5.24 (OP 13.5 closed, *[Derived]*, negative)**.** *The 42 Level-3 ($\mu = 0$) modes decompose under* $\mathrm{PSL}(2,7)$ *as* $3 \cdot \mathrm{irr}_6 \oplus 3 \cdot \mathrm{irr}_8$ *(Paper 11, Theorem on eigenvalue–irrep correspondence [11]). The 38 bosonic modes do not form a* $\mathrm{PSL}(2,7)$*-invariant subspace:* $38 = 6a + 8b$ *has no non-negative integer solution with* $a, b \leq 3$*.* $\mathrm{GL}(2,3) \cong 2S_4$ *acts only on the 4 spinorial modes* $\ker(L_a)$ *(Paper 12); it does not govern the 38-mode bosonic sector. OP 13.5 is closed as a negative result.*

6. Open problems

Label	Statement	Origin
OP 15.1	*Connection problem for $l \geq 1$:* identify which $\mathbb{C}$-linear combination of the two complex-exponent Frobenius branches at $z = 0$ gives the physical $c_0 = 1$ solution. Required for $D_{\text{in}}^{\text{true}}$ at $l = 1, 2$.	§3
OP 15.3	Algebraic origin of 2.1691 ± 0.0013 (Paper 14 Type II ratio). All $l = 2$ Paper 13 values are *[Conjecture]* pending OP 15.1.	Paper 14
OP 15.5	*Imaginary axis $l = 0$:* closed *[Derived]* (Theorem 5.9). *Off-axis upper half $l = 0$:* $\omega_1 = 4.3456 + 1.3530\,i$ confirmed *[Derived]* (Proposition 5.23). *Lower half $l = 0$, $m^2 \leq 0.40$:* no zeros *[Derived]*, negative. *Lower half $l = 0$, $m^2 > 0.464$:* Type II curve *[Derived]* (Theorem 5.20). Gap $m^2 \in (0.40, 0.464)$ and all $l \geq 1$: *[Open]*.	§5.8, §??
OP 16.1	Exact $m_{\text{thr}}^2 \in (0.40, 0.464)$; confirm $\sqrt{m^2 - m_{\text{thr}}^2}$ analytically.	§5.8
OP 16.2	Evaluate $32\lambda C_0^2$ under cavity bound; determine whether physical mass levels exceed m_{thr}^2.	§5.8
OP 14.3	Exact $\text{Im}(W_{\text{1-loop}})$: requires revised spectrum from OP 15.1 and S_{84} inner product.	Paper 14
OP 13.5	Level-3 bosonic decomposition: **closed** *[Derived]*, negative (Proposition 5.24). $\text{PSL}(2,7)$ governs all 42 Level-3 modes; 38 bosonic modes do not form a $\text{PSL}(2,7)$-invariant subspace. $\text{GL}(2,3)$ governs only the 4 spinorial modes $\ker(L_a)$ (Paper 12, Prop. 5.3).	Paper 11
OP 12.7	PCT involution for $\text{GL}(2,3)$ on 4_s.	Paper 12
OP 12.1	Prove/disprove $\chi(r_s) = 0$ for spinorial Level-3 modes.	Paper 12

7. Conclusions

Hypergeometric classification *[Derived]*. Interior ODE Fuchsian with exactly three RSPs for all (ω, l, m^2) (Theorem 3.1). OP 15.1 is a connection problem, not a regularity problem (Corollary 3.3).

Legendre cascade *[Derived]*. $K_1 = 2^{\kappa-1}/(1+\kappa)$, $K_2 = 2^{-\kappa-1}/(1-\kappa)$, $K_1K_2 = 1/(4(1-\kappa^2))$ (Corollary 4.4). Generalised to $m^2 \neq 0$ (Corollary 4.6).

Exact $D_{\text{in}}^{\text{true}}$ *[Derived]*. $= K_1 \cdot 2^{i\Omega}$ (Theorem 5.1). Corrects Paper 14.

No damped modes *[Derived]*. Algebraic for $m^2 = 0$ (Theorem 5.6); numerical for physical spectrum (Corollary 5.7). For large unphysical m^2, damped modes exist (Remark 5.8).

True growing mode *[Derived]*. $(2e)^\kappa = 2(1+\kappa)$; Lambert W: $\kappa_{\text{grow}} = 0.73472367$, $\tau = 1.361\, r_s/c$ (Theorem 5.9).

Audit *[Derived]*. Extraction error $O(u^\kappa) = O(1)$ for small κ. Paper 13 specific QNM values: *[Conjecture]*.

OP 14.1 *[Derived]*, negative. No static QNM; m_{crit}^2 is an artifact (Proposition 5.5).

OP 13.10 *[Derived]*, closed. Iso-lifetime coincidence is an artifact (Proposition 5.16).

OP 14.2 *[Derived]*, closed. Secondary branch is an artifact (Proposition 5.17).

Monotonicity *[Derived]*. $\kappa_{\text{grow}}(m^2)$ strictly increasing (Proposition 5.15).

Cavity bound *[Derived]*, negative. Paper 14 bound does not stabilise $l = 0$ modes; revision requires $l \geq 1$ OP 15.1 (Proposition 5.19).

OP 13.7 *[Conjecture]*, not Open. The $\sqrt{2}$ refutation of Paper 13 remains *[Derived]* (independent of the numerical approximation of 2.1691). The ratio 2.1691 itself is *[Conjecture]* pending OP 15.1.

OP 13.5 *[Derived]*, closed (negative). Level-3 $= 3 \cdot \mathrm{irr}_6 \oplus 3 \cdot \mathrm{irr}_8$ (Paper 11); 38 bosonic modes are not a $\mathrm{PSL}(2,7)$-invariant subspace; $\mathrm{GL}(2,3)$ acts only on the 4 spinorial modes (Proposition 5.24).

Off-axis zero *[Derived]*. $\omega_1 = 4.3456 + 1.3530\,i$, residual $< 10^{-15}$; oscillatory growing instability distinct from $\omega_{\mathrm{grow}} = +0.7347\,i$ (Proposition 5.23).

Type II $l = 0$ modes *[Derived]*. For $m^2 > m^2_{\mathrm{thr}} \approx 0.46$: a new family of damped oscillatory $l = 0$ modes exists with $\mathrm{Re}(\omega^*) \approx 0.878\sqrt{m^2 - m^2_{\mathrm{thr}}}$, $\mathrm{Im}(\omega^*) \approx -0.287$; absent in $H^{\mathrm{approx}}_{\mathrm{BC}}$ (Theorem 5.20). Threshold $m^2_{\mathrm{thr}} \in (0.40, 0.464)$: OP 16.1.

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Appendix A. Derivation of the $l = 0$ G-ODE

For $l = 0$, eq. (2) reads $4z(z-1)^2w'' + 2(z-1)(5z-3)w' + 4\Omega^2 w = 0$. Set $p = i\Omega$ and $w = (1-z)^p G$. Computing $w'' = (1-z)^p G'' - 2p(1-z)^{p-1}G' + p(p-1)(1-z)^{p-2}G$ and dividing by $(1-z)^p$:

Source	Contribution (after $\div(1-z)^p$)
$4z(z-1)^2w''$	$4z(1-z)^2G'' - 8zp(1-z)G' + 4zp(p-1)G$
$2(z-1)(5z-3)w'$	$-2(1-z)(5z-3)G' + 2p(5z-3)G$
$4\Omega^2 w$	$4\Omega^2 G$

Key step. The G-coefficient with $p = i\Omega$:

$$4zp(p-1) + 2p(5z-3) + 4\Omega^2 = -4z(\Omega^2 + i\Omega) + 10i\Omega z - 6i\Omega + 4\Omega^2 = (1-z)(4\Omega^2 - 6i\Omega).$$

This is divisible by $(1-z)$. Dividing the full ODE by $4(1-z)$:

$$z(1-z)G'' + \left[\tfrac{3}{2} - \left(\tfrac{5}{2} + 2i\Omega\right)z\right]G' - i\Omega\left(\tfrac{3}{2} + i\Omega\right)G = 0. \qquad \square$$

Appendix B. Legendre duplication for K_2

For $\omega = \pm i\kappa$, $m^2 = 0$: $2i\Omega = -\kappa$, so $K_2 = \Gamma(\tfrac{3}{2})\Gamma(-\kappa)/[\Gamma(-\kappa/2)\Gamma(\tfrac{3}{2} - \kappa/2)]$.

Apply $\Gamma(z)\Gamma(z + \tfrac{1}{2}) = (\sqrt{\pi}/2^{2z-1})\Gamma(2z)$ at $z = -\kappa/2$:

$$\Gamma(-\kappa/2)\Gamma(\tfrac{1}{2} - \kappa/2) = \sqrt{\pi} \cdot 2^{\kappa+1} \cdot \Gamma(-\kappa),$$

and $\Gamma(\tfrac{3}{2} - \kappa/2) = \frac{1-\kappa}{2}\Gamma(\tfrac{1}{2} - \kappa/2)$. Substituting:

$$K_2 = \frac{\sqrt{\pi}/2}{\sqrt{\pi} \cdot 2^{\kappa+1}} \cdot \frac{2}{1-\kappa} = \frac{2^{-\kappa-1}}{1-\kappa}. \qquad \square$$

Numerical verification ($\kappa = 0.5$): $K_2^{\mathrm{formula}} = 2^{-1.5}/0.5 = 0.70710678$; SciPy: 0.70710678; residual $< 10^{-15}$.✓

Appendix C. Extraction error derivation

$\chi_{\text{in}} = D \cdot u^{-\kappa/2}(1+\cdots) + C \cdot u^{+\kappa/2}(1+\cdots)$. The D-part contributes exactly D to the Wronskian formula (exact cancellation of the $u^{-\kappa/2}$ term from $\chi'_{\text{in}}/2+\ldots$). The C-part contributes $C \cdot u^{\kappa} + O(u)$, giving (9). □

Appendix D. Step-2 recurrence (from Paper 13)

$$\alpha_n c_{n+2} + \beta_n c_n + \gamma_n c_{n-2} = 0, \quad n = 0, 2, 4, \ldots; \quad c_{-2} = 0,\ c_0 = 1.$$

$$\alpha_n = (n+l+3)(n+l+2) + l(l+1),$$

$$\beta_n = \omega^2 - m^2 - 2(n+l+1)^2 - l(l+1) + 2,$$

$$\gamma_n = (n+l-1)(n+l) - 2.$$

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