THE INFORMATIONAL IDENTITY BOUNDARY: ALGEBRAIC FOUNDATIONS AND PHYSICAL CONJECTURES

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Abstract. The Cayley–Dickson algebras $\mathcal{A}_n$ over $\mathbb{R}$ form a tower

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \mathbb{S} \subset \cdots$$

in which each doubling step eliminates one algebraic property. At $n = 4$, three independent results — the failure of the Born Rule, the emergence of zero-divisors, and the first non-trivial kernel of left multiplication — mark a structural discontinuity that we call the *Informational Identity Boundary* (IIB). This paper develops the physical interpretation of that boundary in four proposals.

The foundational results are unconditional [Derived]: the zero-divisor pair graph $ZD^*(\mathcal{A}_4)$ has 84 vertices and 168 edges with automorphism group G_{84} of order 2688; the kernel module $\ker(L_a)$ has dimension 4 at $n = 4$, spanned exactly by the imaginary units non-incident to a in the Fano plane $PG(2, \mathbb{F}_2)$; and the Fano plane governs $\mathcal{A}_3$ multiplication and $\mathcal{A}_4$ annihilation by a precise dual role (Fano bifrontality).

The main quantitative result [Derived] (conditional on Sakharov–Adler) is Newton's gravitational constant in closed form:

$$G_N^{-1} = \frac{16\lambda\, N(C_0)}{12\pi}\Big[164 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 220 \ln 2 - 42 \ln 3\Big],$$

where the coefficient $164 = 168 - 4$ has two independent algebraic origins: $168 = |\mathrm{Aut}(PG(2, \mathbb{F}_2))|$ (bipartite spectral sum of S_{84}) and $4 = \dim \ker(L_a)|_{n=4}$ (Fano non-incidence count). The Planck mass is an output of the ZD algebraic structure and the Black Circle mass M.

Every physical statement carries an explicit epistemic label. The IIB programme is a conjecture with precise algebraic foundations and a clear account of what it has not proved.

Contents

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1. Introduction

1.1. **The boundary.** The Cayley–Dickson doubling construction produces, at each level n, an algebra that is strictly less structured than its predecessor. At $n = 1$ commutativity is lost; at $n = 2$, associativity; at $n = 3$, alternativity and the division property; at $n = 4$, the first zero-divisors appear, and with them a cascade of structural discontinuities that have no analogue at any earlier level.

Paper 1 of this series establishes three unconditional results at $n = 4$ [Derived]:

(i) *Born Rule failure.* The axioms (M),(D),(OA) of quantum probability admit a unique solution $f = N$ for $n \leq 3$ and no solution for $n \geq 4$. The quadratic norm N cannot function as a probability assignment beyond the octonions.

(ii) *Zero-divisor emergence.* The sedenions $\mathcal{A}_4$ contain 84 zero-divisor units (in signed pairs), counted by $|ZD(n)| = 336 \cdot \binom{n-1}{3}_2$, with $336 = 2|PSL(2,7)|$ encoding the Fano geometry of the zero-divisor set.

(iii) *Analytical Mean Theorem.* Despite Born Rule failure, the identity $\mathbb{E}_v[N(av)] = N(a)$ holds for all $n \geq 1$, including $n \geq 4$. The norm N remains statistically universal even when it can no longer serve as a probability rule.

These three facts define the *Informational Identity Boundary*: the level at which quantum-probabilistic structure collapses while the norm — the unique structure preserved by every doubling step — survives.

1.2. **What this paper proposes.** We develop the physical interpretation of the IIB in four proposals of increasing speculative reach.

Proposal 1 (IIB-I) [Conjecture]. The transition $\mathcal{A}_3 \to \mathcal{A}_4$ marks the algebraic boundary beyond which standard quantum-probabilistic structure cannot be maintained. Physical fields can be consistently defined over $\mathcal{A}_n$ for $n \leq 3$; at $n = 4$ the Born Rule obstruction is not a pathology to be resolved but a structural signal.

Proposal 2 (IIB-II) [Conjecture]. The quadratic norm N is the unique algebraic structure that survives every doubling step. We conjecture that this universality is the algebraic signature of gravity. Gauge interactions are intrinsic to specific Cayley–Dickson levels ($n = 1, 2, 3$); the gravitational interaction, associated with N rather than any multiplication structure, persists at every level. Under IIB-II, Newton's gravitational constant G_N is generated by the sedenion scalar field theory S_{84} via the Sakharov–Adler mechanism (Papers 26–28).

Proposal 3 (IIB-III) [Conjecture]. Degrees of freedom in $\mathcal{A}_4 \setminus \mathcal{A}_3$ carry no Standard Model quantum numbers and are gravitationally active via N. Their gauge-invisibility is algebraic, not dynamical: a dark sector whose darkness follows from the Fano non-incidence structure of $\mathcal{A}_4$.

Proposal 4 (IIB-IV) [Speculation]. Within Penrose's Conformal Cyclic Cosmology, black holes in a preceding aeon leave a non-localised residue in $\mathcal{A}_4 \setminus \mathcal{A}_3$ that cannot integrate into the octonionic structure of the succeeding aeon. This residue constitutes the dark-matter component. Its gravitational activity but gauge-invisibility follow from Proposals 1–3. This proposal requires a QFT over $\mathcal{A}_4$ and is not currently falsifiable.

1.3. **The quantitative advance.** Prior presentations of the IIB programme established the structural motivation for IIB-II — the universality of N across doubling steps, the Frobenius identity $\|L_a\|_F^2 = 2^n N(a)$, the Analytical Mean Theorem — without a quantitative consequence. Papers 26–28 change this.

Under IIB-II and the Sakharov–Adler mechanism, integrating out the 84 scalar fields and 4 Majorana fermions of S_{84} in a curved spacetime background yields Newton's constant in closed form [DERIVED] (conditional on Sakharov–Adler; Paper 26, Thm 4.2):

$$G_N^{-1} = \frac{16\lambda\, N(C_0)}{12\pi}\left[164\ln\frac{\Lambda^2}{16\lambda N(C_0)} - 220\ln 2 - 42\ln 3\right].$$

The coefficient $164 = 168 - 4$ is not a numerical accident. It has two independent algebraic origins established in Papers 29 and 34:

- $168 = |\mathrm{Aut}(PG(2,\mathbb{F}_2))|$ is the bipartite spectral invariant of S_{84}: the weighted sum $\sum_i (2+\mu_i)\cdot \mathrm{mult}_i$ over all levels, in which the 14 massless Level-2 modes contribute exactly zero [DERIVED] [Paper 26, Lem 3.1; Paper 34, Thm 3.2].
- $4 = \dim\ker(L_a)|_{n=4}$ is the Fano non-incidence count: the kernel of left multiplication by any zero-divisor unit a is spanned precisely by the 4 imaginary units non-incident to a in $PG(2,\mathbb{F}_2)$ [DERIVED] [Paper 29, Thm 3.1]. These 4 directions are exactly the 4 physical Majorana modes correcting the bosonic sum from 168 to 164.

Both terms trace to the same combinatorial object — the Fano plane — via its two faces: incidence (governing $\mathcal{A}_3$ multiplication) and non-incidence (governing $\mathcal{A}_4$ zero-divisor annihilation). This is the *Fano bifrontality* of Paper 32.

The conditionality on Sakharov–Adler is never suppressed. It appears explicitly every time the G_N formula is invoked.

1.4. **Epistemic policy.** Every mathematical or physical statement in this paper carries one of the following labels:

- [DERIVED] — complete proof, no physical assumption beyond those of the cited paper.
- [DERIVED, COND. X] — complete proof given assumption X, stated in full.
- [DERIVED-NEGATIVE] — proves the negative of a proposed statement.
- [CONJECTURE] — precise statement with identified gap.
- [SPECULATION] — structurally motivated, not currently falsifiable.
- [OBSERVATION] — computationally verified but not proved as theorem.

The following conditionalities are enforced throughout without exception:

Claim	Condition that must appear
G_N formula	conditional on Sakharov–Adler
Physical readings of norm results	conditional on IIB-II
Conjecture A ($M = m_{4_s}$ identification)	not derived from IIB-II; stated explicitly
All SU(3) decompositions	conditional on (A2): $\mathcal{A}_3 \leftrightarrow \mathrm{SU}(3)_{\mathrm{color}}$

No [CONJECTURE] is promoted to [DERIVED] without proof. No GAP computation is presented as an analytic proof unless an analytic proof is separately available.

1.5. **Structure of this paper.** Section 2 develops the algebraic foundations. Section 3 describes the Black Circle. Section 4 is the main quantitative content: Newton's constant from the Cayley–Dickson norm. Section 5 addresses Black Circle thermodynamics and holography. Section 6 covers the matter sector. Section 7 states and discusses the four IIB proposals. Section 8 collects all open problems. Section 9 concludes.

2. Algebraic Foundations

All results in this section are [Derived] from Papers 1–4 and 29–35 of the series, with no physical assumptions.

2.1. **The Cayley–Dickson tower.** The Cayley–Dickson doubling construction begins with $\mathcal{A}_0 = \mathbb{R}$. For each $n \geq 0$, the algebra $\mathcal{A}_{n+1}$ consists of pairs (a, b) with $a, b \in \mathcal{A}_n$, equipped with

$$(a, b)(c, d) = (ac - \bar{d}b,\ da + b\bar{c}), \qquad \overline{(a, b)} = (\bar{a}, -b).$$

The quadratic norm $N((a, b)) = N(a) + N(b)$ is positive definite for all n. The first four algebras are the classical normed division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. At $\mathcal{A}_4 = \mathbb{S}$ (sedenions, dimension 16) Hurwitz's theorem is exhausted: the norm ceases to be multiplicative, zero-divisors appear, and alternativity fails.

Two properties of N are immediate from the definition and hold at every level n without exception:

(a) The inner product $\langle x, y\rangle = \frac{1}{2}(N(x + y) - N(x) - N(y))$ is positive definite.
(b) N is preserved by every doubling step: $N((a, b)) = N(a) + N(b)$.

No other algebraic property of the tower has this unconditional universality. Property (b) is the algebraic foundation of IIB-II.

2.2. **Norm universality: the Frobenius identity and AMT.**

Theorem 2.1 (Frobenius identity [Derived])**.** [[1], Thm 5.6] *For every $a \in \mathcal{A}_n$ and every $n \geq 0$:*

$$\|L_a\|_F^2 = 2^n \cdot N(a),$$

where $L_a : \mathcal{A}_n \to \mathcal{A}_n$ is $x \mapsto ax$ and $\|\cdot\|_F$ denotes the Frobenius norm. The total operator weight of multiplication by a is determined by $N(a)$ alone, at every level.

Theorem 2.2 (Analytical Mean Theorem [Derived])**.** [[1], Thm 5.8] *For every $a \in \mathcal{A}_n$ and every $n \geq 1$:*

$$\mathbb{E}_{v\sim S^{2^n-1}}[N(av)] = N(a).$$

This holds for all n, including $n \geq 4$ where $N(ab) = N(a)N(b)$ fails. The norm N is not locally conserved for $n \geq 4$: zero-divisors give $N(ab) = 0$ with $N(a) = N(b) \neq 0$. But averaged over all directions, $N(av)$ equals $N(a)$ exactly.

Physical reading [Conjecture] *[IIB-II]: the norm N couples to every element universally and isotropically, in the same sense that the gravitational coupling is direction-independent. The mathematical statement is* [Derived]*; the physical reading is* [Conjecture] *[IIB-II].*

2.3. **The Born Rule theorem.**

Theorem 2.3 (Born Rule [Derived])**.** [[1], Thm 10.2] *Let $f : \mathcal{A}_n \to \mathbb{R}_{\geq 0}$ satisfy*

(M) $f(ab) = f(a)f(b)$*;*
(D) $f(x) > 0$ *for* $x \neq 0$*;*
(OA) $f(a + b) = f(a) + f(b)$ *whenever* $\mathrm{Re}(\bar{a}b) = 0$.

Then: for $n \leq 3$ the unique solution is $f = N$; for $n \geq 4$ no solution exists. A fourth axiom (I) $f(\lambda x) = \lambda^2 f(x)$ *is redundant: it follows from* (M),(D),(OA) *for all $n \geq 1$. The three axioms are independent.*

Remark 2.4. The failure of the Born Rule at $n = 4$ does not mean N ceases to exist on $\mathcal{A}_4$. What fails is the relation $f(ab) = f(a)f(b)$ — the multiplicativity that makes $f = N$ a valid probability assignment. N itself remains well-defined on all $\mathcal{A}_n$. This distinction is the algebraic mechanism of IIB-III: N persists (and gravitates, under IIB-II), but its compatibility with $\mathcal{A}_3$-gauge probability structure does not.

Furey's programme [32, 33] identifies $n = 3$ as the level accommodating Standard Model gauge symmetries. IIB-I does not compete with this: Furey marks $n = 3$ as the level accommodating the Standard Model; IIB-I marks $n = 3$ as the *last* level at which quantum-probabilistic coherence holds. Both identify the same algebra as fundamental, from complementary directions.

2.4. Zero-divisor structure of $\mathcal{A}_4$.

Theorem 2.5 (Zero-divisor count [Derived])**.** [[1], Cor 7.4] *For all $n \geq 4$:*

$$|ZD(n)| = 336 \cdot \binom{n-1}{3}_2,$$

where $\binom{n-1}{3}_2$ is the Gaussian binomial coefficient and $336 = 2|PSL(2,7)|$. At $n = 4$: $|ZD(4)| = 336$, forming 84 signed pairs ($ZD^*(\mathcal{A}_4)$ has 84 vertices).

Theorem 2.6 (Spectral decomposition [Derived])**.** [[1], Thm 5.17] *For every zero-divisor unit $a \in \mathcal{A}_n$ ($n \geq 4$):*

$$\mathrm{Spec}(M_a) = \{0^{2^{n-2}},\ 1^{2^{n-1}},\ 2^{2^{n-2}}\},$$

where $M_a = L_a^T L_a$ is the Gram operator. At $n = 4$: $\dim\ker(L_a) = 4$.

Theorem 2.7 ($\{0, 4, 8\}$ Trichotomy [Derived])**.** [[1], Thm 3.6] *For any $a, b \in \mathcal{A}_n$ ($n \geq 4$): $N(ab) \in \{0, 4, 8\}$. The zero-divisor condition is $N(ab) = 0$ iff $\alpha = 0$ AND $\beta = 0$, where α, β are the octonionic split-decomposition invariants.*

Theorem 2.8 (Zero-divisor pair graph [Derived])**.** [[2, 3]] *The graph $ZD^*(\mathcal{A}_4)$ on 84 signed zero-divisor units is 4-regular with 168 edges. Its automorphism group*

$$G_{84} = \mathrm{Aut}(ZD^*(\mathcal{A}_4)), \quad |G_{84}| = 2688,$$

satisfies $G_{84} \cong (\mathbb{Z}/2\mathbb{Z})^4 \cdot GL(3, \mathbb{F}_2)$. The centre $Z(G_{84}) = C_2$, generated by the global sign-flip involution π ($x \mapsto -x$). Under IIB-II this is interpreted as algebraic CPT symmetry [Observation, Paper 35; Speculation, IIB-II].
Critical distinction. $GL(3, \mathbb{F}_2) \cong PSL(2, 7)$, of order 168, acts on the 7 lo_3-classes. It is NOT a subgroup of G_{84} acting on the 84 vertices of $ZD^*(\mathcal{A}_4)$. Conflating these two actions is an error; they operate on different sets.

2.5. Fano geometry and bifrontality.

The Fano plane $PG(2, \mathbb{F}_2)$ has 7 points and 7 lines, each line containing 3 points and each point on 3 lines. Its automorphism group is $GL(3, \mathbb{F}_2) \cong PSL(2, 7)$, of order 168.

The Fano plane appears on both sides of the IIB boundary, governing structurally opposite operations [Derived] [Paper 32, Thm 2.3]:

Level	Face	Algebraic role
$\mathcal{A}_3$ (octonions)	Multiplication face	Incidence: $e_i \cdot e_j \neq 0$ iff $\{i, j, k\}$ is a Fano line
$\mathcal{A}_4$ (sedenions)	Annihilation face	Non-incidence: $a \cdot b = 0$ iff lo_3-classes of a, b are not on a comm

The transition $\mathcal{A}_3 \to \mathcal{A}_4$ is the point where the Fano plane inverts its role: from multiplication table to annihilation map. We call this **Fano bifrontality** [Derived] [Paper 32].

Remark 2.9 ([CONJECTURE] [IIB-II).] The $\mathcal{A}_3$-face organises gauge forces (multiplication structure, level-specific, three forces from three Hurwitz doublings); the $\mathcal{A}_4$-face organises the gravitational sector (annihilation structure, born at the Born Rule failure). The algebraic bifrontality is [DERIVED]; the identification of its two faces with Standard Model forces and gravity is [CONJECTURE] [IIB-II].

2.6. The kernel module theorem.

Theorem 2.10 (Kernel–Fano isomorphism [DERIVED])**.** [[25], Thm 3.1; GAP 4.15.1 + analytic] *For every zero-divisor unit $a \in \mathcal{A}_4$, let $\ell_a \subset PG(2, \mathbb{F}_2)$ be the unique Fano line corresponding to the* lo_3*-class of a, and let*

$$V_{ni}(a) = \mathrm{span}\{e_k : \textit{Fano point of } e_k \notin \ell_a\}.$$

Then:

$$\ker(L_a) \cong V_{ni}(a) \quad \textit{as } \mathrm{Stab}_{G_{84}}(a)\textit{-modules.}$$

$V_{ni}(a)$ has dimension 4 and is irreducible as a Stab-module.

Corollary 2.11 (Triple convergence of 4 [DERIVED])**.** [[1]; [25]; cond. Papers 19, 24, 25] *At $n = 4$, the number 4 has three independent algebraic origins that Theorem 2.10 identifies as the same module:*

Origin	*Statement*	*Status*
Algebraic	$\dim\ker(L_a) = 2^{n-2}\|_{n=4} = 4$	[DERIVED] *[[1], Thm 5.17]*
Fano-geometric	$\|\bar{\ell}_a\| = 7 - 3 = 4$ *non-incident points per line*	[DERIVED] *[[25], Thm 3.1]*
Physical	$FS(4_s) = +1$*: 4 Majorana modes*	[DERIVED] *cond. [Papers 19, 24]*

The convergence occurs only at $n = 4$. At $n \geq 5$: $\dim\ker = 2^{n-2} \geq 8$ while the Fano non-incidence count remains 4.

Corollary 2.12 (Structural origin of $\mathcal{S}_{\mathrm{eff}} = 164$ [DERIVED])**.** [[25], Cor 3.2] *The effective spectral invariant $\mathcal{S}_{\mathrm{eff}} = 164$ appearing in the G_N^{-1} formula (§ 4) decomposes as*

$$\mathcal{S}_{\mathrm{eff}} = |\mathrm{Aut}(PG(2, \mathbb{F}_2))| - \dim\ker(L_a)|_{n=4} = 168 - 4 = 164.$$

Both terms have independent Fano-combinatorial proofs. This upgrades Paper 28 Observation 6.4 from [OBSERVATION] *to* [DERIVED]*.*

2.7. The ZD zeta function and $\mathcal{A}_3$-memory.

Definition 2.13 (ZD zeta function [DERIVED])**.** [[2]]

$$Z(x) = \sum_{n \geq 4} |ZD(n)|\, x^n = \frac{336x^4}{(1-x)(1-2x)(1-4x)(1-8x)}.$$

Equivalently [DERIVED] *[[31]]: $Z(x) = 336x^4 \cdot (x; 2)_4^{-1}$, a q-hypergeometric series with $q = 2$ (the CD doubling base).*

Proposition 2.14 (Weil functional equation [DERIVED])**.** [[2]] $Z(x) = 64x^4\, Z(1/8x)$*, with residue pairing $R_k = -2^{9-6k} R_{3-k}$.*

Theorem 2.15 ($\mathcal{A}_3$-memory [DERIVED])**.** [[30], §4] *The normalised residue vector of $Z(x)$ at its four poles $x_k = 2^{-k}$ is*

$$(s_0, s_1, s_2, s_3) = (1, 7, 14, 8),$$

where $s_0 = \dim \mathbb{R} = 1$, $s_1 = |PG(2, \mathbb{F}_2)| = 7$, $s_2 = \dim G_2 = \dim \mathrm{Aut}(\mathbb{O}) = 14$, $s_3 = \dim \mathbb{O} = 8$. *The generating function of the post-boundary ZD count has its pole structure entirely determined by the dimensions of the four Hurwitz algebras* $\mathcal{A}_0$ *through* $\mathcal{A}_3$.

Note: $s_1 = 7 = m(\mu = -4)$ *(Level-1 spectral multiplicity) and* $s_2 = 14 = m(\mu = -2)$ *(Level-2 multiplicity), connecting the analytic structure of* $Z(x)$ *to the adjacency spectrum of* G_{84} [Observation, Paper 34].

Remark 2.16 (Double structural coincidence [Observation]). The residues satisfy $s_2 = \dim G_2 = \dim \mathrm{Aut}(\mathbb{O}) = 14$ and $s_3 = \dim \mathbb{O} = 8$. Since Furey's programme [32, 33] identifies $\mathrm{SU}(3)_{\mathrm{color}}$ from $\mathbb{O}$ and $\dim \mathrm{SU}(3) = 8 = s_3$, the $\mathcal{A}_3$-memory residues encode both the automorphism group of $\mathcal{A}_3$ (s_2) and the gauge dimension of its SM-gauge subgroup (s_3). Each identification is [Derived] [[30]]; whether the double coincidence is the fingerprint of a Sakharov–Adler loop at level $k = 3$ with G_2-carrier and $\mathrm{SU}(3)$-substructure is [Speculation].

Remark 2.17 ([Open]). The G_2-categorical proof that Weil factorisation necessarily extracts $(1, 7, 14, 8)$ — rather than recovering them a posteriori — remains open [OP 34.1, § 8].

Corollary 2.18 (Alternating sum [Derived])**.** [[29]] $\sum_{k=0}^{3}(-1)^k s_k = 1 - 7 + 14 - 8 = 0$. *This follows from* $Z(x)$ *being a proper rational function (degree-0 numerator, degree-4 denominator) and the residue theorem on* $\mathbb{P}^1(\mathbb{C})$.

Theorem 2.19 ($28 = 4 \times 7$ [Derived])**.** [[31], Thm 3.1] *The number of non-collinear triples in* $PG(2, \mathbb{F}_2)$ *is*

$$28 = \dim \ker(L_a)|_{n=4} \cdot |PG(2, \mathbb{F}_2)| = 4 \times 7.$$

Three independent characterisations converge:

- $28 = C(\dim \mathbb{O}, 2) = C(8, 2) = \dim \mathfrak{so}(8)$ [Derived] *[*[31]*, Prop 3.5];*
- $28 = |$*non-collinear triples of* $PG(2, \mathbb{F}_2)|$ [Derived] *(combinatorial);*
- $28 = \dim \ker \cdot |\mathrm{Fano}|$ [Derived] *above.*

The Fano bifrontality partition gives $7 + 28 = 35$*: the* $C(7, 3) = 35$ *triples of 7 Fano points split as 7 collinear + 28 non-collinear.*

3. The Black Circle

The Black Circle is the spherically symmetric spacetime geometry conjectured, under IIB-I/II, to be sourced by the $\mathcal{A}_4$ zero-divisor field content.

3.1. The metric ansatz [Conjecture] [IIB-I/II].

Conjecture 3.1 (Black Circle geometry [Conjecture] [IIB-I/II)**.** *] Under IIB-I and IIB-II, the sedenion zero-divisor field content sources a spherically symmetric spacetime with metric function:*

$$f_{\mathrm{out}}(r) = 1 - r_s/r, \quad r > r_s \quad \textit{(Schwarzschild exterior)},$$
$$f_{\mathrm{in}}(r) = r^2/r_s^2 - 1, \quad r < r_s \quad \textit{(de Sitter interior)},$$

joined at $r_s = 2G_N M$*, where* M *is the Black Circle mass. The de Sitter radius equals the Schwarzschild radius. No free parameters beyond* r_s.

This ansatz is not derived from first principles; it is the simplest geometry consistent with IIB-I/II. No junction condition derivation from $\mathcal{A}_4$ *field equations currently exists [Open, OP* 8.1*].*

The de Sitter interior has Hubble radius $H^{-1} = r_s$ and positive effective cosmological constant $\Lambda_{\text{eff}} = 3/r_s^2$. Under IIB-II, this is sourced by the condensate C_0 of S_{84}: the stress-energy is [DERIVED] [[9], Thm 3.2]:

$$T_{\mu\nu}|_{C_0} = -g_{\mu\nu} \cdot \frac{3}{r_s^2}.$$

3.2. **The S_{84} field theory.**

Definition 3.2 (S_{84} action [DERIVED])**.** [[8], Cor 3.3] *The sedenion scalar field theory S_{84} is the quartic action on the 84 active zero-divisor modes $\{\varphi_A\}_{A\in ZD^*(\mathcal{A}_4)}$:*

$$S_{84} = \int d^4x\, \sqrt{g}\Big[\tfrac{1}{2}\sum_A (\partial\varphi_A)^2 - 4\lambda \sum_{(A,B)\in ZD} \varphi_A^2\varphi_B^2\Big],$$

with quartic coupling λ. The coupling λ is the unique *quartic coupling consistent with the Fano-$PSL(2,7)$ symmetry of $ZD^*(\mathcal{A}_4)$ and the CSS code structure of $RM(1,n)^\perp$ (Papers 5–8* [4, 5, 6, 7]*)* [DERIVED].

The field content is 84 real scalars plus 4 Majorana fermions from the 4_s sector (Papers 19, 24, 25; see § 6). The one-loop β-function is [DERIVED] [[15]]:

$$\beta(\lambda) = \frac{63\lambda^2}{\pi^2} > 0,$$

where 63 is computed from $\mathrm{Tr}(A^2) = 336$ (4-regularity of $ZD^*(\mathcal{A}_4)$). The Landau pole is at $\mu_L = \mu_0 \exp(\pi^2/63\lambda)$, exponentially large for $\lambda \ll 1$.

3.3. **Mass spectrum.**

Theorem 3.3 (S_{84} mass spectrum [DERIVED])**.** [[9], Thm 4.5; Python + GAP] *The complete mass spectrum of S_{84} around the condensate C_0, with $PSL(2,7)$-module structure, is:*

Level	μ_i	$PSL(2,7)$*-irreps*	$2+\mu_i$	*Mult*	m_i^2 *(*$\xi=0$*)*
L_1	-4	irr_7	-2	*7*	$-32\lambda N(C_0)$
L_2	-2	$\mathrm{irr}_6 \oplus \mathrm{irr}_8$	$\mathbf{0}$	*14*	$\mathbf{0}$
L_3	0	$3\cdot\mathrm{irr}_6 \oplus 3\cdot\mathrm{irr}_8$	2	*42*	$+32\lambda N(C_0)$
L_4	$+2$	$2\cdot\mathrm{irr}_7$	4	*14*	$+64\lambda N(C_0)$
L_5	$+4$	$\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*}$	6	*7*	$+96\lambda N(C_0)$

where $m_i^2 = 16\lambda N(C_0)(2+\mu_i)$ and $\xi = 0$ (minimal coupling). Three structural features:

(i) Level-2 masslessness under triple algebraic protection [DERIVED] *[Papers 11, 21, 22-B]: the factor $(2+\mu_2) = 0$ follows from the $PSL(2,7)$-eigenvalue $\mu_2 = -2$ being fixed by the ZD graph adjacency spectrum. Three independent mechanisms — PSL representation theory, G_2-Goldstone exclusion* [DERIVED-NEGATIVE] *[GAP, Paper 21], and SM-Higgs exclusion* [DERIVED-NEGATIVE] *[Paper 21] — protect Level-2 masslessness.*

(ii) Level-4 and Level-5 are not mirror images of Levels 1 and 2 [DERIVED] *[*[9]*, Thm 4.5]: Level-4 carries $2\cdot\mathrm{irr}_7$ (two copies, dimension 14), while Level-1 carries irr_7 (dimension 7). Level-5 carries $\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*}$ (dimension 7). The SU(3)-content of Levels 1 and 5 is the same under (A2), $\mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$ [Derived, Paper 21, SU(3)-mirror symmetry].*

(iii) Level-3 = 3×Level-2 as $PSL(2,7)$-modules [DERIVED] *[Papers 11, 21, 22-B]: $3\cdot\mathrm{irr}_6 \oplus 3\cdot\mathrm{irr}_8 = 3\cdot(\mathrm{irr}_6 \oplus \mathrm{irr}_8)$, with the factor 3 fixed by the characteristic polynomial of*

the G_{84}-adjacency matrix [Derived] *[*[18]*]. The factor 3 is the Fano incidence number: every Fano point lies on exactly 3 lines.*

3.4. The condensate.

Theorem 3.4 (Condensate value [Derived])**.** [[9], Thm 5.4; conditional on Conjecture 3.1] *The self-consistent condensate satisfies $C_0^4 = 3/(512\pi G_N \lambda r_s^2)$, hence $N(C_0) = C_0^2 = \sqrt{3/(512\pi G_N\lambda)} \cdot r_s^{-1}$. The BF stability bound* [Derived] *[*[10]*]:*

$$\lambda G_N M^2 < 2.14 \times 10^{-3}.$$

One-loop stability [Conjecture, IIB-II; Paper 20]*: C_0 is a saddle point of V_{eff} with 7 tachyonic directions (Level 1, irr_7), 14 massless (Level 2), and 63 stable (Levels 3–5). The gravitational backreaction under IIB-II dominates the direct Coleman–Weinberg correction by a factor of 54* [Derived] *[*[15]*]. The first quantitative one-loop prediction of IIB-II is* [Conjecture, IIB-II; [15], Thm 5.8]*: $\langle N\rangle_{1\text{-loop}}/C_0^2 \approx -8.41\lambda < 0$.*

3.5. Quasi-normal modes.

The QNM spectrum through Papers 13–17 has been revised in Papers 30–31 as follows.

Growing modes ($l = 0, 1$) [Derived] [[11, 12]]:

$$\kappa_{\text{grow}}(l=0) = 0.73472367, \quad \tau_{\text{grow}}(l=0) = 1.361\, r_s/c, \quad \tau_{\text{grow}}(l=1) = 1.357\, r_s/c.$$

Quasi-universality: τ_{grow} agrees to 0.33% at $l = 0, 1$.

Damped mode ($l = 2$, corrected) [Derived] [[26], 50 decimal places]:

$$\omega^*_{\text{damp}} = 1.925250870913304 75\ldots - 0.41596770795216527\ldots\, i, \quad |\text{residual}| = 6.68\times 10^{-51}.$$

Errata. The value $1.9252222467 - 0.415997679i$ from Paper 17 §5.3 is incorrect from the fifth significant digit (incorrect starting guess for Newton's method); it is superseded.

Two growing modes at $l = 2$ [Derived] [[26]]:

$$\omega_1 = 5.9730891600 + 2.8427523204i, \quad \omega_2 = 4.7367434016 + 3.8456935744i.$$

Quasi-universality fails at $l = 2$ [Derived-Negative] [[26]]: $\tau_{\text{grow}}(l = 2) \approx 0.26$–$0.35\, r_s/c$, a factor ~ 4 below $l = 0, 1$.

Conjecture 3.5 (QNM ratio [Conjecture] [IIB-I/II)**.** *[*[[26], Conj 6.3] $|\omega^*_{\text{damp}}|/|\omega_{\text{Sch}}| \approx 2.00 \pm 0.003$*. Proof requires the closed form of A_{part}; no Gamma-function ratio exists at $l \geq 1$* [Derived-Negative] *[*[27]*, Thm 4.2].*

4. Newton's Constant from Algebra

This section is the quantitative core of the paper. Every formula carries the conditionality on the Sakharov–Adler mechanism explicitly; this conditionality is never suppressed.

4.1. The causal chain.

Step	Source	Result	Status
Spectrum	[9], Thm 4.5	$m_i^2 = 16\lambda N(C_0)(2 + \mu_i)$	[Derived]
Bosonic sum	[22], Prop 3.3	$\mathcal{S}_{\text{bos}} = 168 = \lvert\mathrm{GL}(3, \mathbb{F}_2)\rvert$	[Derived]
Main formula	[22], Thm 4.2	G_N^{-1} in closed form	[Derived], cond. SA
$\mathcal{S}_{\text{eff}} = 164$	[25], Cor 3.2	$\ker(L_a) \cong V_{ni}(a) \Rightarrow \mathcal{S}_{\text{eff}} = 164$	[Derived]
Branch selection	[23], Thm 2.3	$G_N^{(1)}$ unique physical branch	[Derived], cond. [10]
Two-loop correction	[23], Thm 4.3	$R \in [0.87, 0.95]$	[Derived], cond. SA
Fermionic backreaction	[24], Thm 5.1	Complete $\delta G_N^{-1}\rvert_{\text{ferm}}$	[Derived], cond. Papers 19,

4.2. **The Sakharov–Adler mechanism.** The Sakharov–Adler induced gravity mechanism [38, 39] states that integrating out massive matter fields in a curved spacetime background generates an Einstein–Hilbert term in the effective action. In the UV cutoff regime, the Frolov–Fursaev supertrace formula gives:

$$G_N^{-1} = \frac{1}{2\pi} \cdot \mathrm{str}\big[k_1 m^2 \ln(\Lambda^2/m^2)\big] + O(\Lambda^0),$$

where $\mathrm{str} = \mathrm{Tr}_{\mathrm{bos}} - \mathrm{Tr}_{\mathrm{ferm}}$ is the supertrace, $k_1 = 1/6$ for minimally coupled scalars ($\xi = 0$), and Λ is the UV cutoff. For S_{84} with 84 bosons and 4 Majorana fermions: $\mathrm{str}(1) = 84 - 2 = 82 \neq 0$, placing us in the *cutoff regime*: $G_N \sim \Lambda^2/m^2 \times (\text{finite})$. We identify $\Lambda = M_{\mathrm{Pl}}$.

The norm-gravity mechanism — deriving G_N from the quadratic norm N via the Sakharov–Adler mechanism applied to the ZD scalar field theory S_{84} — is, on current evidence, original. The conditionality on Sakharov–Adler makes the comparison with any alternative programme most meaningful at the level of what is actually computed [Derived] rather than what is structurally proposed.

4.3. **The bosonic spectral invariant $\mathcal{S} = 168$.**

Proposition 4.1 (Bosonic spectral sum [Derived])**.** [[22], Prop 3.3]

$$\mathcal{S}_{\mathrm{bos}} = \sum_i d_i(2 + \mu_i) = 2|ZD^*(\mathcal{A}_4)| + \mathrm{Tr}(A) = 2 \times 84 + 0 = 168 = |\mathrm{GL}(3, \mathbb{F}_2)|.$$

Level-2 contributes exactly zero: $(2 + \mu_2) = 0$.

Lemma 4.2 (Level-2 gravitational decoupling [Derived])**.** [[22], Lem 3.1] *The 14 massless Level-2 modes ($\mu_2 = -2$, $\mathrm{irr}_6 \oplus \mathrm{irr}_8$) contribute exactly zero to G_N^{-1}: the Frolov–Fursaev weight $m_i^2 \ln(\Lambda^2/m_i^2)$ vanishes when $m_i^2 = 0$. The factor $(2 + \mu_2) = 0$ that confers Level-2 masslessness simultaneously decouples Level-2 from gravity at one loop.* One algebraic mechanism, two physical consequences.

4.4. **The main formula.**

Theorem 4.3 (One-loop induced G_N [Derived] (cond. Sakharov–Adler))**.** [[22], Thm 4.2] *Applying the Frolov–Fursaev supertrace to the S_{84} mass spectrum (84 bosons, 4 Majorana fermions with $m_{4_s}^2 = 32\lambda N(C_0)$):*

$$\boxed{G_N^{-1} = \frac{16\lambda\, N(C_0)}{12\pi}\left[164 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 220 \ln 2 - 42 \ln 3\right].}$$

[Derived, conditional on Sakharov–Adler.]

(i) $G_N^{-1} \propto N(C_0)$ [Derived] *[[22], Cor 4.4]: Newton's constant is determined by the Cayley–Dickson norm of the condensate. IIB-II is supported at one loop in the cutoff regime; OP 20.3 closed at leading-log order.*

(ii) Level-2 contributes zero (Lemma 4.2): the 14 massless modes are gravitationally neutral.

(iii) The coefficient 164 is Fano-structural. See § 4.5.

4.5. **Algebraic origin of $\mathcal{S}_{\mathrm{eff}} = 164$.** Paper 28 Observation 6.4 identified $\mathcal{S}_{\mathrm{eff}} = |GL(3, \mathbb{F}_2)| - \dim\ker(L_a)|_{n=4} = 168 - 4 = 164$ as an [Observation]. Paper 29 closes the gap via Corollary 2.12: both terms are now independently [Derived], making the coefficient a Fano-structural consequence.

Term	Value	Algebraic origin	Status
168	$\lvert\mathrm{Aut}(PG(2,\mathbb{F}_2))\rvert$	Bipartite spectral sum $\mathcal{S}_{\mathrm{bos}}$	[DERIVED]
4	$\dim\ker(L_a)\vert_{n=4}$	Fano non-incidence count (Thm 2.10)	[DERIVED]
164	$168-4=\mathcal{S}_{\mathrm{eff}}$	Fano bifrontality in the Sakharov–Adler formula	[DERIVED]

The Sakharov–Adler formula encodes Fano non-incidence geometry directly. The coefficient of G_N^{-1} counts the complement of the Fano incidence structure of $\mathcal{A}_3$ as seen from $\mathcal{A}_4$.

4.6. Fermionic correction.

Theorem 4.4 (Fermionic backreaction on G_N^{-1} [DERIVED])**.** [[24], Thm 5.1; cond. Papers 19, 26] *The complete fermionic correction from the 4 Majorana modes is:*

$$\delta G_N^{-1}|_{\mathrm{ferm}} = -\frac{16\lambda N(C_0)}{3\pi}\ln\frac{\Lambda^2}{32\lambda N(C_0)}.$$

Bosonic $(168\cdot\log)$ + *fermionic* $(-4\cdot\log)$ = *Theorem 4.3* $(164\cdot\log)$ [DERIVED] *[[24], Cor 5.2].*

Errata note. Paper 26 eq. (5.3) v1 contains a typographical error: $-(7/6)\sqrt{3\lambda/2\pi G_N}\cdot r_s^{-1}$ *should read* $-(7/12)\sqrt{3\lambda/2\pi G_N}\cdot r_s^{-1}$*. Corrected in Paper 27, Thm 1.1* [DERIVED]*; the ratio* $\mathrm{Im}(W)|_{\mathrm{tach}}/\mathrm{Im}(G_N^{-1})|_{L_1} = 744.22/7 \approx 106.32$ *is a pure number independent of all parameters.*

4.7. The self-consistency equation: Planck scale as output.

Theorem 4.5 (Self-consistency equation [DERIVED])**.** [[22], Thm 4.6; cond. SA + [9]] *Substituting* $N(C_0)=\alpha G_N^{-3/2}$ *into Theorem 4.3:*

$$g(G_N) := G_N^{1/2} - \frac{16\lambda\alpha}{12\pi}(P + 246\ln G_N) = 0,$$

with $P = 164\ln(\Lambda^2/16\lambda\alpha) - 220\ln 2 - 42\ln 3$*. For physical parameters: exactly two positive roots* $G_N^{(1)} < G_N^{*} < G_N^{(2)}$*. The Planck mass* $M_{\mathrm{Pl}}^2 = G_N^{-1}$ *is an* output *of the ZD algebraic structure — not an input.*

4.8. Branch selection.

Theorem 4.6 (Branch selection [DERIVED])**.** [[23], Thm 2.3; cond. [10]] *At* $(M,\Lambda) = (0.1, 1)$ *in Planck units, for all* $\lambda \in (0, \lambda_{\mathrm{IR}})$*:*

- *(i)* $G_N^{(2)}$ *violates the cavity bound* $\lambda G_N M^2 < 2.14\times 10^{-3}$ *by at least* $234\times$*. The large branch is universally excluded.*
- *(ii)* $G_N^{(1)}$ *is the unique physical branch, satisfying the cavity bound for* $\lambda < \lambda_c^{(\mathrm{SC})} = 0.07531$ *at* $(M,\Lambda) = (0.1,1)$*; for* $\lambda > \lambda_c^{(\mathrm{SC})}$ *no physical Black Circle solution exists at these parameter values.*
- *(iii) The separation* $G_N^{(2)}/G_N^{(1)} \in [385, 2380]$ *makes branch identification unambiguous.*

Conjecture 4.7 (Conjecture A [CONJECTURE] [IIB-II + Conjecture A])**.** *] Under the identification* $M = m_{4_s} = \sqrt{32\lambda}\,C_0$ *(not derived from IIB-II; stated explicitly every time it is invoked): the system reduces to a unique closed-form solution* $G_N^{(A)}(\lambda) = \exp((1/K(\lambda) - P_0(\lambda))/164)$*, cavity-physical window* $\lambda < \lambda_c^{(A)} = 0.01197$*;* $G_N^{(A)} \in [2.10, 2.39]$ *(Planck units) across* $\lambda \in [0.05, 0.46]$ [CONJECTURE] *[IIB-II + Conjecture A; [23], Thm 5.1].*

4.9. **Two-loop correction.**

Theorem 4.8 (Complete two-loop β-function [Derived]). [[23], Thm 3.6; see also [41, 42]]

$$\beta^{(2\text{-}loop)}(\lambda) = \frac{63\lambda^2 + 4\lambda - 16}{\pi^2} + \frac{-189\lambda^3 - 12\lambda + 48}{4\pi^4}.$$

Two-loop IR zero: $\lambda^{(2)}_{\rm IR} = 0.46373$ *(-2.0% shift from the one-loop value 0.47321).*

Theorem 4.9 (Two-loop G_N correction [Derived]). [[23], Thm 4.3; cond. SA + [17]] *With two-loop running coupling $\lambda^{(2)}(\Lambda)$ inserted into the Frolov–Fursaev formula:*

$$\frac{G_N^{-1,(2\text{-}loop)}}{G_N^{-1,(1\text{-}loop)}} = R \in [0.87,\, 0.95] \quad \text{at } T = 0.1.$$

$R < 1$*: two-loop running increases G_N by 5–15%.*

4.10. **Route B: geometric derivation.**

The ZD set of $\mathbb{S}$ carries a natural Riemannian structure [40]: $ZD(\mathbb{S}) \cong V_2(\mathbb{R}^7) = G_2/SU(2)$, an 11-dimensional Stiefel manifold. The sigma-model on $V_2(\mathbb{R}^7)$ induces an Einstein–Hilbert term from the coset curvature.

Proposition 4.10 (Route B induced G_N [Derived]). [[22], §6; cond. [40] normalisation] *At $r = 2/3$ in the G_2-invariant family on $V_2(\mathbb{R}^7)$:* $\mathrm{scal}(r = 2/3) = 45 > 0$*, giving via the heat-kernel a_1 coefficient:* $G_N^{-1}|_{\text{Route B}} \approx 0.0339$ *($M_{\rm Pl} = 1$ units). Both Route A and Route B give* $G_N^{-1} \propto N(C_0)$ [Derived] *(cond.). The Einstein locus $r = 5/9$ gives* $G_N^{-1}|_{r=5/9} \approx 0.0315$ *(7.6% reduction) and would imply a bare cosmological constant* $\Lambda_{\rm bare} = 25/6\, M_{\rm Pl}^2$*; whether S_{84} dynamics selects $r = 5/9$ is open [OP 26.1, § 8].*

4.11. **IIB-II-strong [Speculation].**

The Sakharov–Adler computation of §§ 4.2–4.9 integrates out S_{84} fields and produces an Einstein–Hilbert term. This is the correct epistemic framing for Papers 26–28. It is not the most radical reading of the result.

Speculation 4.11 (IIB-II-strong [Speculation]). N is not derived from matter; N is primary. Zero-divisors of $\mathcal{A}_4$ are not objects that happen to make N fail to compose; they *are* the failure of $N(ab) = N(a)N(b)$ at $n = 4$. Matter — condensate C_0, ZD^* graph, S_{84} spectrum — emerges at $n = 4$ from the first level where $N(ab) \neq N(a)N(b)$. *Gravity does not emerge from matter. Matter emerges from the imperfection of gravity.*

Structural supports (consistency only; none proves IIB-II-strong):

- AMT: $\mathbb{E}_v[N(av)] = N(a)$ for all n — N does not notice the Born Rule failure [Derived] [[1]].
- Frobenius: $\|L_a\|_F^2 = 2^n N(a)$ — operator norm universally N-determined [Derived] [[1]].
- Backreaction 54:1 [Derived] [[15]]: gravitational response dominates the direct Coleman–Weinberg correction.
- $S_{BH} \propto \|L_{C_0}\|_F^2 \cdot M^2$ [Conjecture] [IIB-II, [28]]: entropy encodes the N-norm of the condensate.

IIB-II-strong does not change the epistemic status of Papers 26–28; they remain [Derived] conditional on Sakharov–Adler. Both framings agree on the formula; they disagree on what the formula means.

5. Black Circle Thermodynamics and Holography

5.1. **Partition of thermodynamic observables.**

Theorem 5.1 (Thermodynamic partition [DERIVED])**.** [[28], Thm 7.1; synthesis of Papers 26, 27, 28] *The thermodynamic observables of the Black Circle partition into three algebraic classes:*

Class I — Kinematic (G_N-independent):

- $T_H \cdot r_s = 1/(4\pi)$ [DERIVED] *[*[24]*, Prop 2.1]*
- *Free energy* $F = M - T_H S_{BH} = M/2$ [DERIVED] *[*[24]*, Cor 2.5]*
- $S_{GHY} = -M$ *for all* $G_N > 0$ [DERIVED] *[*[24]*, Prop 3.1]*
- $\delta S_{GHY} = 0$ *identically* [DERIVED-NEGATIVE] *[*[24]*, Cor 3.2]*

Class II — Norm-dependent (through $G_N \propto N(C_0)$):

- $S_{BH} = 4\pi G_N M^2$ [DERIVED] *(Bekenstein–Hawking* [37]*)*
- $S_{BH}/S_{BH}^{(\mathrm{Pl})} = G_N^{(1)}$ [DERIVED] *[*[24]*, Cor 2.3]*

Class III — Spectral (full $ZD^*(\mathcal{A}_4)$ data):

- *Two-loop IR zero* $\lambda_{\mathrm{IR}}^{(2)} = 0.46373$ [DERIVED] *cond. [*[23]*]*
- *Two-loop correction* $R \in [0.87, 0.95]$ [DERIVED] *cond. [*[23]*]*

The Class I universality $T_H \cdot r_s = 1/(4\pi)$ holds regardless of G_N: the G_N in $r_s = 2G_N M$ and in $T_H = 1/(8\pi G_N M)$ cancel exactly. Under IIB-II, horizon geometry is set by N alone.

Corollary 5.2 ([DERIVED])**.** *Level-2 contributes zero to Class II and all Class III spectral invariants. Level-2 belongs entirely to Class I: it is the* kinematic sector *of the spectrum, with $(2 + \mu_2) = 0$ as the single algebraic mechanism underlying its kinematic status.*

5.2. **Entropy, temperature, and the first law.** With $G_N^{(1)}(M, \lambda)$ from the SC equation (§ 4.7), $S_{BH} = 4\pi G_N^{(1)} M^2$ and $T_H = 1/(8\pi G_N^{(1)} M)$. First law verification [DERIVED]: $T_H\, dS_{BH} = dM$. Numerical values at $M = 0.1$, $\Lambda = 1$ (Planck units):

λ	$G_N^{(1)}$	S_{BH}	T_H	$T_H S_{BH}$
0.05	2.5993	0.3266	0.1531	0.0500
0.10	3.0371	0.3817	0.1310	0.0500
0.20	3.6190	0.4548	0.1099	0.0500
0.30	4.0365	0.5072	0.09852	0.0500

$T_H S_{BH} = M/2 = 0.0500$ is exact at all λ, confirming the first law to six significant figures.

5.3. **The GHY term: a structural negative.**

Proposition 5.3 (GHY G_N-independence [DERIVED])**.** [[24], Prop 3.1] $S_{GHY}(G_N) = -r_s/(2G_N) = -M$ *for all* $G_N > 0$*.*

Corollary 5.4 ([DERIVED-NEGATIVE])**.** [[24], Cor 3.2] $\delta S_{GHY} := S_{GHY}(G_N + \delta G_N) - S_{GHY}(G_N) = 0$ *identically, at all orders in δG_N, for fixed M. The correct computation of the genuine quantum GHY term requires the boundary Seeley–DeWitt coefficient for S_{84} in the Black Circle background — a distinct heat-kernel problem [Open, OP 8.8, § 8].*

5.4. **Holographic formula.**

Conjecture 5.5 (Holographic entropy formula [CONJECTURE] [IIB-II)**.** *[* [[28], Conj 8.2; first explicit statement synthesising Papers 1, 26, 28]

$$S_{BH} = \frac{\pi}{4} \cdot \|L_{C_0}\|_F^2 \cdot M^2,$$

where $\|L_{C_0}\|_F^2 = 16N(C_0)$ *by Theorem 2.1 at* $n = 4$.
Chain: $S_{BH} = 4\pi G_N M^2$ [Derived] *(Bekenstein–Hawking [37]);* $G_N \propto N(C_0)$ [Derived] *cond. SA [[22], Cor 4.4];* $\|L_{C_0}\|_F^2 = 16N(C_0)$ [Derived] *[[1], Thm 5.6 at* $n = 4$*]. Three independently derived results; their combination is* [Conjecture] *[IIB-II].*
Physical reading [Conjecture] *[IIB-II]: The Frobenius norm* $\|L_{C_0}\|_F^2$ *assigns zero weight to every direction in* $\ker(L_{C_0})$*: the 4 non-incident Fano directions are invisible to* $\|L_{C_0}\|_F^2$*. The horizon entropy is therefore a trace over the 12 active (on-line Fano) directions on which* L_{C_0} *acts non-trivially — the complement of* $\ker(L_{C_0})$.

5.5. **Majorana threefold isolation.**

Observation 5.6 (Majorana threefold isolation [Observation + Conjecture, IIB-II]). *[[28], Obs 3.3]* The 4 Majorana modes of $\mathrm{irr}_8 = 4_s$ are simultaneously isolated in three independent senses, all traceable to $\ker(L_a) \cong V_{ni}(a)$:

(1) *Algebraic isolation* [Derived] cond. [Papers 19, 24, 25, 29]: $\ker(L_a) \cong V_{ni}(a)$ as Stab-modules.
(2) *Gravitational isolation* [Derived] [[22], Lem 3.1]: $(2 + \mu_2) = 0$ decouples irr_8 from the Sakharov–Adler sum defining G_N^{-1}.
(3) *Entropic isolation* [Conjecture] [IIB-II; Conj 5.5]: $\|L_{C_0}\|_F^2$ assigns zero weight to every direction in $\ker(L_{C_0}) \cong V_{ni}(C_0)$; the 4 Majorana directions contribute zero entropy weight.

Three isolations, one geometric cause: Fano non-incidence.

5.6. **The Black Circle as logical mirror of $\mathcal{A}_3$.**

Conjecture 5.7 (Logical mirror [Conjecture] [IIB-I/II)**.** */* [[28], Conj 9.1] *Under IIB-I and IIB-II, the Black Circle is the* logical mirror *of* $\mathcal{A}_3$*: the algebraic boundary structure that provides* $\mathcal{A}_3$ *with its gravitational dynamics. Without* G_N *— exclusively an* $\mathcal{A}_4$ *object generated by the ZD structure via the annihilation face of the Fano bifrontality — the matter of* $\mathcal{A}_3$ *has no gravitational field equations. The Black Circle is not an object* inside $\mathcal{A}_4$*: it is the condition for* $\mathcal{A}_3$ *to be physically complete.*

6. The Matter Sector

6.1. **Spinor module and Majorana condition.**

Proposition 6.1 (irr_8 is the $Cl(0,7)$-spinor module [Derived])**.** [[20], Prop 3.1] *The Level-2 off-line sector* $\mathrm{irr}_8 = 4_s$ *is the* $Cl(0,7)$*-spinor module — the unique real 8-dimensional irreducible of* $Cl(0,7)$*, restricted to* $\mathrm{Stab}_{G_{84}}(a) \cong GL(2,3)$ *as the 4-dimensional irrep* 4_s.

Theorem 6.2 (Frobenius–Schur indicator [Derived])**.** [[13], Thm 5.1; GAP 4.15.1]

$$FS(4_s) = \frac{1}{|GL(2,3)|} \sum_{g \in GL(2,3)} \chi_{4_s}(g^2) = \frac{48}{48} = +1.$$

Three consequences: (i) $4_s \cong \overline{4_s}$ *(real structure); (ii) charge conjugation* $C : 4_s \to 4_s$ *with* $C^2 = +1$*; (iii) the Majorana condition* $\psi = C\psi$ *is representation-theoretically consistent.*

Theorem 6.3 (Majorana condition [Derived] (cond. IIB-I/II + $FS(4_s) = +1$))**.** [[14], Thm 7.1] *The four* 4_s *modes are Majorana. The derivation proceeds via:*

- *OS2 fails structurally* [Derived-Negative] *[[14]]: no Euclidean half-space exists in the compact de Sitter interior; the standard Wick rotation is unavailable.*

- *Bunch-Davies vacuum identified with global dS_4 regularity* [DERIVED] *[*[14]*, Thm 5.3].*

  Fermionic sector: the full action $S_{84}^{(f)}$ with Dirac operator [DERIVED] *[*[19]*] and Yukawa coupling* $\pm 2iC_0$ [DERIVED] *[*[21]*] are developed in Papers 23–25.*
- *Bros–Epstein–Moschella PCT theorem applies to the Bunch-Davies vacuum on dS_4* [DERIVED] *[*[14]*, Thm 5.5].*
- *The four 4_s modes are Majorana* [DERIVED] *(cond.) from above.*

Errata note [[20]*, Rem 3.2]. Paper 24 Remark 3.2 wrote* $\ker(L_a) = V_{\bar\ell} \oplus V_{\bar\ell}^{(8)}$ *which would imply* $\dim\ker = 8$. *The GAP computation of Paper 29 shows* $\dim\ker = 4$*: the kernel is a 4-dimensional* diagonal *subspace, not a direct sum of two 4-dimensional spaces. Paper 24 Rem 3.2 describes the ambient space, not the kernel. No result of Papers 18–28 that uses* $\dim\ker = 4$ *is affected.*

6.2. **The 4_s modes as dark-matter candidates.**

Conjecture 6.4 (4_s dark-matter candidates [CONJECTURE] [IIB-III, under (A1) and (A2)). *] The 4 Majorana spinors of 4_s are simultaneously:*

(i) Gauge-invisible: they live in $\mathcal{A}_4 \setminus \mathcal{A}_3$; no SM quantum numbers under (A2) [[DERIVED] *cond.].*
(ii) Gravitationally active: they source C_0 and contribute to $T_{\mu\nu}$ via IIB-II, and provide the Majorana correction to G_N^{-1} (§ 4.6).
(iii) Majorana-compatible: $FS(4_s) = +1$, $C^2 = +1$ (Theorems 6.2, 6.3).
(iv) Below the QNM threshold: $m_{4_s}^2 = 32\lambda N(C_0) < 0.128\, r_s^{-2}$ under the cavity bound, below the Type II threshold $m_{\rm thr}^2 = 0.4605600022$ [DERIVED] *[*[12]*].*

6.3. **The generation problem.** The Standard Model has 3 generations; 4_s produces 4 Majorana modes. Three resolutions are structurally available [Open, OP 8.7, § 8]:

(a) *Preferred hypothesis*: one mode is a sterile right-handed neutrino, giving 3 observed SM generations plus one unobserved. Falsifiable via KATRIN, JUNO, and neutrinoless double beta decay.
(b) A symmetry-breaking mechanism projects $4_s \to 3$ modes. No such mechanism has been identified.
(c) The 4-mode count is fundamental and the SM three-generation count requires re-examination.

6.4. **Quadruple protection of Level-2.**

Theorem 6.5 (Quadruple protection [DERIVED]). [[28], Thm 3.1; synthesis of Papers 11, 19, 21, 26] *The Level-2 eigenspace ($\mu_2 = -2$, $\mathrm{irr}_6 \oplus \mathrm{irr}_8$, 14 modes) is simultaneously:*

(i) Massless: $m_2^2 = 16\lambda N(C_0)(2 + \mu_2) = 0$ [DERIVED] *[*[9]*, Thm 4.5].*
(ii) Gravitationally decoupled: contribution to G_N^{-1} exactly zero [DERIVED] *[*[22]*, Lem 3.1].*
(iii) Gauge-invisible (under (A2)): no SM quantum numbers [DERIVED] *cond. [*[16]*, §2].*
(iv) Contains 4 Majorana modes: irr_8 sub-sector has $FS(4_s) = +1$ [DERIVED] *cond. [Papers 18, 19].*

The single algebraic mechanism underlying all four properties is $(2 + \mu_2) = 0$.

Three mechanisms for giving Level-2 a positive mass have been rigorously excluded [DERIVED-NEGATIVE] *[GAP 4.15.1,* [16]*]: G_2-Goldstone* $(\mathrm{adj}(G_2)|_{PSL(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*} \oplus \mathrm{irr}_8 \neq \mathrm{irr}_6 \oplus \mathrm{irr}_8)$*; SM-Higgs under (A2); ξ-splitting at $PSL(2,7)$ and S_4 levels. The curvature coupling $\xi \neq 0$ gives a uniform mass shift $12\xi/r_s^2$ for all 14 Level-2 modes without symmetry breaking [Open, OP 8.6].*

6.5. Fano partition of Level-2.

Proposition 6.6 (Fano partition [Derived] (module))**.** [[20], Prop 4.1 and cond. physical reading] *Level-2* $= \mathrm{irr}_6 \oplus \mathrm{irr}_8$ *where:*

- *On-line sector (*irr_6*, 6 bosonic modes): lies outside* $\ker(L_a)$ *for any* $a \in ZD^*(\mathcal{A}_4)$*.* $\mathrm{irr}_6 \cong \mathrm{Sym}^2(\mathrm{irr}_3)$ [Derived] *[*[20]*]. SU(3) content under (A2):* **6** *(colour sextet).*
- *Off-line sector (*irr_8*, 4 Majorana modes): no equivariant embedding in* V_{ZD} [Derived-Negative] *[*[20]*]; lies in* $\ker(L_a)$ *for every* $a \in ZD^*(\mathcal{A}_4)$*. SU(3) content under (A2):* **8** *(colour octet).*

Proposition 6.7 (S_4-resolution [Derived])**.** [GAP 4.15.1; [16]]

$$\text{Level-2}|_{S_4} = \mathrm{irr}_1 \oplus 2 \cdot \mathrm{irr}_2 \oplus 2 \cdot \mathrm{irr}_3 \oplus \mathrm{irr}_{3'}.$$

The irr_1 *singlet is the unique fully* S_4*-symmetric Level-2 mode.*

Proposition 6.8 (SU(3)-mirror symmetry [Derived])**.** [[16]] *Level-1*$|_{\mathrm{SU(3)}}$ = *Level-5*$|_{\mathrm{SU(3)}}$ $= \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$ *under (A2).*

6.6. The irr_1 singlet as CCC candidate.

Speculation 6.9 (irr_1 as CCC messenger [Speculation] [IIB-IV).] *[*[28]*, Cor 5.3]* The irr_1 singlet of Level-2 satisfies three necessary conditions for crossing the CCC conformal boundary:

(i) Massless: $m_1^2 = 0$ (Level-2 protection) [Derived].
(ii) At the $SO(1,4)$ complementary series boundary [Derived] [[9], Cor 5.9].
(iii) Fully S_4-symmetric [Derived] [GAP, [16]].

Sufficiency requires: (a) conformal weight analysis; (b) Bunch-Davies BC consistency; (c) $\mathrm{irr}_{1'}$ exclusion as CCC parity constraint [Open, OP 8.5, § 8].

6.7. One-loop effective potential and backreaction.

Theorem 6.10 (Saddle-point structure [Derived])**.** [[15], Thm 3.4] $V_{\text{classical}}(C) = 64\lambda C^4$*;* C_0 *is not a stationary point of* $V_{\text{classical}}$ *but satisfies the Einstein equation* [Derived] *[*[15]*, Prop 3.2]. One-loop effective potential:* $V_{1\text{-}loop}(C_0) \approx -233.4\lambda^2 C_0^4$*.* C_0 *is a saddle point of* V_{eff}*: 7 tachyonic, 14 massless, 63 stable directions.*

Theorem 6.11 (Backreaction dominance [Derived])**.** [[15]]

$$\frac{|\delta m_1^2|_{\text{back}}}{|\delta m_1^2|_{\text{direct}}} = \frac{58.35}{1.08} \approx 54.$$

Total effective Level-1 mass: $m_{1,\text{eff}}^2 = -32\lambda C_0^2(1 + 1.79\lambda + O(\lambda^2)) < 0$*. Case (a) — metastability enhanced, not cured — is confirmed at one loop* [Derived]*;* μ*-independent confirmation via two-loop RGE is* [Derived] *cond. [*[17]*; CLOSED, OP 20.1].*

Theorem 6.12 (First IIB-II loop prediction [Conjecture] [IIB-II)**.** */* [[15], Thm 5.8]

$$\frac{\langle N\rangle_{1\text{-}loop}}{C_0^2} \approx -8.41\lambda < 0 \quad \textit{at } \mu^2 = 32\lambda C_0^2.$$

Under IIB-II: $\delta G_{\text{eff}}/G \approx -8.41\lambda$ [Conjecture] *[IIB-II; OP 20.3 partially closed]. AMT compatibility* [Derived] *[*[15]*, Prop 5.9]:* $\langle N\rangle_{1\text{-}loop} \neq 0$ *does not violate Theorem 2.2; they operate at distinct levels.*

Remark 6.13 (One-loop imaginary part [Conjecture])*. [*[13]*, Thm 3.2]* $\mathrm{Im}(W_{1\text{-loop}}) \approx (70.848 - 31.009\, m_4^2) \cdot r_s^{-1} \in (62.9, 70.9)\, r_s^{-1}$ for $m_4^2 \in (0, 0.256)$. $\mathrm{Im}(W) > 0$ confirms a metastable condensate and is the first quantum contribution toward testability of IIB-II.

7. The Four Proposals: Current State

The four proposals are of increasing speculative reach. IIB-I has an unconditional algebraic basis and requires one physical interpretation step. IIB-II has a quantitative consequence but depends on Sakharov–Adler throughout. IIB-III depends on two physical assumptions. IIB-IV requires a QFT that does not yet exist.

7.1. **Logical dependencies.**

- IIB-I is the physical reading of the Born Rule theorem.
- IIB-II uses IIB-I to motivate the norm-gravity identification and adds the Sakharov–Adler computation.
- IIB-III uses IIB-I (Born Rule failure = gauge-invisibility mechanism) and IIB-II (N couples universally = gravitational activity).
- IIB-IV uses IIB-II and IIB-III plus the CCC framework.

Falsifying IIB-I falsifies all four proposals.

7.2. **IIB-I: The Born Rule boundary [Conjecture].**

Conjecture 7.1 (IIB-I [Conjecture]). *The transition $\mathcal{A}_3 \to \mathcal{A}_4$ marks the algebraic boundary beyond which standard quantum-probabilistic structure cannot be maintained. Physical fields can be consistently defined over $\mathcal{A}_n$ for $n \leq 3$; at $n = 4$ the Born Rule obstruction is not a computational artifact to be resolved but a structural signal marking the boundary of coherent quantum measurement.*

Algebraic basis. The Born Rule theorem (Theorem 2.3) establishes unconditionally that no probability assignment $f : \mathcal{A}_n \to \mathbb{R}_{\geq 0}$ satisfying (M),(D),(OA) exists for $n \geq 4$. The physical claim — that this algebraic failure corresponds to a physical boundary rather than merely indicating that sedenions are not useful for standard quantum mechanics — is the conjecture.
What IIB-I does not claim. It does not claim that $\mathcal{A}_4$ fails to support a physical theory; it claims that $\mathcal{A}_4$ supports a theory of a qualitatively different type, in which the Born Rule failure is a feature rather than a bug. No such theory has been constructed [Open, OP 8.1].
Relation to Furey. Furey's programme [32, 33] derives gauge symmetry groups $U(1)_Y \times SU(2)_L \times SU(3)_C$ from $\mathcal{A}_3 = \mathbb{O}$. IIB-I does not compete: Furey marks $n = 3$ as the level accommodating the Standard Model; IIB-I marks $n = 3$ as the last level at which quantum-probabilistic coherence holds. Both identify the same algebra as fundamental, from complementary directions.

Remark 7.2 (Measurement-boundary reading [Conjecture] [IIB-I).] The Born Rule failure is not a statement about the complexity of $\mathcal{A}_4$ as a mathematical object. It is a statement about what can be measured. No measurement protocol that respects axioms (M),(D),(OA) can assign a quantum number to a state in $\mathcal{A}_4 \setminus \mathcal{A}_3$: not because such states lack structure, but because the Born-rule-compatible probability framework is algebraically insufficient to reach them. Under IIB-I, the $\mathcal{A}_3 \to \mathcal{A}_4$ transition is a *measurement boundary*, not merely an algebraic one. This reading is what IIB-III will require.

7.3. **IIB-II: The norm as gravity [Conjecture].**

Conjecture 7.3 (IIB-II [Conjecture]). *The quadratic norm N is the unique algebraic structure that survives every Cayley–Dickson doubling step. We conjecture that this universality is the algebraic signature of gravity: gauge interactions are intrinsic to specific*

Cayley–Dickson levels ($n = 1, 2, 3$); the gravitational interaction, associated with N rather than any multiplication structure, persists at every level.

What is now [DERIVED] *(cond. Sakharov–Adler).* Under IIB-II and the Sakharov–Adler mechanism, Newton's gravitational constant is (Theorem 4.3):

$$G_N^{-1} = \frac{16\lambda N(C_0)}{12\pi}\left[164\ln\frac{\Lambda^2}{16\lambda N(C_0)} - 220\ln 2 - 42\ln 3\right],$$

with $G_N^{-1} \propto N(C_0)$ [[22], Cor 4.4]. The Planck scale is an output [[22], Thm 4.6]. The coefficient 164 is fully grounded in Fano combinatorics [[25], Cor 3.2]. The conditionality on Sakharov–Adler is never suppressed.

Algebraic supports for IIB-II (not proofs):

Support	Statement	Status
AMT	$\mathbb{E}_v[N(av)] = N(a)$ — gravity-like universality	[DERIVED]
Frobenius	$\|L_a\|_F^2 = 2^n N(a)$ — norm determined at every level	[DERIVED]
Backreaction 54:1	Gravitational dominates gauge correction	[DERIVED] [[15]]
G_N formula	$G_N^{-1} \propto N(C_0)$ explicitly computed	[DERIVED] cond. SA
Level-2 decoupling	$(2 + \mu_2) = 0$: massless AND gravity-neutral	[DERIVED]

None of these proofs IIB-II. The claim that N *is* gravity — rather than *correlating with* gravity — is the conjecture.

Remark 7.4 (The logical gap between IIB-II and IIB-II-strong). The Sakharov–Adler formula of § 4 is a consequence of IIB-II, not its content. IIB-II conjectures that N *sources* gravity; the SA computation shows that a specific mechanism — integrating out S_{84} in curved spacetime — produces $G_N^{-1} \propto N(C_0)$. If SA were the wrong mechanism and a different one were found that still gives $G_N^{-1} \propto N(C_0)$, IIB-II would be supported, not refuted. IIB-II-strong (Speculation 4.11) is the deeper claim: not that N sources gravity through some mechanism, but that N *is* gravity — that matter, condensate, and the ZD^* graph emerge from the first level at which $N(ab) \neq N(a)N(b)$, and that the SA formula is what was always going to result from any consistent loop computation over that algebra. The distinction matters for falsification: a failure of SA as a mechanism falsifies the SA computation but not, by itself, IIB-II. It would falsify IIB-II only if no alternative mechanism can produce $G_N^{-1} \propto N(C_0)$. It would not touch IIB-II-strong, whose content is in § 4.11.

The CD tower and the four forces [SPECULATION] *[IIB-II-strong].*

n	Algebra	Property lost	Conjectured force	Status
1	$\mathbb{C}$	Ordering	Electromagnetism	[SPECULATION]
2	$\mathbb{H}$	Commutativity	Weak interaction	[SPECULATION]
3	$\mathbb{O}$	Associativity	Strong interaction	[SPECULATION]
4	$\mathbb{S}$	Born Rule	Gravity	[CONJECTURE] [IIB-II]

The attribution of EM, weak, and strong interactions to $n = 1, 2, 3$ is [SPECULATION] [IIB-II-strong]: structurally motivated, not derived from any calculation. The attribution of gravity to $n = 4$ is [CONJECTURE] [IIB-II], supported by the G_N formula. The entries at $n = 1, 2, 3$ have no quantitative consequence in the current series.

Gauge coupling extension [CONJECTURE FOR $k = 3$; SPECULATION FOR $k = 1, 2$; OP 26.7, SHARPENED IN PAPERS 33–34]. The proposal that the gauge coupling constants g_1^2, g_2^2, g_3^2 emerge from N at levels $n = 1, 2, 3$ via the Sakharov–Adler mechanism, with $\mathrm{Aut}(\mathcal{A}_k)$ replacing $ZD^*(\mathcal{A}_4)$, is supported structurally by the residue identity $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$ (§ 2.7). For $k = 3$: $s_2 = 14 = \dim G_2 = \dim \mathrm{Aut}(\mathcal{A}_3)$ makes the connection precise enough to be a conjecture [Conjecture, IIB-II-strong; OP 33.1]. For $k = 1, 2$: the matter fields executing the Sakharov–Adler loop on $\mathrm{Aut}(\mathcal{A}_k)$ have not been identified [Open, OP 8.10]. For $k = 3$: the identification $s_2 = 14 = \dim G_2$ establishes a structural match [Derived] [[30]]; the matter fields executing the SA loop at level $k = 3$ have likewise not been identified. The conjecture label for $k = 3$ reflects structural precision, not partial construction of the loop.

7.4. **IIB-III: The sedenionic dark sector [Conjecture].**

Conjecture 7.5 (IIB-III [Conjecture] [under (A1) and (A2)). *] Degrees of freedom in $\mathcal{A}_4 \setminus \mathcal{A}_3$ carry no Standard Model quantum numbers and are gravitationally active via N. Their gauge-invisibility is algebraic, not dynamical: a dark sector whose darkness follows from the Fano non-incidence structure of $\mathcal{A}_4$.*

Physical assumptions stated in full:

- **(A1):** *The $\mathcal{A}_3 \to \mathcal{A}_4$ boundary is physically realised: sedenions are the relevant algebraic structure.*
- **(A2):** *The Standard Model gauge forces are confined to $\mathcal{A}_3$-compatible structures: the octonionic path embedding $\mathcal{A}_3 \leftrightarrow \mathrm{SU}(3)_{\mathrm{color}}$ is physical.*

Without (A1), IIB-III has no physical content. Without (A2), gauge-invisibility does not follow from the Born Rule failure.

Why algebraic darkness differs from dynamical darkness. In every standard dark sector model, gauge-invisibility is imposed: new fields are defined with SM charges set to zero by construction. The Lagrangian is chosen to make them dark. In IIB-III, gauge-invisibility is a consequence that cannot be removed by rewriting the Lagrangian. Under (A2), a Standard Model apparatus is a quantum device whose probability assignments respect axioms (M),(D),(OA) over $\mathcal{A}_3$. The Born Rule theorem (Theorem 2.3) establishes unconditionally that no such assignment can be extended to $\mathcal{A}_4 \setminus \mathcal{A}_3$ degrees of freedom: the Born-rule-compatible measurement framework of $\mathcal{A}_3$ is algebraically insufficient to assign SM quantum numbers to $\mathcal{A}_4 \setminus \mathcal{A}_3$ states. The darkness is not a property of the dark sector — it is a property of the measurement boundary. It cannot be removed by a coupling redefinition because it does not originate from a coupling.
Algebraic basis. At $n = 4$, an $\mathcal{A}_3$-observer cannot perform a Born-rule-consistent measurement distinguishing sedenionic components. Under (A2), this is gauge-invisibility: no SM quantum number can be assigned to $\mathcal{A}_4 \setminus \mathcal{A}_3$ states because no SM measurement protocol reaches them. The information is present in $\mathcal{A}_4 \setminus \mathcal{A}_3$; what fails is the capacity of $\mathcal{A}_3$-apparatus to access it.
Dark sector stratification [Conjecture] [IIB-III]. The four orbit types of the $\mathrm{Aut}(\mathcal{A}_n)$-action on $ZD(n)$ — orbits of size 2, 14, 84, 336 — stratify the dark sector by degree of Fano coherence with $\mathcal{A}_3$:

Orbit size	Fano content	Interpretation
336	Full $PG(2, \mathbb{F}_2)$	Maximal $\mathcal{A}_3$-coherence
84	One Fano line	Intermediate
14	One Fano point	Minimal coherence
2	None	Maximally alien to $\mathcal{A}_3$

Remark 7.6 (No free coupling in the dark sector [Derived] + [Conjecture] [IIB-III).] *[*[8]*, Cor 3.3;* [Conjecture] *[IIB-III]]* The quartic coupling λ of S_{84} is the unique coupling consistent with the Fano-$PSL(2,7)$ symmetry of $ZD^*(\mathcal{A}_4)$. Under IIB-III, the dark sector's self-interaction strength is not a free parameter: it is fixed by the combinatorial geometry of $\mathcal{A}_4$. This distinguishes IIB-III structurally from models with adjustable dark coupling constants.

Current observational constraint. The 4_s dark-matter candidate mass is $m_{4_s}^2 = 32\lambda N(C_0)$ [Derived] [[14]], giving $m_{4_s}^2 < 0.128\, r_s^{-2}$ under the cavity bound — the first concrete mass range for IIB-III candidates, in Black Circle units.

7.5. **IIB-IV: The CCC–sedenionic residue [Speculation].**

Conjecture 7.7 (IIB-IV [Speculation] [under (A1),(A2),(A3) + IIB-II + IIB-III). *] (A3): The universe undergoes a sequence of aeons in Penrose's sense* [36]*: each aeon ends in a conformally flat state, and non-localised field configurations can cross the conformal boundary.*

Under (A1)–(A3) and IIB-II/III:

(i) In the terminal phase of aeon k, Hawking evaporation does not produce a configuration lying entirely in $\mathcal{A}_3$. A non-localised sedenionic residue in $\mathcal{A}_4 \setminus \mathcal{A}_3$ persists at the conformal boundary.
(ii) This residue crosses into aeon $k+1$; it cannot integrate into $\mathcal{A}_3^{(k+1)}$ for the algebraic reason of IIB-III.
(iii) The residue is gravitationally active and gauge-invisible; it constitutes the dark-matter component of aeon $k+1$.

Remark 7.8 ([Speculation] [IIB-IV + QFT over $\mathcal{A}_4$).] Successive aeons have monotonically decreasing dark-matter fractions, consistent with the combinatorial ZD density $\varrho(n) \to 0$ as $n \to \infty$ [Speculation]; the conversion from combinatorial density to physical dark-matter fraction requires the normalisation map that is currently open (§ 9.3).

A dimensional mass estimate: under three unproven identifications — CCC boundary maps ZD elements to dark-sector degrees of freedom; each ZD pair carries energy $\varepsilon = m^4$; the Planck-epoch energy density applies — $m_{\text{dark}} \approx 0.168\, M_{\text{Pl}}$ [Speculation] [IIB-IV + QFT over $\mathcal{A}_4$]. This estimate rests on three underived steps and is a dimensional plausibility check, not a prediction.

Epistemic status. IIB-IV requires: (a) a QFT over $\mathcal{A}_4$ defining states, observables, and $T_{\mu\nu}$; (b) a model of Hawking radiation termination as a sedenionic residue; (c) a model of CCC conformal crossing for $\mathcal{A}_4$ fields. None of (a)–(c) is currently available.

7.6. **Connections to prior art.** Furey's programme [32, 33] derives gauge *symmetry groups* from $\mathcal{A}_3 = \mathbb{O}$. The present conjecture addresses gauge *coupling constants*. The two are complementary: Furey determines which groups appear; IIB-II-strong asks what controls their coupling strength. Both remain [Conjecture] or [Speculation].

Günaydin and Gürsoy [35] identify $G_2 = \mathrm{Aut}(\mathbb{O})$ with quark colour; Dixon [34] develops a framework using $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. The IIB programme differs: it identifies the SM gauge sector with $n = 1, 2, 3$ and the gravitational sector with $n = 4$, and *derives* G_N from the $n = 4$ ZD structure via induced gravity. Prior programmes associating gravity with non-associative algebra use different gravity carriers (associators, gauge structures, or star-product deformations of the metric) and do not derive G_N in closed form.

7.7. **What the programme has not proved.** For completeness and to prevent misreading:

- No IIB proposal (I–IV) has been proved.
- The G_N formula is [Derived] conditional on Sakharov–Adler. The QFT over $\mathcal{A}_4$ that would justify the Sakharov–Adler application has not been constructed.
- The $\mathcal{A}_3 \leftrightarrow \mathrm{SU}(3)_{\mathrm{color}}$ identification (A2) has not been derived.
- Conjecture A ($M = m_{4_s}$) has no algebraic derivation.
- The CCC conformal crossing of $\mathcal{A}_4 \setminus \mathcal{A}_3$ degrees of freedom has not been modelled.

The IIB programme will be falsified if any of the following is established: (i) the Born Rule theorem admits an extension to $n \geq 4$; (ii) the Sakharov–Adler mechanism applied to S_{84} does not produce $G_N^{-1} \propto N(C_0)$; (iii) a QFT over $\mathcal{A}_4$ is constructed and $\mathcal{A}_4 \setminus \mathcal{A}_3$ degrees of freedom are found to carry SM gauge charges. Each falsification condition is algebraically precise.

8. Open Problems

We collect all open problems, stratified by type. Closed problems are listed in § 8.4.

8.1. **Primary structural open problem.**

Open Problem 8.1 (QFT over $\mathcal{A}_4$). Construct a QFT over $\mathcal{A}_4$ defining states, observables, and the stress-energy tensor $T_{\mu\nu}$. This is the primary obstacle: without it, neither IIB-II (in the sense of a dynamical $G = f(\langle N \rangle)$) nor IIB-III (as a falsifiable dark-sector claim) nor IIB-IV (as a CCC mechanism) can be tested. The Sakharov–Adler computation of Papers 26–28 is a partial step in this direction: it integrates out the S_{84} spectrum in a fixed curved background, but does not constitute a QFT over $\mathcal{A}_4$.

8.2. **Algebraic open problems.**

Open Problem 8.2 (Lie group with Level-2 content). Does there exist a compact Lie group G with $\dim G - 14 = 0$ (so that $\dim G = 14$) and $\mathrm{adj}(G)|_{PSL(2,7)} \cong \mathrm{irr}_6 \oplus \mathrm{irr}_8$? G_2 is excluded [Derived-Negative] [GAP, [16]]. Resolution would identify the symmetry-breaking pattern responsible for Level-2 masslessness and close OP 8.6.

Open Problem 8.3 (Categorical Weil proof). Prove that Weil factorisation $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(PG(3, \mathbb{F}_2), x)$ necessarily extracts $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$. Each individual identification is [Derived] [[30]]. The G_2-equivariant proof that Weil factorisation is the mechanism requires: Schubert cell structure of $PG(3, \mathbb{F}_2)$; the chain $PSL(2, 7) \subset SU(3) \subset G_2$; and G_2-equivariant interpretation of the Frobenius eigenvalue $4 = \dim \mathcal{A}_2$ at the $\mathbb{H}$-Schubert stratum.

Open Problem 8.4 (Twisted cocycle = non-split extension class). Does the octonionic 2-cocycle c_{oct} define a twisted cocycle of $(\mathbb{Z}/2\mathbb{Z})^3$ whose twisting class equals the lo-bit projection of the non-split extension class $[\mathrm{Aut}(\mathcal{A}_4)] \in H^2(PSL(2, 7), (\mathbb{Z}/2\mathbb{Z})^4)$? Both sides are classified by a single $\mathbb{F}_2$-element. GAP computation of $H^2(PSL(2, 7), (\mathbb{Z}/2\mathbb{Z})^4)$ is feasible.

Open Problem 8.5 (irr_1 CCC crossing condition). Give necessary and sufficient conditions for irr_1 to cross the CCC conformal boundary. Three necessary conditions are [Derived]: massless; at $SO(1, 4)$ complementary series boundary; fully S_4-symmetric. Sufficiency requires: (a) conformal weight analysis; (b) Bunch-Davies BC consistency; (c) $\mathrm{irr}_{1'}$ exclusion as CCC parity constraint.

8.3. **Physical open problems.**

Open Problem 8.6 (Level-2 mass mechanism). Identify a mechanism giving the 14 massless Level-2 modes a positive mass consistent with $PSL(2,7)$-symmetry. Three mechanisms are excluded [DERIVED-NEGATIVE] [Paper 21]: G_2-Goldstone; SM-Higgs under (A2); ξ-splitting at $PSL(2,7)$ and S_4 levels. The curvature coupling $\xi \neq 0$ gives a uniform mass shift without symmetry breaking [not yet analysed].

Open Problem 8.7 (Generation problem). Derive the three-generation count of the Standard Model from 4_s-sector content, or exclude this derivation. Three resolutions are structurally available (§ 6.3); none is forced by current algebra.

Open Problem 8.8 (Quantum GHY correction). Compute the genuine quantum GHY term for S_{84} in the Black Circle background. The boundary Seeley–DeWitt coefficient for the de Sitter interior with Schwarzschild exterior has not been computed. The classical GHY perturbation $\delta S_{GHY} = 0$ (Corollary 5.4) is not the quantity sought.

Open Problem 8.9 ($\delta G_{\text{eff}}/G$ prediction). Derive a scheme-independent prediction for $\delta G_{\text{eff}}/G$ under IIB-II. Currently: $\delta G_{\text{eff}}/G \approx -8.41\lambda$ [CONJECTURE] [IIB-II; Paper 20] at $\mu^2 = 32\lambda C_0^2$. A scheme-independent combination using the complete two-loop running is the next step.

Open Problem 8.10 (Gauge couplings from N at levels $n = 1,2,3$). Construct the matter fields Φ_k executing the Sakharov–Adler loop on $\mathrm{Aut}(\mathcal{A}_k)$ for $k = 1,2,3$, yielding gauge coupling constants $g_k^2 \propto \dim \mathrm{Aut}(\mathcal{A}_k)$. For $k = 3$: $s_2 = 14 = \dim G_2$ establishes a structural match [DERIVED] [[30]]; the matter field has not been identified. For $k = 1,2$: no structural match yet known.

Open Problem 8.11 (Conjecture A derivability). Derive or exclude the identification $M = m_{4_s} = \sqrt{32\lambda}\, C_0$ from the algebra of the series. If derived: OP 27.1 [CONJECTURE] [IIB-II + Conjecture A] would be promoted to [DERIVED] conditional on IIB-II and Sakharov–Adler.

Open Problem 8.12 (Non-perturbative two-loop SC equation). Solve the full non-perturbative self-consistency equation $g(G_N) = 0$ (Theorem 4.5) at two-loop order via numerical ODE integration of the RG equation. The leading-log approximation is under control; the full equation remains open [[23]].

Open Problem 8.13 (SU(3) charges and IIB-III). Determine the SU(3) quantum numbers of the sedenionic dark sector under assumption (A2) and derive the phenomenological consequences for dark matter direct detection and collider searches.

Open Problem 8.14 (Does S_{84} quantisation fix $\hbar$ algebraically?). [SPECULATION]: if G_N encodes the Fano non-incidence count (the $\mathcal{A}_3 \to \mathcal{A}_4$ non-incidence structure), $\hbar$ may encode the incidence structure *lost* at the transition — the Fano plane itself, which lives in $\mathcal{A}_3$. Under this speculation: both fundamental constants of quantum gravity (G_N and $\hbar$) emerge from the same algebraic transition, with G_N from the non-incidence face and $\hbar$ from the incidence face of the Fano bifrontality. No calculation supports this.

8.4. **Closed problems: selected register.** Selected closures since Paper 20 (the endpoint of IIB v5):

OP	Closed by	Status	Summary
20.1	Paper 22-A	[DERIVED] cond.	μ-independent metastability at all RG scales
23.1	Paper 25	[DERIVED] cond.	$c_{\text{ferm}} = 8/7$, $d_{\text{ferm}} = 20/7$
23.4	[25], Thm 4.1	[DERIVED]	V_{ni} irreducible as Stab-module
25.2	[23], Thm 3.6	[DERIVED]	Two-loop β; $\lambda^{(2)}_{\text{IR}} = 0.46373$
26.3	[23], Thm 1.1	[DERIVED]	Ratio $\text{Im}(W)/\text{Im}(G_N^{-1}) = 106.32$ (pure number)
26.5	[23], Thm 2.3	[DERIVED] cond.	$G_N^{(2)}$ universally excluded
26.6	[23], Thm 4.3	[DERIVED] cond.	Two-loop G_N^{-1}; $R \in [0.87, 0.95]$
28.1	[25], Thm 3.1	[DERIVED]	$\ker(L_a) \cong V_{ni}(a)$; $\mathcal{S}_{\text{eff}} = 164$ grounded

9. Conclusions

9.1. What this paper has established. The algebraic backbone of the IIB programme is complete and unconditional. The following results require no physical assumption:

- The Born Rule theorem: no $f : \mathcal{A}_n \to \mathbb{R}_{\geq 0}$ satisfying (M),(D),(OA) exists for $n \geq 4$; for $n \leq 3$ the unique solution is $f = N$.
- $|ZD(n)| = 336 \cdot \binom{n-1}{3}_2$ and $\text{Spec}(M_a) = \{0^{2^{n-2}}, 1^{2^{n-1}}, 2^{2^{n-2}}\}$.
- Fano bifrontality: $PG(2, \mathbb{F}_2)$ governs $\mathcal{A}_3$ multiplication and $\mathcal{A}_4$ annihilation by dual roles.
- $\ker(L_a) \cong V_{ni}(a)$ as $\text{Stab}_{G_{84}}(a)$-modules.
- The spectral invariant $\mathcal{S}_{\text{bos}} = 168 = |\text{Aut}(PG(2, \mathbb{F}_2))|$.
- The coefficient $164 = 168 - 4 = \mathcal{S}_{\text{bos}} - \dim \ker(L_a)|_{n=4}$ is Fano-structural.
- $28 = 4 \times 7$ and its three independent characterisations.
- $\mathcal{A}_3$-memory residues $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$.
- Level-2 quadruple protection: masslessness, gravitational decoupling, gauge-invisibility, Majorana compatibility — all from $(2 + \mu_2) = 0$.
- The two-loop β-function, branch selection, and fermionic correction.

The following results are [DERIVED] conditional on Sakharov–Adler:

- G_N^{-1} in closed form (Theorem 4.3).
- $G_N^{-1} \propto N(C_0)$: Newton's constant is determined by the Cayley–Dickson norm of the condensate.
- The Planck mass $M_{\text{Pl}}^2 = G_N^{-1}$ is an output, not an input.
- Two-loop correction factor $R \in [0.87, 0.95]$.

The conditionality on Sakharov–Adler is never removed and never will be until the QFT over $\mathcal{A}_4$ that motivates it is constructed.

9.2. What has changed since the initial submission. The initial submission of this paper could say: the norm N appears to be the algebraic correlate of gravity for structural reasons. The current version can say something sharper: the norm N generates Newton's constant in a computable closed form, with a coefficient 164 that is fully grounded in the Fano non-incidence geometry of $\mathcal{A}_4$. The coefficient is not a numerical coincidence — it is a count of Fano combinatorics.

The Planck scale is an output of the algebra, not a free parameter. This is a qualitative change in the character of IIB-II, not merely an addition of detail.

9.3. The minimal standard of evidence. The logical sequence required to convert the IIB programme from conjecture to result:

(1) **Construct a QFT over $\mathcal{A}_4$** (Open, OP 8.1). This is the primary obstacle. Without it, neither IIB-II nor IIB-III nor IIB-IV can be tested.
(2) **Show that this QFT admits a universal N-sourced interaction** and no $\mathcal{A}_3$-gauge coupling for $\mathcal{A}_4 \setminus \mathcal{A}_3$ degrees of freedom.
(3) **Derive a prediction for the dark-matter power spectrum or direct-detection cross-section** numerically distinct from standard WIMP/axion predictions.

Until step 1 is completed, the IIB is a conjecture with a precise algebraic backbone, a structured set of quantitative physical consequences, and a clear account of what it has and has not proved.

9.4. **The programme's position.** The IIB programme sits at the intersection of three lines of inquiry that have not previously converged: the structure theory of Cayley–Dickson algebras (Papers 1–9, 29–35), the semiclassical physics of the Black Circle geometry (Papers 10–21, 30–31), and the induced gravity programme in the Sakharov–Adler tradition (Papers 22–28, 32–34).

The algebraic results of Papers 1–9 and 29–35 are independent of all physical conjecture: they are theorems in algebra and combinatorics. The Black Circle results of Papers 10–21 and 30–31 are conditional on the metric ansatz (Conjecture 3.1) but not on the form of G_N. The G_N derivation of Papers 22–28 depends on both the metric ansatz and the Sakharov–Adler mechanism. The stacking of conditionalities is explicit and stratified; no result from a higher stratum is silently used to support a claim in a lower one.

The IIB programme will be falsified if any of the following is established: (i) the Born Rule theorem admits an extension to $n \geq 4$ (refuting IIB-I); (ii) the Sakharov–Adler mechanism applied to S_{84} does not produce $G_N^{-1} \propto N(C_0)$ (refuting IIB-II at its first quantitative test); (iii) a QFT over $\mathcal{A}_4$ is constructed and $\mathcal{A}_4 \setminus \mathcal{A}_3$ degrees of freedom are found to carry SM gauge charges (refuting IIB-III). Each falsification condition is algebraically precise. The programme invites falsification on these terms.

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