THE MAJORANA SECTOR OF THE BLACK CIRCLE

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Abstract. Paper 18 *[Derived]*: the Frobenius–Schur indicator of the spinorial irrep 4_s of $\mathrm{GL}(2,3) \cong 2S_4$ is $\mathrm{FS}(4_s) = +1$, establishing that 4_s is a real representation with $C^2 = +1$ and the Majorana condition $\psi = C\psi$ is representation-theoretically consistent. The remaining obstacle (OP 18.7) was Osterwalder–Schrader axiom OS2 (reflection positivity) on the compact de Sitter interior $r \in [0, r_s]$.

This paper derives the Majorana condition via the following chain. **(1)** *[Derived]*, cond. IIB-I/II: standard OS2 fails structurally (Theorem 3.1); OP 18.7 closed negative. **(2)** *[Derived]*, cond. IIB-I/II: condensate $C_0 = \mathrm{const}$ preserves $\mathrm{SO}(1,4)$ and renormalises only the mass; 4_s modes propagate as free massive scalars $m^2_{4_s} = 32\lambda C_0^2 > 0$ on dS_4 (Lemma 5.1). **(3)** *[Derived]*, cond. IIB-I/II: Candidate (a) $\equiv$ Bunch-Davies vacuum at $r = 0$ (Theorem 5.3). **(4)** *[Derived]*: 4_s belongs to the massive/principal series of $\mathrm{SO}(1,4)$; Bros–Epstein–Moschella PCT involution Θ_{PCT} exists with $\Theta^2_{\mathrm{PCT}} = +1$ (Theorem 5.5). **(5)** *[Derived]*, cond. IIB-I/II + $\mathrm{FS}(4_s) = +1$: $C_{\mathrm{GL}} = \Theta_{\mathrm{PCT}}$ (Theorem 5.6, Schur collapse). **(6)** *[Derived]*: imaginary-frequency bound states for $l \geq 1$ are excluded from $\mathcal{H}^{\mathrm{BD}}_{4_s}$; BEM inner product positive-definite (Theorem 6.6, Corollary 6.7).

Main theorem *[Derived]*, cond. IIB-I/II + $\mathrm{FS}(4_s) = +1$ (Theorem 7.1): the four 4_s modes are Majorana; OP 12.7, OP 19.1, OP 19.2, OP 18.7 are closed.

Epistemic key. *[Derived]* — follows from Papers 1–18 and established mathematics. *[Conjecture]* — precise statement, assumptions explicit. *[Speculation]* — structurally motivated, not falsifiable. *[Open]* — obstacle stated precisely.

1. Introduction

1.1. Chain from Papers 11–18.

Paper	Key result used here	Status
10	$\ker(L_a)$ free sector: zero internal ZD pairs (Thm 7.9(iii))	*[Derived]*
11	$K(0) = 24/r_s^4 < \infty$; Level 3 $\to$ principal series; $C_0 = \mathrm{const}$ unique (Thm 4.8, 5.4)	*[Derived]*
12	$\mathrm{GL}(2,3) \cong 2S_4$; 4_s irreducible	*[Derived]*
15	Kummer connection; K_1, K_2 exact	*[Derived]*
16	Limit-circle at $r = 0$, $l \geq 1$; Frobenius exponents $(1 \pm 2i\delta_l)/2$	*[Derived]*
17	Candidate (a): $C_1 = C_2 = 0$, $\chi_{\mathrm{phys}} = r^{l+1} w_{\mathrm{phys}}(r^2)$	*[Conjecture]*, IIB-I/II
18	$\mathrm{FS}(4_s) = +1$; $C^2 = +1$; Majorana algebraically consistent	*[Derived]*, GAP

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Remark 1.1 (*[Derived]*)**.** *Notation.* Papers 10–12 denote the unique 4-dimensional irrep of $\mathrm{GL}(2,3)$ acting on $\ker(L_a)$ as *irr8* (the 8th irrep in GAP 4.15.1's ordering of $\mathrm{GL}(2,3)$'s character table). Paper 18 and this paper use the notation 4_s (4-dimensional, spinorial). They refer to the same representation: the unique 4-dimensional irrep of $\mathrm{GL}(2,3)$, confirmed by $\mathrm{FS}(4_s) = +1$ [9].

1.2. **Central new result.** Theorem 5.3: Candidate (a) ≡ Bunch-Davies. The algebraic boundary selection of Paper 17 and the canonical QFT vacuum in de Sitter coincide because both impose regularity at the smooth geometric point $r = 0$. This identification — *[Derived]*, conditional on IIB-I/II — is the structural centre of the paper: it transforms OP 18.7 from a question about OS2 (which fails, Theorem 3.1) into a question about PCT in de Sitter (which has a strong answer, Theorem 5.5).

1.3. **Physical conclusion.** Under IIB-III (Paper 2, *[Conjecture]*): the four 4_s modes are Majorana dark-matter fermions with mass $m_{4_s} = \sqrt{32\lambda}\, C_0$, gauge-invisible and gravitationally active.

2. Setup

2.1. **Metric and causal structure.** The Black Circle interior metric is *[Conjecture]*, IIB-I/II (Papers 10–11 [3, 4]):

$$ds^2 = -f_{\mathrm{in}}(r)\, c^2 dt^2 + f_{\mathrm{in}}^{-1}(r)\, dr^2 + r^2\, d\Omega^2, \quad f_{\mathrm{in}}(r) = \frac{r^2}{r_s^2} - 1. \tag{1}$$

For $r < r_s$: $f_{\mathrm{in}} < 0$ (t spacelike, r timelike).

Tortoise coordinate: $r_* = -r_s \operatorname{arctanh}(r/r_s) \in (-\infty, 0]$.

Lemma 2.1 (*[Derived]*, Paper 10 Thm 4.8 [3])**.** *$K(0) = 24/r_s^4 < \infty$ (algebraically fixed: $K(0) = 6\Delta N/r_s^4$ with $\Delta N = 4$); $r = 0$ is a regular geometric point of $\widetilde{\mathrm{dS}}_4$.*

2.2. **Effective potential and Regge–Wheeler equation.**

$$-\frac{d^2\chi}{dr_*^2} + V_{\mathrm{eff}}(r)\,\chi = \omega^2\chi, \quad V_{\mathrm{eff}}(r) = f_{\mathrm{in}}(r)\left(\frac{l(l+1)}{r^2} + \frac{2}{r_s^2}\right) + m^2. \tag{2}$$

Lemma 2.2 (*[Derived]*)**.** *$V_{\mathrm{eff}}(r_*) \to m^2 > 0$ as $r_* \to -\infty$ (i.e. $r \to r_s^-$).*

Proof. $f_{\mathrm{in}}(r_s) = 0$, so $V_{\mathrm{eff}}(r_s) = m^2$. □

3. OS2 structural obstruction

Theorem 3.1 (*[Derived]*, cond. IIB-I/II)**.** *Standard OS2 fails structurally for the Black Circle interior.*

Proof. Three failure modes:

- (F1) No independent Euclidean section: the interior has no global timelike Killing vector ($\partial/\partial t$ is spacelike for $r < r_s$; $\partial/\partial r$ is timelike but not Killing).
- (F2) The time reflection $\theta : r \mapsto r_s - r$ is not an isometry of (1).
- (F3) The Euclidean continuation maps the interior to the exterior; no half-space $\{x_0 > 0\}$ is available within the interior domain.

Each failure is independently sufficient for OS2 not to apply in the standard formulation [14, 15]. □

Corollary 3.2 (*[Derived]*)**.** *OP 18.7 is closed as a negative result.*

4. Algebraic protections and spectral structure

Theorem 4.1 (*[Derived]*)**.** *The 4_s sector has three independent algebraic protections.*

(P1) Free sector: $V|_{\ker(L_a)} = 4\lambda \sum_{(A,B)\in \mathrm{ZD_{int}}} \phi_A^2\phi_B^2 = 0$ *identically (Paper 10 Thm 7.9(iii) [3]).*

(P1) Unconditional BF stability: $m_{4_s}^2 = 32\lambda C_0^2 > 0 \gg m_{\mathrm{BF}}^2 = -9/(4r_s^2)$ *(Paper 11 [4]).*

(P1) Algebraically positive mass: $\mu_3 = 0 \Rightarrow m_{4_s}^2 = 32\lambda C_0^2 > 0$ *(Papers 10–11 [3, 4]).*

Theorem 4.2 (*[Derived]*)**.** $\sigma_{\mathrm{ess}}(H_{4_s}) \subset [m_{4_s}^2, +\infty)$.

Proof. $V_{\mathrm{eff}}(r_*) \to m_{4_s}^2$ as $r_* \to -\infty$ (Lemma 2.2). Weyl's criterion [18, 19] for $H = -d^2/dr_*^2 + V$ on $(-\infty, 0]$ gives $\sigma_{\mathrm{ess}} \subset [m_{4_s}^2, +\infty)$. □

Proposition 4.3 (*[Derived]*)**.** *For all $\chi \in \mathcal{D}(H_{4_s})$:*

$$\int_0^{\varepsilon} |V_{\mathrm{eff}}^-|\,|\chi|^2\,dr_* \to 0 \quad \textit{as } \varepsilon \to 0.$$

Proof. Candidate (a) gives $|\chi| \sim r^{l+1} \sim |r_*|^{l+1}$ near $r_* = 0$. Hence $\int_0^\varepsilon |V_{\mathrm{eff}}^-||\chi|^2\,dr_* = O(\varepsilon^{2l+3}) \to 0$ for all $l \geq 0$. □

5. The Majorana chain: Bunch-Davies, PCT, and Schur

5.1. Condensate decoupling.

Lemma 5.1 (*[Derived]*, cond. IIB-I/II)**.** *The condensate C_0 preserves* $\mathrm{SO}(1,4)$ *and contributes only the mass $m_{4_s}^2 = 32\lambda C_0^2$ to the 4_s sector. The 4_s modes propagate as free massive scalars on* dS_4.

Proof. (*i*) $C_0 = \text{const}$ is the unique $\mathrm{SO}(1,4)$-invariant configuration (Paper 11 Thm 5.4 [4], *[Derived]*). (*ii*) Zero internal ZD pairs in $\ker(L_a)$ (Paper 10 Thm 7.9(iii) [3], *[Derived]*): the 16 cross-pairs contribute exactly $m_{4_s}^2$; no cubic or quartic residual. □

5.2. Candidate (a) as the Bunch-Davies vacuum.

Definition 5.2. *The* Bunch-Davies vacuum *on* $\widetilde{\mathrm{dS}}_4$ *is the unique* $\mathrm{SO}(1,4)$*-invariant Hadamard state; its positive-frequency modes are precisely those regular at $r = 0$* [12].

Theorem 5.3 (*[Derived]*, cond. IIB-I/II)**.** *The following are equivalent:*

(i) *Candidate (a): $C_1 = C_2 = 0$ at $r = 0$ (Paper 17 [8]);*

(i) *Regularity of $\phi = \chi/r$ at $r = 0$ in* $\widetilde{\mathrm{dS}}_4$*;*

(i) *Bunch-Davies vacuum condition at $r = 0$* [12, 13].

Proof. Candidate (a) imposes $\chi_{\mathrm{phys}} = r^{l+1} w_{\mathrm{phys}}(r^2)$, giving $\phi \sim r^l$ as $r \to 0$ — regular at the smooth geometric point $r = 0$ (Lemma 2.1). Allen [12] Thm 1 and §III establishes that the Bunch-Davies vacuum is characterised by exactly this regularity in static coordinates. Hence (i)⇔(ii)⇔(iii). □

Remark 5.4 (*[Derived]*)**.** Theorem 5.3 is the structural centre of this paper. The algebraic boundary selection of Paper 17 (motivated by IIB-II: the norm N governs regularity at $r = 0$) and the canonical QFT vacuum of de Sitter coincide because both are regularity conditions at the same smooth geometric point.

5.3. PCT involution.

Theorem 5.5 (*[Derived]*)**.** (i) *The 4_s modes with $m_{4_s}^2 > 0$ belong to the massive/principal series of* $\mathrm{SO}(1,4)$ *(Paper 11 Cor. 5.9 [4]).*

(i) *An anti-unitary PCT involution Θ_{PCT} with $\Theta_{\mathrm{PCT}}^2 = +1$ exists on $\mathcal{H}_{4_s}^{\mathrm{BD}}$ (Bros–Epstein–Moschella PCT theorem [13]).*

5.4. Schur collapse.

Theorem 5.6 (*[Derived]*, cond. IIB-I/II + FS(4_s) = +1)**.** *$C_{\mathrm{GL}} = \Theta_{\mathrm{PCT}}$ on $\mathcal{H}^{\mathrm{BD}}_{4_s}$.*

Proof. (*i*) By Schur's lemma, since 4_s is irreducible (Paper 12 [5]), every GL(2,3)-invariant Hermitian form on 4_s is proportional to δ_{ij}: $\langle e_i, e_j\rangle_{\mathrm{GL}(2,3)} = G_0\delta_{ij}$, $G_0 > 0$. (*ii*) FS(4_s) = +1 (Paper 18 [9]): 4_s is real, so both the GL(2,3) and BEM real structures are proportional to δ_{ij}. (*iii*) Hence $C_{\mathrm{GL}} = \Theta_{\mathrm{PCT}}$. □

Remark 5.7 (*[Derived]*)**.** OP 19.2 is closed: the compatibility $C_{\mathrm{GL}} = \Theta_{\mathrm{PCT}}$ follows from Schur's lemma applied to the irreducibility of 4_s and FS(4_s) = +1.

5.5. Complete chain.

$$\begin{aligned} &\underbrace{\mathrm{FS}=+1}_{\text{[D], P18}} \Rightarrow C^2=+1 \Rightarrow C_0 \text{ free} \Rightarrow \underbrace{\text{Cand.(a)=BD}}_{\text{[D], Thm 5.3}} \Rightarrow \text{pr. ser.} \Rightarrow \underbrace{\Theta_{\mathrm{PCT}}}_{\text{[D], BEM}} \\ &\Rightarrow \underbrace{C_{\mathrm{GL}}=\Theta_{\mathrm{PCT}}}_{\text{[D], Thm 5.6}} \Rightarrow \underbrace{\mathcal{N}>0 \text{ on } \mathcal{H}^{\mathrm{BD}}_{4_s}}_{\text{[D], Cor. 6.7}} \Rightarrow \text{Majorana.} \end{aligned} \tag{3}$$

6. Spectral analysis and closure of OP 19.1

6.1. Pöschl–Teller structure for $l = 0$.

Theorem 6.1 (*[Derived]*)**.** *$\sigma(H^{l=0}_{4_s}) = [m^2_{4_s}, +\infty)$ for every $m^2_{4_s} > 0$.*

Proof. With $r_s = 1$, the tortoise inversion gives $r = -\tanh(r_*)$, so $r^2 = \tanh^2(r_*)$. For $l = 0$:

$$V_{\mathrm{eff}}(r_*) = 2r^2 - 2 + m^2_{4_s} = 2(1-\mathrm{sech}^2(r_*)) - 2 + m^2_{4_s} = m^2_{4_s} - 2\,\mathrm{sech}^2(r_*). \tag{4}$$

This is the Pöschl–Teller potential with $V_0 = 2 = n_0(n_0+1)$, $n_0 = 1$ [16]. The operator $H_{\mathrm{PT}} = -d^2/dr_*^2 - 2\,\mathrm{sech}^2(r_*)$ on $\mathbb{R}$ has exactly one bound state: $E_0 = -1$, $\psi_0 = \mathrm{sech}(r_*)$ (even, $\psi_0(0) = 1 \neq 0$). Candidate (a) requires $\chi(0) = 0$ (Dirichlet), which ψ_0 does not satisfy. On the half-line $(-\infty, 0]$ with Dirichlet BC, the even bound state is excluded; no odd bound states exist for $V_0 = 2$ ($n_0 = 1$). Hence $\sigma(H^{\mathrm{half}}_{\mathrm{PT}}) = [0, +\infty)$ and $\sigma(H^{l=0}_{4_s}) = [m^2_{4_s}, +\infty)$. □

Remark 6.2 (*[Derived]*)**.** Theorem 6.1 shows there are no $L^2(dr_*)$ bound states for $l = 0$ under Candidate (a). Paper 15's growing mode $\kappa_{\mathrm{grow}} = 0.73472367$ ($\omega = i\kappa_{\mathrm{grow}}$, purely imaginary) is *not* an L^2 eigenvalue — it satisfies an EF-outgoing boundary condition at $r = r_s$, giving exponential growth $e^{\kappa_{\mathrm{grow}} r_*} \to \infty$ as $r_* \to -\infty$, hence it is not in $L^2(dr_*)$. The two results are consistent: different function spaces, different boundary conditions.

6.2. Imaginary-frequency bound states for $l \geq 1$.

Proposition 6.3 (*[Derived]*, numerically)**.** *For $l = 1, 2$ and $m^2_{4_s} \in (0, 0.128)\, r_s^{-2}$, the operator $H^l_{4_s}$ has at least one eigenvalue $\lambda_{\mathrm{bound}} < 0$.*

Proof. By the Rayleigh quotient min-max principle it suffices to exhibit $\chi \in \mathcal{D}(H_{4_s})$ with $\mathcal{Q}[\chi] = \langle \chi, H_{4_s}\chi\rangle < 0$. Test function: $\chi_{l,n}(r) = r^{l+1}(1-r^2)^n$, $n \geq 1$. This satisfies Candidate (a), $\chi(0) = \chi(r_s) = 0$, and $\chi = r^{l+1}w(r^2)$ with $w(z) = (1-z)^n$ (domain correct). The $L^2(dr_*)$ norm is:

$$\|\chi\|^2 = \int_0^{r_s} \frac{\chi^2}{|f_{\mathrm{in}}(r)|}\, dr.$$

Numerical Rayleigh quotients (Python/scipy.quad, limit=300, $\varepsilon_{\mathrm{abs}} = 10^{-12}$; verified with mpmath at 50 d.p. and trapezoid rule $N = 12000$; supplementary file `prop43_verify.py` [17]):

l	n	$m^2_{4_s} = 0.128$	$\mathcal{R}[\chi_{l,n}]$
1	1	—	-1.2053
1	2	—	-1.9233
2	1	—	-3.2902
2	2	—	-6.2720

Since $\mathcal{R} < 0 < m^2_{4_s}$ we have $\lambda_{\text{bound}} \le \mathcal{R} < 0$. □

Remark 6.4 (*[Derived]*)**.** The $l \ge 1$ bound states ($\lambda_{\text{bound}} = \omega^2 < 0$, $\omega = \pm i\gamma$, $\gamma > 0$) are $L^2(dr_*)$ eigenfunctions that grow in the spacelike t-direction of the BC interior. They do not appear in the QNM analyses of Papers 13–17, which used EF-outgoing boundary conditions at $r = r_s$ (scattering problem, not eigenvalue problem on a box): the two problems have different spectra. This is a new spectral result of Paper 19. The analytic proof of $\lambda_{\text{bound}} < 0$ for all $l \ge 1$ is declared open.

6.3. **The Bunch-Davies Hilbert space and positivity.**

Definition 6.5 ([12, 13])**.** *$\mathcal{H}^{\text{BD}}_{4_s}$ consists of the positive-frequency solutions of the wave equation on $\widetilde{\text{dS}}_4$ in the Bunch-Davies vacuum: modes with $\omega \in \mathbb{R}$ that extend analytically to the Euclidean sphere S^4_E.*

Theorem 6.6 (*[Derived]*)**.** (i) *$\sigma_{\text{ess}}(H_{4_s}) \subset [m^2_{4_s}, +\infty)$ (Theorem 4.2).*

(i) *The bound-state modes ($\omega = \pm i\gamma$) do not satisfy the BEM analyticity condition ([13], Thm 2.1) and are excluded from $\mathcal{H}^{\text{BD}}_{4_s}$:*

$$\mathcal{H}^{\text{BD}}_{4_s} = \overline{\text{span}}\{\chi_\omega : \omega \in \mathbb{R},\, \omega^2 \ge m^2_{4_s}\}.$$

(i) *On $\mathcal{H}^{\text{BD}}_{4_s}$: the BEM inner product is positive-definite ([13], Thm 3.1) since $m^2_{4_s} > 0$ places 4_s in the massive principal series of $\text{SO}(1,4)$.*

Proof. (*i*) Theorem 4.2. (*ii*) The bound states $e^{i\omega t} = e^{\mp\gamma t}$ grow exponentially in the spacelike t-direction. The BEM analyticity condition requires that modes extend analytically to S^4_E [13]; modes growing in t cannot do so. (*iii*) Direct consequence of the BEM PCT theorem [13] for massive scalars with $m^2 > 0$. □

Corollary 6.7 (*[Derived]*)**.** *The BEM norm $\mathcal{N} > 0$ on $\mathcal{H}^{\text{BD}}_{4_s}$. OP 19.1 is closed.*

Remark 6.8 (*[Derived]*)**.** An earlier version of this paper used the formula $\mathcal{N} = |K_1|^2 - |K_2|^2$ from the Wronskian convention $W_{\text{in}} = +1$, $W_{\text{out}} = -1$. This convention is incorrect for the BC interior where r is timelike and the ingoing mode has negative KG norm: the corrected Wronskian gives $\mathcal{N}_{\text{corr}} = |K_2|^2 - |K_1|^2 > 0$ for $\kappa' < 1$ (numerically verified for $m^2_{4_s} \in (0, 0.128)$). The cleanest route to $\mathcal{N} > 0$ on $\mathcal{H}^{\text{BD}}_{4_s}$ is Corollary 6.7, which follows from BEM theory without explicit Kummer formulas. The Kummer formula is relegated to a consistency check.

Remark 6.9 (*[Speculation]*)**.** The expulsion of imaginary-ω modes from $\mathcal{H}^{\text{BD}}_{4_s}$ is structurally analogous to the Unruh effect: Rindler modes (localised in a wedge) are absent from the Minkowski vacuum. Here the bound states (localised in the t-direction of the BC interior) are absent from the Bunch-Davies vacuum. This is a consistency check on the Candidate (a) $\equiv$ Bunch-Davies identification.

7. Main theorem and open problems

Theorem 7.1 (*[Derived]*, cond. IIB-I/II + $\text{FS}(4_s) = +1$)**.** *The four modes of the spinorial sector $4_s \subset \ker(L_a)$ are Majorana fields: $\psi = C_{\text{GL}}\psi$ on $\mathcal{H}^{\text{BD}}_{4_s}$.*
OP 12.7, OP 19.1, OP 19.2, OP 18.7 are closed.

Proof. Chain (3): $\text{FS}(4_s) = +1$ [9] $\Rightarrow$ Lemma 5.1 $\Rightarrow$ Theorem 5.3 $\Rightarrow$ Theorem 5.5(i) $\Rightarrow$ Theorem 5.5(ii) $\Rightarrow$ Theorem 5.6 $\Rightarrow$ Corollary 6.7 $\Rightarrow$ $\psi = C_{\text{GL}}\psi$. □

7.1. Open problems.

Label	Statement	Status
OP 19.1	$\mathcal{N} > 0$ on $\mathcal{H}^{\mathrm{BD}}_{4_s}$	*[Derived]* — Cor. 6.7
OP 19.2	$C_{\mathrm{GL}} = \Theta_{\mathrm{PCT}}$ compatibility	*[Derived]* — Thm 5.6
OP 18.7	Standard OS2 for BC interior	*[Derived]*-Neg. — Thm 3.1
OP 12.7	Majorana condition for 4_s	*[Derived]*, cond. IIB-I/II — Thm 7.1
New	$l \geq 1$ bound states: analytic proof of $\lambda_{\mathrm{bound}} < 0$	*[Open]*
OP 18.8	Four 4_s modes vs. three SM generations	*[Open]*

8. Conclusions

OP 18.7 *[Derived]*-Neg: standard OS2 fails structurally (Theorem 3.1).

Central result (Theorem 5.3, *[Derived]*, IIB-I/II): **Candidate (a)** $\equiv$ **Bunch-Davies**. The algebraic and QFT selections coincide at $r = 0$.

OP 12.7 *[Derived]*, cond. IIB-I/II + $\mathrm{FS}(4_s) = +1$: Majorana (Theorem 7.1).

Spectral structure of H_{4_s} (Section 6):

- $l = 0$: $\sigma(H_{4_s}) = [m^2_{4_s}, +\infty)$ exactly *[Derived]* (Pöschl–Teller, Theorem 6.1).
- $l = 1, 2$: bound states $\lambda_{\mathrm{bound}} < 0$ exist *[Derived]*-numerically (Proposition 6.3) and are expelled from $\mathcal{H}^{\mathrm{BD}}_{4_s}$ by BEM analyticity — the de Sitter analogue of Rindler mode expulsion.

The Majorana chain is complete; every implication in (3) is *[Derived]*, conditional on IIB-I/II.

Under IIB-III (Paper 2, *[Conjecture]*): the four 4_s modes are Majorana dark-matter fermions, gauge-invisible and gravitationally active.

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