THE PHYSICAL BOUNDARY CONDITION AT $r = 0$ FOR THE BLACK CIRCLE

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Abstract. Paper 16 closed OP 15.1 negatively: for $l \geq 1$ the physical series w_{phys} satisfies the nonhomogeneous equation $L_w[w_{\text{phys}}] = 2l(l+1)/z$, so $w_{\text{phys}} \notin \ker(L_w)$. Both Frobenius branches are $L^2(0, \varepsilon)$ — a limit-circle singularity requiring a self-adjoint extension to select the physical domain [18, 19].

This paper determines the physical boundary condition (BC) at $r = 0$ and derives the first QNM spectrum for $l \geq 1$.

Three candidates, converging. (c) Norm conservation via IIB-II *[Conjecture]*: the oscillating cross-term $2\text{Re}(C_1\bar{C}_2 \cdot r^{2i\delta_l}) \to \infty$ forces $C_1C_2 = 0$. (b) Hartle–Hawking *[Conjecture]*: outgoing KG flux $J^r(\chi_2)|_{r\to 0} = +2\delta_l$ requires $C_2 = 0$. (a) Global dS$_4$ regularity *[Conjecture]*: the homogeneous contributions $\phi_j^{\text{hom}} = \chi_j^{\text{hom}}/r \sim r^{-1/2} \to \infty$ are singular; any finiteness condition at $r = 0$ forces $C_1 = C_2 = 0$. Hierarchy: (a) $\subset$ (b) $\subset$ (c).

QNM condition corrected. *[Derived]* Paper 16 Remark 5.3 omitted the particular-solution coefficient A_{part}. The complete condition is $H_{\text{BC}}^{\text{full}} = [A_{\text{part}} + C_1\tilde{K}_1^{(\varrho_1,l)} + C_2\tilde{K}_1^{(\varrho_2,l)}] \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0$.

QNM spectrum for $l = 1$, $m^2 = 0$, Candidate (a). *[Conjecture]* Two modes found: a damped mode $\omega^* = 1.29788 - 0.00762i$ ($\tau = 131\, r_s/c$) and a **growing mode** $\omega^* = 1.6449 + 0.7372i$ ($\tau = 1.357\, r_s/c$). The growing mode persists for all physical masses $m^2 \leq 0.384$. Quasi-universality: $\tau_{\text{grow}}(l = 1) \approx \tau_{\text{grow}}(l = 0)$ to 0.33% (OP 18.6).

Falsifiability. Candidates (a) and (b) give $l = 1$ growing modes at $\omega^* \approx 1.645 + 0.737i$ and $\omega^* \approx 5.052 + 0.410i$ respectively: distinguishable by over 3 units in $\text{Re}(\omega)$.

Secondary. OP 17.3 closed *[Derived]*: $m^2_{\text{thr}} = 0.4605600022$ (10 sig. figs.). All Paper 13 $l \geq 1$ QNM values are artifacts under Candidate (a) *[Conjecture]*.

Epistemic key. *[Derived]* — follows from Papers 1–16 and established mathematics. *[Conjecture]* — precise; additional assumptions stated explicitly. *[Speculation]* — structurally motivated; not currently falsifiable. *[Open]* — obstacle stated precisely; result not yet obtained.

1. Introduction

1.1. **IIB-II and the norm-gravity connection.** The physical core of the series is IIB-II (Paper 2, *[Conjecture]*): N is the unique structure preserved at every Cayley–Dickson doubling step, and gravity is associated with N rather than with any multiplication structure. The Frobenius identity $\|L_a\|_F^2 = 2^n N(a)$ for all n (Papers 1–2, *[Derived]*) is the operator-level manifestation; the growing mode at $l = 0$ ($\kappa_{\text{grow}} = 0.73472367$, Paper 15) and the $T_{\mu\nu}|_{C_0} = -g_{\mu\nu} \cdot 3/r_s^2$ (Paper 11) are IIB-II rendered concrete. Paper 17 is the first paper in which IIB-II enters the calculation explicitly, as Candidate (c) for the BC at $r = 0$.

1.2. **What Papers 13–16 established.** Paper 12: Frobenius obstruction, exponent ratio $2 = \|L_a\|_F^2/\|L_a\|_F^2|_{n=3}$, QNM condition $H_{\text{BC}}(\omega) = D_{\text{in}}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$. Paper 15: $D_{\text{in}}^{\text{true}}(l = 0) = K_1 \cdot 2^{i\Omega}$; no damped $l = 0$ modes; true growing mode; Type II $l = 0$ curve. Paper 16: for $l \geq 1$,

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$r = 0$ is a regular singular point with complex Frobenius exponents $\varrho_{1,2} = (1 \pm 2i\delta_l)/2$, limit-circle; $L_w[w_{\rm phys}] = 2l(l+1)/z$ *[Derived]*; Kummer coefficient $\tilde{K}_1^{(\varrho_2,l)}$ *[Derived]*.

1.3. **What Paper 13 assumed (incorrectly for $l \geq 1$).** Paper 13 Prop. 4.1 stated "$r = 0$ is an ordinary point because $f_{\rm in}(0) \neq 0$". For $l \geq 1$ this is incorrect: the centrifugal term $l(l+1)/r^2$ creates a regular singular point independently of the metric (Paper 16, Prop. 2.1, *[Derived]*). The step-2 recurrence of Paper 13 remains valid as a particular solution of the inhomogeneous w-ODE; the error was in the geometric interpretation.

2. Algebraic and geometric foundation

Interior RW ODE ($r_s = 1$, $\xi = 0$) and w-ODE via $\chi = r^{l+1}w(r^2)$ are as in Papers 15–16. Papperitz–Riemann symbol [Paper 15, *[Derived]*]:

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ \varrho_1 & +i\Omega & l/2 & ;\, z \\ \varrho_2 & -i\Omega & (l+3)/2 & \end{Bmatrix},\ \varrho_{1,2} = \frac{1 \pm 2i\delta_l}{2},\ \delta_l = \frac{\sqrt{4l(l+1)-1}}{2},\ \Omega = \frac{\sqrt{\omega^2 - m^2}}{2}. \tag{1}$$

Limit-circle at $r = 0$ for $l \geq 1$: both $|\chi_j^{\rm hom}| \sim r^{1/2} \in L^2(0,\varepsilon)$ (Paper 16, Prop. 2.2, *[Derived]*).

The general solution: $\chi_{\rm full} = \chi_{\rm part} + C_1\chi_1^{\rm hom} + C_2\chi_2^{\rm hom}$, where $\chi_{\rm part} = r^{l+1}w_{\rm phys}(r^2)$ is the step-2 series with $c_0 = 1$ (particular solution of the inhomogeneous ODE), and $\chi_j^{\rm hom} = r^{l+1}u_{\varrho_j}(r^2)$ are solutions of the homogeneous ODE.

3. Candidate (c): the IIB-II norm condition

3.1. The AMT is automatic.

Proposition 3.1 (*[Derived]*). *For any $(C_1, C_2) \in \mathbb{C}^2$ and $r \in (0, r_s)$: $\mathbb{E}_v[N(\chi_{\rm phys}(r)\cdot e_0 \cdot v)] = N(\chi_{\rm phys}(r)\cdot e_0) = |\chi_{\rm phys}(r)|^2$.*

Proof. AMT [Paper 2, Thm. 2.4, *[Derived]*] applied to $a = \chi_{\rm phys}(r)\cdot e_0 \in \mathcal{A}_4$. Unconditional for all $n \geq 1$. □

Remark 3.2. The AMT conserves N in the *mean*. The physically relevant condition is *pointwise* convergence of $N(\chi_{\rm phys})$ as $r \to 0^+$: a strictly stronger requirement.

3.2. Frobenius behaviour of the homogeneous branches.

Proposition 3.3 (*[Derived]*). *For $l \geq 1$, the homogeneous solutions satisfy:*

$$\chi_j^{\rm hom}(r) \sim r^{1/2 \pm i\delta_l}\cdot G_j(r), \quad r \to 0^+, \tag{2}$$

with G_j holomorphic and non-zero at $r = 0$. In particular $|\chi_j^{\rm hom}|^2 \sim r \to 0$ with no oscillation.

Proof. $\chi = r^{l+1}w(r^2)$ with $w \sim z^{\varrho_j}$, $\varrho_j = [-(2l+1) \pm 2i\delta_l]/4$. Then $\chi_j^{\rm hom} \sim r^{l+1+2\varrho_j} = r^{(1\pm 2i\delta_l)/2} = r^{1/2\pm i\delta_l}$. The factor $r^{\pm i\delta_l} = e^{\pm i\delta_l \ln r}$ has modulus 1. □

Proposition 3.4 (*[Derived]*). *For (C_1, C_2) with $C_1C_2 \neq 0$:*

$$N(C_1\chi_1^{\rm hom} + C_2\chi_2^{\rm hom}) \sim r\big[|C_1|^2 + |C_2|^2 + 2\,{\rm Re}(C_1\bar{C}_2 \cdot r^{2i\delta_l})\big]. \tag{3}$$

The interference term $2{\rm Re}(C_1\bar{C}_2\cdot r^{2i\delta_l})$ oscillates without bound as $r \to 0^+$ (since $\delta_l > 0$ is real).

Proof. From (2): $\chi_1^{\rm hom}\overline{\chi_2^{\rm hom}} = r^{1/2+i\delta_l}\cdot r^{1/2+i\delta_l} = r\cdot r^{2i\delta_l}$. Hence $N(C_1\chi_1 + C_2\chi_2) = r[|C_1|^2 + |C_2|^2 + 2{\rm Re}(C_1\bar{C}_2\cdot r^{2i\delta_l})]$. □

3.3. The IIB-II selection and its limit.

Theorem 3.5 (*[Conjecture]*, IIB-II)**.** *Under the norm-gravity conjecture IIB-II, the physical BC requires:*

$$\boxed{C_1 C_2 = 0.}$$

Proof. By Proposition 3.4, $C_1C_2 \neq 0$ implies $N(\chi_{\text{full}})$ oscillates without bound as $r \to 0^+$. IIB-II requires N to determine gravity pointwise; unbounded oscillation is inadmissible. Hence $C_1C_2 = 0$. The oscillation in (3) is *[Derived]*; the step to physical inadmissibility requires IIB-II. ☐

Remark 3.6 (*[Derived]*)**.** The partial solution dominance near $r = 0$:

Solution	Behaviour	Exponent
χ_j^{hom}	$r^{1/2\pm i\delta_l} \to 0$ slower	$1/2$
χ_{part}	$r^{l+1} \to 0$ faster	$l+1 \geq 2$

Near $r = 0$, the homogeneous branches **dominate** χ_{part}. The IIB-II condition acts on the dominant (homogeneous) sector, not on χ_{part}. Theorem 3.5 reduces the self-adjoint extension from a $U(1)$ family to a binary choice $\{C_1 = 0, C_2 \neq 0\}$ or $\{C_1 \neq 0, C_2 = 0\}$. Candidates (a) and (b) resolve this choice.

Remark 3.7 (*[Speculation]*)**.** The failure of associativity $[x, y, z] \neq 0$ in $\mathcal{A}_4$ and the limit-circle singularity at $r = 0$ for $l \geq 1$ both appear at the same algebraic threshold $\mathcal{A}_3 \to \mathcal{A}_4$. Structural parallel; no derivation.

4. Candidates (a) and (b)

4.1. Candidate (a): global dS_4 regularity.

Proposition 4.1 (*[Derived]*)**.** *For $l \geq 1$: the homogeneous field $\phi_j^{\text{hom}} = \chi_j^{\text{hom}}/r \sim r^{-1/2\pm i\delta_l} \cdot G_j(r)$. Since $r^{-1/2} \to +\infty$ as $r \to 0^+$, ϕ_j^{hom} is **singular** at $r = 0$ (not L^2 in the radial measure). The particular field $\phi_{\text{part}} = \chi_{\text{part}}/r = r^l w_{\text{phys}}(r^2)$ is C^∞ at $r = 0$ (convergent power series in r^2).*

Theorem 4.2 (*[Conjecture]*, IIB-I/II)**.** *Under the Black Circle metric ansatz (IIB-I/II) and any regularity condition at $r = 0$ (global C^∞ on dS_4, C^1, or simply pointwise finiteness):*

$$\boxed{C_1 = C_2 = 0.}$$

The physical solution is $\chi_{\text{phys}} = \chi_{\text{part}} = r^{l+1} w_{\text{phys}}(r^2)$.

Proof. By Proposition 4.1, $\phi_j^{\text{hom}} \sim r^{-1/2} \to \infty$: any non-zero C_j makes ϕ singular. ϕ_{part} is C^∞ and finite. Any regularity condition forces $C_1 = C_2 = 0$. (*Argument strengthened relative to draft v1: the modes are singular, not merely non-smooth.*) ☐

Remark 4.3. Epistemic chain: ϕ_j^{hom} singular at $r = 0$: *[Derived]*. ϕ_{part} is C^∞ at $r = 0$: *[Derived]*. Black Circle metric is correct geometry: *[Conjecture]* (IIB-I/II). Conclusion $C_1 = C_2 = 0$: *[Conjecture]* (IIB-I/II).

Remark 4.4. Theorem 4.2 implies $C_1C_2 = 0$, consistent with Theorem 3.5. Candidate (a) is strictly stronger: it resolves the binary choice left by Candidate (c). For $l = 0$: no centrifugal singularity, regular endpoint, Candidate (a) is not operative; Paper 15 analysis applies.

4.2. **Candidate (b): Hartle–Hawking.**

Proposition 4.5 (*[Derived]*)**.** *In the Black Circle interior (r timelike, $r = 0$ past Cauchy surface), the KG current satisfies:*

$$J^r(\chi_j)\big|_{r\to 0} = \mp 2\delta_l, \quad j = 1\ (-, \textit{ingoing}), \quad j = 2\ (+, \textit{outgoing}).$$

Proof. $J^r = i(\chi^*\partial_r\chi - \chi\partial_r\chi^*)$. With $\chi_j \sim r^{\varrho_j}$: $J^r_j = -2\,\mathrm{Im}(\varrho_j) = \mp 2\delta_l$. □

Theorem 4.6 (*[Conjecture]*, BC timelike)**.** *The Hartle–Hawking condition* [20] *(no outgoing flux from $r = 0$) requires $C_2 = 0$.*

The QNM condition under Candidate (b) ($C_1 = 1$, $C_2 = 0$):

$$H^{(b)}_{\mathrm{BC}} = \left[A_{\mathrm{part}} + \tilde{K}_1^{(\varrho_1,l)}\right] \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0. \tag{4}$$

Result for $l = 1$, $m^2 = 0$: $\omega^{*(b)} = 5.052080812 + 0.4104389i$, $\tau = 2.44\, r_s/c$.

4.3. **Hierarchy and falsifiability.** Candidate (a) ⇒ Candidate (b) ⇒ Candidate (c). The two candidates that resolve the binary choice give observationally distinct spectra:

Candidate	$\omega^*(l = 1,\, m^2 = 0)$	τ	Type
(a) growing	$1.6449 + 0.7372i$	$1.357\, r_s/c$	growing
(a) damped	$1.2979 - 0.0076i$	$131\, r_s/c$	damped
(b)	$5.052 + 0.410i$	$2.44\, r_s/c$	growing

Candidates (a) and (b) differ by > 3.4 units in $\mathrm{Re}(\omega)$: the first **experimentally falsifiable** prediction of the series.

5. The complete QNM condition for $l \geq 1$

5.1. **The particular-solution coefficient A_{part}.**

Lemma 5.1 (*[Derived]*)**.** *The variation-of-parameters particular solution satisfies $w_{\mathrm{vp}}(0) = -1$ (independent of l).*

Proof. At $z \to 0$: contributions from the two Wronskian integrals give $w_{\mathrm{vp}}(0) = \frac{l(l+1)}{-2i\delta_l} \cdot \frac{i\delta_l}{l(l+1)/2} = -1$, using $(\varrho_1 - \varrho_2) = i\delta_l$ and $(\varrho_1 + \frac{2l+1}{2})(\varrho_2 + \frac{2l+1}{2}) = l(l+1)/2$. □

The normalised particular solution is $w_{\mathrm{phys}} = -w_{\mathrm{vp}}$, giving $w_{\mathrm{phys}}(0) = 1$.✓

Corollary 5.2 (*[Derived]*)**.**

$$A_{\mathrm{part}}(\omega; l, m^2) = \tilde{K}_1^{(\varrho_2,l)} \cdot I_1 - \tilde{K}_1^{(\varrho_1,l)} \cdot I_2, \quad I_j = \int_0^1 u_{\varrho_j}(t) \cdot \frac{l(l+1)\, t^{(2l-1)/2}}{-2i\delta_l(1-t)}\, dt. \tag{5}$$

5.2. **The corrected QNM condition.**

Theorem 5.3 (*[Derived]* — corrects Paper 16, Remark 5.3)**.** *The QNM condition for $l \geq 1$ with general (C_1, C_2):*

$$\boxed{H^{\mathrm{full}}_{\mathrm{BC}}(\omega; l, m^2) = \left[A_{\mathrm{part}} + C_1\tilde{K}_1^{(\varrho_1,l)} + C_2\tilde{K}_1^{(\varrho_2,l)}\right] \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0.} \tag{6}$$

Remark 5.4. Paper 16 Remark 5.3 wrote $[\cdots]$ without A_{part}. That formula is valid only if C_1, C_2 denote the full-solution coefficients in the homogeneous basis, but Paper 16 Theorem 4.1 proved $w_{\mathrm{phys}} \notin \ker(L_w)$: the correction A_{part} is necessary.

Corollary 5.5 (*[Conjecture]*, IIB-I/II)**.** *Under Candidate (a) ($C_1 = C_2 = 0$):*

$$H^{(a)}_{\mathrm{BC}}(\omega; l, m^2) = A_{\mathrm{part}}(\omega; l, m^2) \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0. \tag{7}$$

Remark 5.6 (*[Derived]*)**.** $l = 0$ consistency: $l(l+1) = 0$ gives $I_1 = I_2 = 0$ and $A_{\mathrm{part}} = 0$; the condition reduces to $K_1 \cdot 2^{i\Omega} = 2^{i\omega/2}e^{i\omega}$, recovering Paper 15, Theorem 5.1.✓

5.3. Initial conditions for w_{phys}.

Lemma 5.7 (*[Derived]*). *The step-2 coefficient c_2 of $w_{\text{phys}} = 1 + c_2 z + c_4 z^2 + \cdots$:*

$$c_2 = \frac{3l(l+1) + 2l - (\omega^2 - m^2)}{2(l^2 + 3l + 3)}, \tag{8}$$

where $2(l^2 + 3l + 3) = \alpha_0 = (l+3)(l+2) + l(l+1)$. For $l = 0$: $c_2 = -(\omega^2 - m^2)/6 = -2\Omega^2/3$ (Paper 15, Lemma 4.1).✓ For $l = 1$, $\omega = 0$, $m^2 = 0$: $c_2 = 8/14 = 4/7$.✓

Correct initialisation: $w_{\text{phys}}(z_0) = 1 + c_2 z_0$, $w'_{\text{phys}}(z_0) = c_2$. Initialising with $w'_{\text{phys}}(z_0) = 0$ introduces error $\sim |c_2| z_0$.

6. QNM spectrum under Candidate (a)

6.1. Method and $l = 0$ consistency check. Wronskian projection at $z_{\text{match}} = 0.75$; analytic u_ϱ derivatives; correct IC from Lemma 5.7; mpmath at 50 d.p. Verification: $A_{\text{part}}/K_1 = 1 + O(2.4 \times 10^{-5})$ for 5 values of ω, all at 50 significant figures.✓

6.2. $l = 1$, $m^2 = 0$ spectrum. Two modes under Candidate (a):

$$\omega^*_{\text{damp}} = 1.29788457826 - 0.00762233175i, \quad |H^{(a)}_{\text{BC}}| = 2.14 \times 10^{-51}, \quad \tau = 131\, r_s/c, \tag{9}$$

$$\omega^*_{\text{grow}} = 1.6449410 + 0.7371871i, \quad |H^{(a)}_{\text{BC}}| < 10^{-38}, \quad \tau = 1.357\, r_s/c. \tag{10}$$

The growing mode was found via parametric continuation (its attraction basin is distant from the initial grid).

Theorem 6.1 (*[Derived]* (numerical), Candidate (a), IIB-I/II)**.** *The Black Circle is unstable in the $l = 1$ sector: a growing mode with* $\text{Im}(\omega^*) > 0$ *exists for all physical masses* $m^2 \leq 0.384$.

6.3. $l = 2$, $m^2 = 0$. First damped mode: $\omega^*_{l=2} = 1.9252222467 - 0.415997679i$, $\tau = 2.4\, r_s/c$.✓

6.4. Mass dependence of the $l = 1$ growing mode.

m^2	$\text{Im}(\omega^*_{\text{grow}})$	τ (r_s/c)
+0.20	+0.6841	1.46
0.00	+0.7372	1.36
−0.10	+0.7695	1.30
−0.25	+0.8047	1.24

Slope: $d(\text{Im}\, \omega^*)/dm^2 \approx -0.2688$. Threshold for stability: $m_c^2 \approx +2.74 \gg m^2_{\text{phys}}$. The growing mode persists throughout the physical spectrum.

6.5. Quasi-universality of τ_{grow}.

$$\tau_{\text{grow}}(l = 0, m^2 = 0) = 1.361\, r_s/c \quad \text{[Paper 15]}, \quad \tau_{\text{grow}}(l = 1, m^2 = 0) = 1.357\, r_s/c \quad \text{[Paper 17]}. \tag{11}$$

Difference: 0.33%. *[Conjecture]* (numerical): the two growing-mode timescales are quasi-universal, likely because both are controlled by the boundary condition at $z = 1$ through $2^{i\omega/2} e^{i\omega}$, independently of l. Algebraic origin: OP 18.6.

7. OP 17.3: exact m_{thr}^2

Theorem 7.1 (*[Derived]*).

$$\boxed{m_{\text{thr}}^2 = 0.4605600022}$$

to 10 significant figures. Coefficient in $\mathrm{Re}(\omega^*) \approx A\sqrt{m^2 - m_{\text{thr}}^2}$*:* $A = 0.87153$ *(5 s.f.).*

Proof. Richardson extrapolation on $[\mathrm{Re}(\omega^*(m^2))]^2 \approx A^2(m^2 - m_{\text{thr}}^2)$; 60 d.p. arithmetic; $\sigma = 1.87 \times 10^{-16}$ on 8 points; verified at $m_{\text{thr}}^2 \pm 10^{-4}$. □

Corollary 7.2 (*[Derived]*). *Under the cavity bound* $\lambda GM^2 < 2.14 \times 10^{-3}$*:* $m_4^2 < 0.256$ *and* $m_5^2 < 0.384$*, both* $< m_{\text{thr}}^2$*. No Type II* $l = 0$ *oscillatory modes exist in the physical Black Circle spectrum.* ***OP 17.3 closed.***

Remark 7.3 (*[Derived]*). Corrections to Papers 15–16: $A = 0.878 \to 0.87153$ (5 s.f.); $m_{\text{thr}}^2 \in (0.40, 0.464) \to 0.4605600022$.

8. OP 17.2: no closed form for $\tilde{K}_1^{(\varrho_2,l)}$

Proposition 8.1 (*[Derived]*). *The Legendre duplication formula does not apply to* $\tilde{K}_1^{(\varrho_2,l)}$ *for* $l \geq 1$*.*

Proof. At $l = 0$ with $\omega = \pm i\kappa$: $c - a - b = \kappa \in \mathbb{R}_{>0}$, duplication applies. For $l \geq 1$: the four Γ-arguments are $1 - i\delta_l$, $-2i\Omega$, $\frac{5}{4} - i(\Omega + \frac{\delta_l}{2})$, $-\frac{1}{4} - i(\Omega + \frac{\delta_l}{2})$. At $\Omega = i\kappa/2$ these become complex with $\mathrm{Im} \neq 0$: no duplication identity applies. □

OP 17.2 *[Open]*: possible routes via DLMF §15.8, §15.12, or Gauss–Kummer contiguous relations.

9. OP 17.4: Paper 13 $l \geq 1$ QNMs under Candidate (a)

Theorem 9.1 (*[Conjecture]*, IIB-I/II). *Under Candidate (a), the Paper 13* $l = 2$ *point* $\omega \approx 2.137 - 0.085i$ *gives* $|H_{\text{BC}}^{(a)}| = 0.375$*: not a QNM. All Paper 13* $l \geq 1$ *QNM values (Type I and Type II) are artifacts of the Wronskian extraction under the incorrect BC (Paper 16, Prop. 4.5).*

Remark 9.2. Paper 16 Prop. 4.5 showed $C_2(z_0)$ varied by $\sim 60\%$ across $z_0 \in [0.2, 0.7]$. This is now explained: under Candidate (a), $C_2 = 0$ exactly; the Wronskian instability was the signal that no stable C_2 exists.

Remark 9.3. Epistemic note: if Candidate (a) fails (IIB-I/II false), OP 17.4 remains open. The label *[Conjecture]* (IIB-I/II) is essential.

10. Open problems

Label	Statement	Status
OP 17.1	BC at $r = 0$ for $l \geq 1$: resolved as $C_1 = C_2 = 0$ under Candidate (a) *[Conjecture]* (IIB-I/II).	Closed *[Conjecture]*
OP 17.2	Closed form for $\tilde{K}_1^{(\varrho_2,l)}$, $l \geq 1$.	*[Open]*
OP 17.3	Exact m_{thr}^2.	Closed: 0.46056 *[Derived]*

Label	Statement	Status
OP 17.4	Paper 13 $l \geq 1$ QNMs.	Closed negative *[Conjecture]* (IIB-I/II)
OP 17.5	$A_{\text{part}}(\omega; l, m^2)$ for $l \geq 1$.	Closed *[Derived]*
OP 18.1	Full functional-analytic self-adjoint extension at $r = 0$.	*[Open]*
OP 18.2	Non-trivial AMT BC map on $\mathcal{A}_4$.	*[Open]*
OP 18.3	QNMs for $l \geq 1$, $m^2 \neq 0$ (Levels 1,4,5).	*[Open]*
OP 18.4	$\mathrm{Im}(W_{\text{1-loop}})$ revised with correct spectrum.	*[Open]* (needs OP 18.3)
OP 18.5	Off-axis zeros for $l = 1$ with $\mathrm{Im}(\omega) > 1$.	*[Open]*
OP 18.6	Algebraic origin of $\tau_{\text{grow}}(l = 0) \approx \tau_{\text{grow}}(l = 1)$ (0.33% difference).	*[Open]*, new
OP 12.7	PCT involution for $\mathrm{GL}(2, 3)$ on 4_s.	*[Open]*

11. Conclusions

Three candidates, hierarchy. (c) $C_1C_2 = 0$ from IIB-II norm oscillation *[Conjecture]*. (b) $C_2 = 0$ from Hartle–Hawking *[Conjecture]*. (a) $C_1 = C_2 = 0$ from dS_4 regularity *[Conjecture]*. Hierarchy (a)⊂(b)⊂(c).

QNM condition corrected *[Derived]*. $H_{\text{BC}}^{\text{full}} = [A_{\text{part}} + C_1\tilde{K}_1^{(\varrho_1)} + C_2\tilde{K}_1^{(\varrho_2)}] \cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega} = 0$ (corrects Paper 16 Remark 5.3, which omitted A_{part}).

$l = 1$ instability *[Derived]* (Candidate (a)). Growing mode $\omega^* = 1.6449 + 0.7372i$, $\tau = 1.357\, r_s/c$. Persists for all $m^2 \leq 0.384$. Quasi-universal: $\tau_{\text{grow}}(l = 1) \approx \tau_{\text{grow}}(l = 0)$ to 0.33%.

Falsifiability. Candidates (a) and (b) give $\mathrm{Re}(\omega^*_{l=1}) = 1.645$ vs 5.052: first experimentally falsifiable prediction of the series.

OP 17.3 closed *[Derived]*. $m^2_{\text{thr}} = 0.4605600022$.

Paper 13 $l \geq 1$ QNMs: artifacts *[Conjecture]*. Under Candidate (a): all $l \geq 1$ values are extraction artifacts. Wronskian instability of Paper 16 is explained.

IIB-II contribution. Candidate (c) is the first explicit use of the norm-gravity conjecture in a calculation. The AMT is automatic; the non-trivial IIB-II condition operates on the homogeneous sector, where the norm oscillates.

Errata from previous papers

Paper 13, Proposition 4.1 *[Derived]*: "$r = 0$ ordinary point" incorrect for $l \geq 1$. See Paper 16 Prop. 2.1.

Paper 13, §6.2 *[Derived]*: "$l = 1$ same framework as $l = 0$, no oscillatory modes" incorrect under the correct BC; $l = 1$ has $\omega^* = 1.298 - 0.008i$.

Paper 13, Type I for $l = 1$ *[Conjecture]* (IIB-I/II): values $\kappa \in \{0.539, 0.590, 0.688, 0.735\}$ are extraction artifacts.

Paper 13, Type II for $l = 2$ *[Conjecture]* (IIB-I/II): $\omega \approx 2.137 - 0.085i$ is an artifact; physical $l = 2$ QNM: $\omega^* = 1.9252 - 0.4160i$.

Paper 14, §4 Track B *[Derived]*: "no oscillatory mode for $l = 1$" incorrect under the correct BC; $\omega^*_{l=1} = 1.298 - 0.008i$ lies in the searched region.

Paper 14, $\mathrm{Im}(W) = -74.800\, r_s^{-1}$ *[Open]* (OP 18.4): computed with Paper 13 $l \geq 1$ values (now artifacts); revision deferred.

Paper 15, A: $0.878 \to 0.87153$ (5 s.f.).

Papers 15–16, m^2_{thr}: $\approx 0.464 \to 0.4605600022$ (10 s.f.).

Paper 16, Remark 5.3 *[Derived]*: QNM formula incomplete (missing A_{part}); corrected by Theorem 5.3.

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