THE LIMIT-CIRCLE OBSTACLE FOR BLACK CIRCLE QUASI-NORMAL MODES

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Abstract. Paper 15 classified the interior Regge–Wheeler ODE as Gauss hypergeometric for all (ω, l, m^2) and derived the exact QNM condition for $l = 0$, leaving OP 15.1 open: find the physical solution $w_{\rm phys}$ for $l \geq 1$ and derive $D_{\rm in}^{\rm true}(l \geq 1)$.

This paper closes OP 15.1 negatively in its original formulation and identifies the precise obstacle.

Algebraic results *[Derived].* The s-homotopic substitution $w = z^{\rho_2}(1-z)^{i\Omega}G$ on the ρ_2 Frobenius branch transforms the w-ODE to the Gauss equation with parameters

$$\tilde{c} = 1 - i\delta_l, \quad \tilde{a} = -\tfrac{1}{4} + i\Big(\Omega - \tfrac{\delta_l}{2}\Big), \quad \tilde{b} = \tfrac{5}{4} + i\Big(\Omega - \tfrac{\delta_l}{2}\Big),$$

where $\delta_l = \sqrt{4l(l+1)-1}/2$ and $\Omega = \sqrt{\omega^2 - m^2}/2$. The Kummer connection formula gives $\tilde{K}_1^{(\rho_2,l)}$ (eq. (6)) as the exact near-r_s coefficient for the ρ_2 Frobenius solution. Setting $\delta_0 = i/2$ formally recovers the $l = 0$ formula of Paper 15 exactly.

Negative result *[Derived].* The step-2 series $w_{\rm phys} = 1 + c_2 z + \cdots$ (Paper 13) satisfies the nonhomogeneous identity

$$L_w[w_{\rm phys}](z) = \frac{2l(l+1)}{z}$$

verified to eight significant figures for all $z \in (0,1)$. Consequently $w_{\rm phys} \notin \ker(L_w)$ and OP 15.1 in its original formulation ("find (A,B) with $w_{\rm phys} = Au_{\rho_1} + Bu_{\rho_2}$") has no solution. Wronskian extraction confirms: $C_2(z_0)$ varies by $\sim 60\%$ across $z_0 \in [0.2, 0.7]$, violating Abel's identity.

Precise obstacle *[Open]*, OP 17.1. The chi-ODE has a regular singular point at $r = 0$ for all $l \geq 1$, with Frobenius exponents $(1 \pm 2i\delta_l)/2$ — both L^2, giving a limit-circle singularity. The physical boundary condition requires a self-adjoint extension whose selection depends on the global dS_4 geometry.

Secondary results *[Derived].* From Paper 11, $m_i^2 = 16\lambda C_0^2(2+\mu_i)$ (eq. (mass), §4.1), giving $m_4^2 = 4\sqrt{6x/\pi}$ and $m_5^2 = 6\sqrt{6x/\pi}$ ($x = \lambda GM^2$). Both lie below $m_{\rm thr}^2 \approx 0.464$ under the cavity bound.

Epistemic key. *[Derived]* — follows from Papers 1–15 and established mathematics. *[Conjecture]* — precise statement; assumptions stated explicitly. *[Open]* — obstacle stated precisely; result not yet obtained.

1. Introduction

1.1. **Context.** Papers 1–9 built the Cayley–Dickson zero-divisor series: ZD zeta function, Fano geometry, PSL(2, 7), Reed–Muller CSS codes. Papers 10–11 constructed the Black Circle geometry and performed canonical quantisation: mass spectrum $\{7, 14, 42, 14, 7\}$, condensate $C_0^4 = 3/(2048\pi G^3M^2\lambda)$, BF bound $\lambda GM^2 < 27\pi/128$. Papers 12–15 developed the QNM analysis. Paper 15 derived the exact QNM condition $H_{\rm BC}^{\rm true}(l=0) = K_1 \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0$ *[Derived]* and left OP 15.1 open.

1.2. **What Papers 13 and 15 assumed for $l \geq 1$.** Paper 13 derived the step-2 recurrence for $\chi = r^{l+1}\sum c_{2k}r^{2k}$ with $c_0 = 1$, claiming "$r = 0$ is an ordinary point because $f_{\rm in}(0) = -1 \neq 0$"

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[Paper 13, Prop. 4.1]. Paper 15, Corollary 3.3 noted the complex Frobenius exponents but stated OP 15.1 as "identifying the $c_0 = 1$ solution from the two complex-exponent branches" — implicitly assuming w_{phys} is in the solution space of the homogeneous w-ODE.

1.3. **The issue.** Paper 13's claim is incorrect for $l \geq 1$. The effective potential $V_{\text{eff}}(r) \approx -l(l+1)/r^2$ as $r \to 0$ (with $f_{\text{in}}(0) = -1$) creates a centrifugal singularity independent of metric smoothness. The chi-ODE has a regular singular point at $r = 0$ with complex Frobenius exponents $(1 \pm 2i\delta_l)/2$, both L^2 — a limit-circle singularity. The BC $\chi \sim r^{l+1}$ does not correspond to either Frobenius branch, and the series w_{phys} satisfies a nonhomogeneous w-ODE with source $2l(l+1)/z$.

1.4. What this paper does.

- Derives the Gauss parameters $(\tilde{a}, \tilde{b}; \tilde{c})$ for the ρ_2 Frobenius branch *[Derived]* (§3).
- Proves $L_w[w_{\text{phys}}] = 2l(l+1)/z$ and closes OP 15.1 negatively *[Derived]* (§4).
- Gives $\tilde{K}_1^{(\rho_2,l)}$ via the Kummer formula *[Derived]* (§5).
- Addresses OP 16.1 and OP 16.2 *[Derived]* (§6).
- States OP 17.1 precisely *[Open]* (§7).

2. Algebraic foundation

2.1. **The ODEs.** Interior RW ODE ($r_s = 1$, $\xi = 0$):

$$(r^2-1)^2\chi'' + 2r(r^2-1)\chi' + \left[\omega^2 - m^2 - (r^2-1)\big(l(l+1)/r^2 + 2\big)\right]\chi = 0. \tag{1}$$

Setting $\chi = r^{l+1}w(r^2)$, $z = r^2$, the w-ODE is [Paper 15, Thm. 3.1, *[Derived]*]:

$$4z(z-1)^2w'' + 2(z-1)\big[(2l+5)z - (2l+3)\big]w' + \left[\frac{l(l+1)(z-1)(z-2)}{z} + 2l(z-1) + (\omega^2 - m^2)\right]w = 0. \tag{2}$$

The Papperitz–Riemann symbol [Paper 15, Thm. 3.2, *[Derived]*]:

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ \rho_1 & +i\Omega & l/2 & ; z \\ \rho_2 & -i\Omega & (l+3)/2 & \end{matrix}\right\}, \quad \Omega = \frac{\sqrt{\omega^2 - m^2}}{2}, \tag{3}$$

with $\rho_{1,2} = [-(2l+1) \pm 2i\delta_l]/4$, $\delta_l = \sqrt{4l(l+1)-1}/2 > 0$ for $l \geq 1$. PR sum $= 1$.✓

The QNM condition [Papers 13/15, *[Derived]*]: $H_{\text{BC}}(\omega) = D_{\text{in}}^{\text{true}}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$, where $D_{\text{in}}^{\text{true}}$ is the coefficient of the growing Frobenius mode $u^{i\Omega}$ ($u = 1 - r$) at r_s.

2.2. **Singular structure at $r = 0$ for $l \geq 1$.**

Proposition 2.1 (RSP at $r = 0$, *[Derived]*)**.** *For $l \geq 1$, $r = 0$ is a regular singular point of the chi-ODE (1) with Frobenius exponents $\varrho = (1 \pm 2i\delta_l)/2$:*

l	*Exponents* ϱ	$\text{Re}(\varrho)$	*Behaviour*
0	1, 0	1, 0	*regular endpoint*
1	$(1 \pm i\sqrt{7})/2$	1/2	*both* $\chi_j \sim r^{1/2} \to 0$
2	$(1 \pm i\sqrt{23})/2$	1/2	*both* $L^2(0, \epsilon)$
$l \geq 1$	$(1 \pm 2i\delta_l)/2$	1/2	*both* $L^2(0, \epsilon)$

Proof. Write (1) in normal form. Near $r = 0$: $f_{\text{in}}(0) = -1$, so $\lim_{r\to 0} rP(r) = 0$ and $\lim_{r\to 0} r^2Q(r) = l(l+1)$. Indicial equation: $\varrho(\varrho-1) + l(l+1) = 0$, giving $\varrho = (1 \pm \sqrt{1-4l(l+1)})/2 = (1 \pm 2i\delta_l)/2$. ☐

Proposition 2.2 (Limit-circle, *[Derived]*)**.** *For all $l \geq 1$: $|\chi_j(r)| = r^{1/2}$, so $\int_0^\epsilon |\chi_j|^2\,dr = \epsilon^2/2 < \infty$. Both Frobenius solutions are $L^2(0, \epsilon)$ — limit-circle in Weyl's classification [17, 18]. A boundary condition is required to select the self-adjoint extension.*

Remark 2.3. For $l = 0$: $Q(r) = O(1)$ near $r = 0$ (no centrifugal singularity), giving real exponents $\{1, 0\}$ and a regular endpoint. Paper 15's $l = 0$ analysis is unaffected.

3. Gauss parameters for the ρ_2 Frobenius branch

Theorem 3.1 (Gauss parameters, *[Derived]*)**.** *The substitution $w = z^{\rho_2}(1-z)^{i\Omega}G$ transforms (2) to the Gauss hypergeometric equation $z(1-z)G'' + [\tilde{c} - (\tilde{a}+\tilde{b}+1)z]G' - \tilde{a}\tilde{b}\,G = 0$ with*

$$\tilde{c} = 1 - i\delta_l, \quad \tilde{a} = -\tfrac{1}{4} + i\Big(\Omega - \tfrac{\delta_l}{2}\Big), \quad \tilde{b} = \tfrac{5}{4} + i\Big(\Omega - \tfrac{\delta_l}{2}\Big). \tag{4}$$

Proof. Three-step algebraic verification.

Step 1 [$1/z$ cancellation]: the coefficient of $1/z$ is $2[2\rho_2^2 + (2l+1)\rho_2 + l(l+1)] = 0$ since ρ_2 satisfies the indicial equation.

Step 2 [G' coefficient]: $4\rho_2 + 2l + 3 = 2 - 2i\delta_l = 2\tilde{c}$✓; $4\rho_2 + 4i\Omega + 2l + 5 = 4 - 2i\delta_l + 4i\Omega = 2(\tilde{a}+\tilde{b}+1)$✓.

Step 3 [G coefficient]: using the algebraic identity $(2l+1)^2 - 4\delta_l^2 = 2$ (from $4\delta_l^2 = (2l+1)^2 - 2$):

$$[2(\rho_2 + i\Omega) + l][2(\rho_2 + i\Omega) + (l+3)] = 4\tilde{a}\tilde{b}. \checkmark$$

PR sum: $i\delta_l + (-2i\Omega) + (1 - i\delta_l + 2i\Omega) = 1$.✓Specific values: $\delta_1 = \sqrt{7}/2$, $\delta_2 = \sqrt{23}/2$. □

4. The nonhomogeneous source and closure of OP 15.1

4.1. The source term.

Theorem 4.1 (Nonhomogeneous source, *[Derived]*)**.** *The step-2 series $w_{\text{phys}}(z) = \sum_{k\geq 0} c_{2k} z^k$ with $c_0 = 1$ satisfies:*

$$L_w[w_{\text{phys}}](z) = \frac{2l(l+1)}{z} \quad \textit{for all } z \in (0,1),\ l \geq 1. \tag{5}$$

Proof. The step-2 recurrence zeros all coefficients of r^{n+l+1} for $n \geq 0$. The power r^{l-1} (index $n = -(l+1) < 0$, outside the recurrence range) has coefficient in the chi-ODE:

$$\big[(r^2-1)^2\chi''\big]_{r^{l-1}} + \big[l(l+1)/r^2 \cdot \chi\big]_{r^{l-1}} = l(l+1)c_0 + l(l+1)c_0 = 2l(l+1).$$

Converting to the w-variable ($\chi = r^{l+1}w$, $z = r^2$): the residual $2l(l+1)r^{l-1} = r^{l+1} \cdot 2l(l+1)/z$, giving $L_w[w_{\text{phys}}] = 2l(l+1)/z$. □

Remark 4.2. For $l = 0$: $l(l+1) = 0$, source $= 0$, consistent with Paper 15. For $l = 1$: source $= 4/z$. For $l = 2$: source $= 12/z$.

Numerical verification ($l = 1$, $\omega = -0.3i$, $N = 600$ terms):

z	$L_w[w_{\text{phys}}] \cdot z$	Expected
0.05	4.00000000	4
0.20	4.00000000	4
0.50	4.00000000	4
0.70	4.00000000	4

Agreement to 8 decimal places for all z.✓

4.2. Closure of OP 15.1.

Corollary 4.3 (OP 15.1 closed, *[Derived]*)**.** *For $l \geq 1$: $w_{\text{phys}} \notin \ker(L_w)$. No finite $(A,B) \in \mathbb{C}^2$ satisfies $w_{\text{phys}} = Au_{\rho_1} + Bu_{\rho_2}$ on $(0,1)$.*

Proof. If $w_{\text{phys}} = Au_{\rho_1} + Bu_{\rho_2}$ with $u_j \in \ker(L_w)$, linearity gives $L_w[w_{\text{phys}}] = 0$, contradicting (5). □

Remark 4.4. OP 15.1 asked for (A,B) with $w_{\text{phys}} = Au_1 + Bu_2$. Corollary 4.3 shows this has no solution — not because the connection formula is hard to compute, but because the equation has no solution at all. The source term $2l(l+1)/z$ traces precisely to the BC $\chi \sim r^{l+1}$ at the limit-circle singular point $r = 0$.

4.3. **Wronskian instability.**

Proposition 4.5 (Wronskian instability, *[Derived]*)**.** *If $w_{\text{phys}} \in \ker(L_w)$, Abel's identity would require $C_2(z_0) = W(u_{\rho_1}, w_{\text{phys}})/W(u_{\rho_1}, u_{\rho_2})$ to be constant. For $l = 1$, $N = 600$ step-2 terms, u_{ρ_j} via mpmath (50 d.p.):*

ω	z_0	$C_2(z_0)$	$\lvert C_2\rvert$
$-0.3i$	0.20	$+0.246 - 0.027i$	0.248
$-0.3i$	0.30	$+0.346 + 0.079i$	0.355
$-0.3i$	0.40	$+0.414 + 0.220i$	0.469
$-0.3i$	0.50	$+0.448 + 0.393i$	0.596
$-0.3i$	0.60	$+0.439 + 0.604i$	0.747
$-0.3i$	0.70	$+0.366 + 0.864i$	0.938
	std/|mean|		0.61

$|C_2|$ varies by factor ~ 3.8 across z_0. Identical instability for $\omega = -0.5i$ and $\omega = 0.5 - 0.3i$.

5. The Kummer formula for the homogeneous branch

Notation. $\tilde{K}_1^{(\rho_j,l)}$ denotes the Kummer coefficient for the ρ_j Frobenius branch ($j = 1, 2$). The shorthand $\tilde{K}_1^{(l)} \equiv \tilde{K}_1^{(\rho_2,l)}$ is used throughout; for the ρ_1 branch, replace $\delta_l \to -\delta_l$ in all formulas.

Theorem 5.1 ($\tilde{K}_1^{(\rho_2,l)}$, *[Derived]*)**.** *For $u_{\rho_2}(z) = z^{\rho_2}(1-z)^{i\Omega}{}_2F_1(\tilde{a}, \tilde{b}; \tilde{c}; z)$, the Kummer connection formula (DLMF §15.8.10) gives:*

$$u_{\rho_2}(z) = \tilde{K}_1^{(\rho_2,l)}\,{}_2F_1\Big(\tilde{a}, \tilde{b}; 1 + 2i\Omega; 1 - z\Big) + \tilde{K}_2^{(\rho_2,l)}(1-z)^{-2i\Omega}{}_2F_1\Big(\tilde{c} - \tilde{a}, \tilde{c} - \tilde{b}; 1 - 2i\Omega; 1 - z\Big)$$

with

$$\boxed{\tilde{K}_1^{(\rho_2,l)} = \frac{\Gamma(1 - i\delta_l)\,\Gamma(-2i\Omega)}{\Gamma\Big(\frac{5}{4} - i\Big(\Omega + \frac{\delta_l}{2}\Big)\Big)\,\Gamma\Big(-\frac{1}{4} - i\Big(\Omega + \frac{\delta_l}{2}\Big)\Big)},} \tag{6}$$

valid for $-2i\Omega \notin \mathbb{Z}$. Near r_s: u_{ρ_2} contributes $\tilde{K}_1^{(\rho_2,l)} \cdot 2^{i\Omega} \cdot u^{i\Omega} + \tilde{K}_2^{(\rho_2,l)} \cdot 2^{-i\Omega} \cdot u^{-i\Omega}$. The ρ_1-branch coefficient $\tilde{K}_1^{(\rho_1,l)}$ is obtained from (6) with $\delta_l \to -\delta_l$.

Corollary 5.2 ($l = 0$ consistency, *[Derived]*)**.** *Setting $\delta_0 = i/2$: $\Gamma(1 - i\delta_0) = \Gamma(3/2)$, $\tilde{c} - \tilde{a} = 3/2 - i\Omega$, $\tilde{c} - \tilde{b} = -i\Omega$, giving*

$$\tilde{K}_1^{(\rho_2,l=0)}\Big|_{\delta_0 = i/2} = \frac{\Gamma(3/2)\,\Gamma(-2i\Omega)}{\Gamma(3/2 - i\Omega)\,\Gamma(-i\Omega)} = K_1^{(l=0)},$$

recovering Paper 15, Corollary 4.2 exactly.✓

Remark 5.3 (Status and future use)**.** Theorem 5.1 is *[Derived]* for the homogeneous solution u_{ρ_j}. Once OP 17.1 determines the physical combination (C_1, C_2), the QNM condition will be:

$$H_{\text{BC}}^{\text{true}}(l \geq 1; \omega, m^2) = \Big[C_1\tilde{K}_1^{(\rho_1,l)} + C_2\tilde{K}_1^{(\rho_2,l)}\Big] \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0.$$

Remark 5.4 (No Legendre cascade, OP 17.2)**.** For $\omega = -i\kappa$, $m^2 = 0$: all four Γ-function arguments in $\tilde{K}_1^{(\rho_2,l)}$ are complex (mixing δ_l and κ). No Legendre duplication applies, unlike the $l = 0$ case. Whether a closed form exists is OP 17.2.

6. Secondary results

6.1. **OP 16.1 — Threshold m^2_{thr}.** From Paper 15, Theorem 5.20 *[Derived]*: $m^2_{\mathrm{thr}} \in (0.40, 0.464)$. Newton iteration on $H^{\mathrm{true}}_{\mathrm{BC}}(l = 0; \omega^*, m^2_{\mathrm{thr}}) = 0$ with $\mathrm{Im}(\omega^*) = 0$ is deferred to Paper 17 (OP 17.3).

6.2. **OP 16.2 — Physical mass levels.**

Proposition 6.1 (*[Derived]*)**.** *From Paper 11, §4.1 (Mass matrix), eq. (mass): $m_i^2 = 16\lambda C_0^2(2+\mu_i)$ at $\xi = 0$. Levels 4 and 5 have $\mu_4 = +2$ and $\mu_5 = +4$, giving $m_4^2 = 64\lambda C_0^2$ and $m_5^2 = 96\lambda C_0^2$. In $r_s = 1$ units ($G = 1$, $M = 1/2$, $G^3M^2 = 1/4$), with $x = \lambda GM^2$:*

$$m_4^2 = 4\sqrt{\frac{6x}{\pi}}, \qquad m_5^2 = 6\sqrt{\frac{6x}{\pi}}. \tag{7}$$

Proof. From Paper 11: $C_0^2 = \sqrt{3/(512\pi\lambda)}$ (using $G^3M^2 = 1/4$). With $\lambda = 4x$:

$$m_4^2 = 64 \cdot 4x \cdot \sqrt{\frac{3}{512\pi \cdot 4x}} = 256\sqrt{\frac{3x}{2048\pi}} = 4\sqrt{\frac{6x}{\pi}},$$
$$m_5^2 = 96 \cdot 4x \cdot \sqrt{\frac{3}{512\pi \cdot 4x}} = 384\sqrt{\frac{3x}{2048\pi}} = 6\sqrt{\frac{6x}{\pi}}.$$

Remark on the ratio $m_5^2/m_4^2 = 3/2$: the mass formula involves $(2 + \mu_i)$, not μ_i directly. Thus $(2 + \mu_5)/(2 + \mu_4) = 6/4 = 3/2$, even though $\mu_5/\mu_4 = 2$. The shift $+2$ comes from the quadratic potential structure of S_{84} (Paper 11, §4.1): the massless level is Level 2 ($\mu = -2$), not Level 3 ($\mu = 0$). □

Proposition 6.2 (Below threshold, *[Derived]*)**.** *Under the Paper 14 cavity bound $x < 2.14 \times 10^{-3}$:*

$$m_4^2 < 4\sqrt{6 \times 2.14 \times 10^{-3}/\pi} \approx 0.256 < m^2_{\mathrm{thr}} \approx 0.464,$$
$$m_5^2 < 6\sqrt{6 \times 2.14 \times 10^{-3}/\pi} \approx 0.384 < m^2_{\mathrm{thr}} \approx 0.464.$$

No Type II $l = 0$ oscillatory modes exist for the physical Black Circle spectrum under the cavity bound.

Proposition 6.3 (Critical couplings, *[Derived]*)**.** *Setting $m_k^2 = m^2_{\mathrm{thr}}$ in (7):*

$$(\lambda GM^2)_4^{\mathrm{crit}} = \frac{\pi(m^2_{\mathrm{thr}})^2}{96} \approx 7.0 \times 10^{-3}\ (3.3 \times \textit{cavity}),$$
$$(\lambda GM^2)_5^{\mathrm{crit}} = \frac{\pi(m^2_{\mathrm{thr}})^2}{216} \approx 3.1 \times 10^{-3}\ (1.5 \times \textit{cavity}).$$

General formula: $(\lambda GM^2)_k^{\mathrm{crit}} = \pi(m^2_{\mathrm{thr}})^2/[6(2 + \mu_k)^2]$. Level 5 is marginal: $1.5\times$ the cavity bound.

7. Open problems

Label	Statement	Origin
OP 17.1	*Physical BC at $r = 0$ for $l \geq 1$*: determine (C_1, C_2) from the global dS_4 geometry. Two candidates: (a) smooth extension to the de Sitter poles; (b) Hartle–Hawking (no outgoing waves from the Cauchy surface $r = 0$). Once known: $H^{\mathrm{true}}_{\mathrm{BC}}(l \geq 1) = [C_1\tilde{K}_1^{(\rho_1,l)} + C_2\tilde{K}_1^{(\rho_2,l)}] \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0$.	§5
OP 17.2	Closed form for $\tilde{K}_1^{(\rho_2,l)}$, $l \geq 1$ (no Legendre cascade).	§5
OP 17.3	Exact $m^2_{\mathrm{thr}} \in (0.40, 0.464)$ via Newton on $H^{\mathrm{true}}_{\mathrm{BC}}(l = 0) = 0$ targeting $\mathrm{Im}(\omega^*) = 0$. Computable.	§6

Label	Statement	Origin
OP 17.4	Status of Paper 13 QNMs for $l \geq 1$: physical or artifacts of the nonhomogeneous BC? Requires OP 17.1.	Paper 13
OP 12.7	PCT involution for $\mathrm{GL}(2,3)$ on 4_s.	Paper 12

8. Conclusions

Algebraic results *[Derived]*. Theorem 3.1 gives the Gauss parameters for the ρ_2 Frobenius branch with complete three-step algebraic verification; Theorem 5.1 gives $\tilde{K}_1^{(\rho_2,l)}$; Corollary 5.2 confirms the $l = 0$ limit.

OP 15.1 closed negatively *[Derived]*. Theorem 4.1 proves $L_w[w_{\mathrm{phys}}] = 2l(l+1)/z \neq 0$ (verified to 8 decimal places). Corollary 4.3 closes OP 15.1. Proposition 4.5 confirms via Wronskian instability ($\sim 60\%$ variation). The source term traces precisely to the BC $\chi \sim r^{l+1}$ at the limit-circle singular point $r = 0$.

Precise obstacle *[Open]*, OP 17.1. The physical self-adjoint extension at $r = 0$ is the missing input. Its resolution completes the QNM analysis for $l \geq 1$ via the formula in Remark 5.3.

Physical stability *[Derived]*. Both mass levels 4 and 5 lie below m^2_{thr} under the cavity bound; Level 5 is marginal at $1.5\times$ the cavity bound (Propositions 6.2–6.3).

Errata on Paper 13, Proposition 4.1 *[Derived]*. Paper 13, Prop. 4.1 stated: "$r = 0$ is an ordinary point because $f_{\mathrm{in}}(0) \neq 0$." For $l \geq 1$ this is incorrect: the centrifugal term $l(l+1)/r^2$ creates a regular singular point at $r = 0$ independently of the metric. The Frobenius exponents are $(1 \pm 2i\delta_l)/2$ (Proposition 2.1). The step-2 recurrence of Paper 13 remains valid *[Derived]* as a particular solution of the *inhomogeneous* w-ODE (Theorem 4.1); the error was in the geometric interpretation, not in the recurrence itself. The QNM values for $l \geq 1$ in Paper 13 were already labelled *[Conjecture]* in Paper 15; their status is OP 17.4.

Errata from previous papers

Paper 13, Proposition 4.1 *[Derived]*: "$r = 0$ is an ordinary point" is incorrect for $l \geq 1$. See Proposition 2.1 and Conclusions above.

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Appendix A. Key identities

A.1. $\delta_l = \sqrt{4l(l+1)-1}/2$: $\delta_1 = \sqrt{7}/2 \approx 1.3229$, $\delta_2 = \sqrt{23}/2 \approx 2.3979$.
A.2. $(2l+1)^2 - 4\delta_l^2 = 2$ (from $4\delta_l^2 = (2l+1)^2 - 2$). Used in Theorem 3.1, Step 3.
A.3. $l = 0$ limit: $\delta_0 = i/2$, $\tilde{K}_1^{(\rho_2,l=0)} = K_1^{(l=0)}$. [Corollary 5.2]
A.4. Source term: $L_w[w_{\mathrm{phys}}] = 2l(l+1)/z$; $l = 1$: $4/z$; $l = 2$: $12/z$.
A.5. Mass formula [Paper 11, §4.1, eq. (mass)]: $m_i^2 = 16\lambda C_0^2(2+\mu_i)$. Levels 4 and 5: $m_4^2 = 64\lambda C_0^2$, $m_5^2 = 96\lambda C_0^2$.
A.6. $(\lambda G M^2)_k^{\mathrm{crit}} = \pi(m^2_{\mathrm{thr}})^2/[6(2+\mu_k)^2]$.

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