THE LEVEL-2 SECTOR OF THE BLACK CIRCLE

GIOVANNI LEVRATTI

Abstract. Paper 11 [4] established that the Black Circle mass spectrum has 14 exactly massless modes at Level 2 ($\mu = -2$, $\xi = 0$). The coincidence $\dim(G_2) = \mathrm{mult}(\text{Level-2}) = 14$, together with $\mathrm{PSL}(2,7) \subset G_2 = \mathrm{Aut}(\mathbb{O})$, raises the question: is Level-2 the adjoint of G_2 restricted to $\mathrm{PSL}(2,7)$, and are the 14 modes Goldstone bosons of a spontaneous G_2-breaking triggered by Level-1 tachyon condensation?

This paper answers the question completely. The adjoint of G_2 restricted to $\mathrm{PSL}(2,7)$ decomposes as $\mathrm{irr}_3 \oplus \mathrm{irr}_{3^*} \oplus \mathrm{irr}_8$, while the actual Level-2 content is $\mathrm{irr}_6 \oplus \mathrm{irr}_8$. These are inequivalent $\mathrm{PSL}(2,7)$-modules: their characters disagree at conjugacy classes 2A and 4A. *[Derived]*, GAP 4.15.1 [10]. Three independent obstructions to the Goldstone hypothesis are given: representational, combinatorial, and structural. *[Derived]*

A positive structural result replaces it. Under the chain $\mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2$ [3], the Level-2 eigenspace carries $\mathrm{SU}(3)$ quantum numbers $8 \oplus 6$. As a consequence, the Standard Model Higgs mechanism cannot generate a renormalisable mass gap for Level-2 under assumption (A2). *[Derived]*, cond. (A2)

The de Sitter symmetry-breaking subgroup $S_4 \subset \mathrm{PSL}(2,7)$ [3] resolves Level-2 into $\mathrm{irr}_1 \oplus 2\,\mathrm{irr}_2 \oplus 2\,\mathrm{irr}_3 \oplus \mathrm{irr}_{3'}$. The curvature coupling ξ assigns uniform mass to all 14 modes, regardless of S_4-multiplet. Open Problem 21.3 is **closed**. *[Derived]*, GAP 4.15.1

A structural observation: Level-3 $= 3\,(\mathrm{irr}_6 \oplus \mathrm{irr}_8) = 3 \times$ Level-2 as $\mathrm{PSL}(2,7)$-modules, with the factor 3 fixed by the characteristic polynomial of the ZD adjacency matrix. Under $\mathrm{SU}(3)$: Level-3 $= 3\,(8 \oplus 6)$. The connection to the three SM generations is *[Speculation]*, cond. (A2).

Four open problems are stated. OP 21.4 analyses four candidates for the minimum Black Circle radius: the "purely Level-2" regime requires $r_s < 1.20\,l_{\mathrm{Pl}}$, universally sub-Planckian and outside the current semiclassical IIB framework. *[Derived]*The Planck-time condition $\tau_{\mathrm{grow}} = t_{\mathrm{Pl}}$ (Paper 18 [6]) is the most IIB-internally motivated sub-Planck candidate. *[Speculation]*

Epistemic key. *[Derived]* — follows from Papers 1–20 and established mathematics. *[Conjecture]* — precise statement, assumptions explicit. *[Speculation]* — structurally motivated, not yet falsifiable. *[Open]* — obstacle stated precisely.

Contents

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1. Introduction

1.1. **Background and motivation.** Paper 11 [4] established the mass spectrum of S_{84}, the 84-mode scalar action on the Black Circle de Sitter interior. The zero-divisor pair graph $\mathrm{ZD}^*(\mathcal{A}_4)$ of $\mathcal{A}_4$ (the sedenions) has 84 vertices and 168 edges [1]. The Hessian at the condensate C_0 has eigenvalues $m_i^2 = 16\lambda C_0^2(2+\mu_i)$ with multiplicities $\{7, 14, 42, 14, 7\}$ at $\mu_i \in \{-4, -2, 0, +2, +4\}$ respectively (Theorem 4.5 of [4]). Level-2 ($\mu = -2$) consists of exactly 14 massless modes at $\xi = 0$: an algebraically exact zero protected by no apparent gauge symmetry.

The coincidence $\dim(G_2) = 14 = \mathrm{mult}(\text{Level-2})$ immediately raises a precise question. The automorphism group $G_2 = \mathrm{Aut}(\mathbb{O})$ of the octonion algebra $\mathcal{A}_3$ has dimension 14, and $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ is a well-known subgroup of G_2 (see e.g. [9]). If Level-2 transforms as the adjoint representation of G_2 restricted to $\mathrm{PSL}(2,7)$, the masslessness would be explained as a symmetry consequence: the 14 modes would be Goldstone bosons of a spontaneous breaking

$$G_2 \longrightarrow \mathrm{PSL}(2,7)$$

triggered by condensation of the Level-1 tachyon (irr_7, $m_1^2 = -32\lambda C_0^2 < 0$, Paper 20 [8]). This would simultaneously explain the masslessness of Level-2, connect OP 8.6 of [2] to the Higgs sector of the Standard Model via identification $\mathcal{A}_3 \leftrightarrow \mathrm{SU}(3)_{\mathrm{color}}$ (assumption (A2) of the IIB programme), and produce concrete predictions for IIB-III dark-matter candidates.

This paper answers the question completely. The answer is negative; the refutation reveals a more precise positive statement about the SU(3) quantum numbers of Level-2.

1.2. **The two $\mathrm{PSL}(2,7) \hookrightarrow G_2$ embeddings.** Paper 10 [3] identified two non-conjugate embeddings of $\mathrm{PSL}(2,7)$ into G_2:

(i) The irr_7 path. $\mathrm{PSL}(2,7)$ acts irreducibly on $\mathbb{R}^7$ via the 7-dimensional standard representation ($m(\mathbf{1}) = 1$, one G_2-invariant 3-form). This path governs Level-1 ($\mu = -4$, tachyonic, $1 \cdot \mathrm{irr}_7$) and Level-4 ($\mu = +2$, $2 \cdot \mathrm{irr}_7$, stable Fano sector): the characteristic polynomial $(x+4)(x-2)^2$ of the matrix M_7 distributes exactly one irr_7 eigenvector to Level-1 and two to Level-4 (Theorem 4.5 and Remark 4.6 of [4]).

(ii) The SU(3) path. $\mathrm{PSL}(2,7)$ embeds as a subgroup of $\mathrm{SU}(3) \subset G_2$ via the holomorphic SU(3) structure of $\mathrm{Im}(\mathbb{O})$, giving $m(\mathbf{1}) = 3$ (three independent G_2-invariant 3-forms: the Cayley form φ, $\mathrm{Re}(\Omega)$, and $\mathrm{Im}(\Omega)$) ([3], Theorem 7.5 and Remark 7.6(ii)). This path governs Level-5 ($\mu = +4$, $\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*}$, condensate direction).

The ZD graph spectrum thus simultaneously instantiates both non-conjugate embeddings at the opposite spectral extremes. The Goldstone hypothesis asks whether Level-2 is governed by the full Lie algebra of G_2. Section 2 shows it is not.

1.3. **Organisation.** Section 2 carries out the representation-theoretic computation: Proposition 2.1 computes $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)}$ via character theory and verifies the result with GAP 4.15.1; Theorem 2.4 gives three independent proofs of the non-existence of the Goldstone mechanism; Proposition 2.6 establishes the $8 \oplus 6$ SU(3) structure; Remark 2.11 states the Level-3 tripling. Section 3 computes the S_4 de Sitter restriction (OP 21.3 closed, OP 8.6 updated). Section 4 states four open problems. Section 5 concludes. Appendix A gives the complete GAP 4.15.1 scripts and raw output.

2. Representation-Theoretic Computation

2.1. **Background: mass spectrum and spectral table.** The PSL(2,7) representation content of the ZD graph eigenspaces is established in [4], Theorem 4.5:

Level	μ_i	Mult.	PSL$(2,7)$-content	m_i^2 $(\xi = 0)$
5	$+4$	7	$\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*}$	$+96\lambda C_0^2$
4	$+2$	14	$2 \cdot \mathrm{irr}_7$	$+64\lambda C_0^2$
3	0	42	$3 \cdot \mathrm{irr}_6 \oplus 3 \cdot \mathrm{irr}_8$	$+32\lambda C_0^2$
2	-2	14	$\mathrm{irr}_6 \oplus \mathrm{irr}_8$	0
1	-4	7	irr_7	$-32\lambda C_0^2$

The six irreducible representations of $\mathrm{PSL}(2,7)$ have dimensions $1, 3, 3, 6, 7, 8$ (with irr_3 and irr_{3^*} a complex conjugate pair over $\mathbb{Q}$; all others are self-conjugate).

2.2. **Computing** $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)}$**.**

Proposition 2.1 (*[Derived]*, GAP 4.15.1 [10])**.** *The restriction of the adjoint representation of G_2 to $\mathrm{PSL}(2,7) \subset G_2$ decomposes as:*

$$\mathrm{adj}(G_2)\big|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*} \oplus \mathrm{irr}_8. \tag{1}$$

The character values in standard order (1A, 2A, 3A, 4A, 7A, 7B) *are:*

$$\chi_{14} = (14,\ -2,\ -1,\ +2,\ 0,\ 0). \tag{2}$$

Proof. The adjoint of G_2 equals $\mathrm{adj}(G_2) = \Lambda^2(7) - 7$ as G_2-representations, where 7 is the standard representation. As a $\mathrm{PSL}(2,7)$-module under the irr_7 embedding: $\chi_{14} = \chi_{\Lambda^2(7)} - \chi_7$. For the 7-dimensional irr_7 of $\mathrm{PSL}(2,7)$, the character of $\Lambda^2(\mathrm{irr}_7)$ on the six conjugacy classes (class sizes $1, 21, 56, 42, 24, 24$) is computed from the formula $\chi_{\Lambda^2}(g) = (\chi_7(g)^2 - \chi_7(g^2))/2$. Subtracting χ_7 yields $\chi_{14} = (14, -2, -1, +2, 0, 0)$.

Inner products against the six irreducibles of $\mathrm{PSL}(2,7)$ (using $|G| = 168$, character table from [11] and verified by GAP 4.15.1 [10]):

$$\begin{aligned}
&\langle \chi_{14}, \chi_1 \rangle = \tfrac{1}{168}(14 - 42 - 56 + 84 + 0 + 0) = 0,\\
&\langle \chi_{14}, \chi_3 \rangle = 1, \quad \langle \chi_{14}, \chi_{3^*} \rangle = 1,\\
&\langle \chi_{14}, \chi_6 \rangle = 0, \quad \langle \chi_{14}, \chi_7 \rangle = 0, \quad \langle \chi_{14}, \chi_8 \rangle = 1.
\end{aligned}$$

Dimension check: $3+3+8 = 14$. ✓All values independently confirmed by GAP 4.15.1 (Appendix A). □

Corollary 2.2 (*[Derived]*, GAP 4.15.1 [10])**.** *The tachyonic representation* irr_7 *— which is the* $\mathrm{PSL}(2,7)$*-content of Level-1 — has multiplicity zero in* $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)}$.

2.3. $\mathrm{Sym}^2(\mathrm{irr}_3)$ and the actual Level-2 content.

Proposition 2.3 (*[Derived]*, GAP 4.15.1 [10])**.** $\mathrm{Sym}^2(\mathrm{irr}_3) = \mathrm{irr}_6$ *as* $\mathrm{PSL}(2,7)$*-modules.*

Proof. The character of $\mathrm{Sym}^2(\mathrm{irr}_3)$ is $\chi_{\mathrm{Sym}^2}(g) = (\chi_3(g)^2 + \chi_3(g^{[2]}))/2$. GAP inner-product decomposition returns multiplicity vector $[0,0,0,1,0,0]$ on irreps of dimensions $[1,3,3,6,7,8]$, giving pure irr_6. □

The character comparison between the Goldstone prediction and the actual Level-2 content:

PSL(2,7)-module	dim	χ(1A)	χ(2A)	χ(3A)	χ(7A)	χ(7B)
$\mathrm{irr}_3 \oplus \mathrm{irr}_{3^*}$	6	6	-2	0	-1	-1
$\mathrm{irr}_6 = \mathrm{Sym}^2(\mathrm{irr}_3)$	6	6	$+\mathbf{2}$	0	-1	-1

The decisive difference is at class 2A (-2 vs. $+2$). The complete comparison between Level-2 and the Goldstone prediction $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)}$:

Module	χ(1A)	χ(2A)	χ(3A)	χ(4A)	χ(7A)	χ(7B)
Level-2 (actual, [4])	$+14$	$+2$	-1	0	0	0
$\mathrm{adj}(G_2)\|_{\mathrm{PSL}(2,7)}$ (Goldstone pred.)	$+14$	-2	-1	$+2$	0	0
Match?	✓	×	✓	×	✓	✓

The mismatch is at 2A and 4A ($21+42 = 63$ out of 168 group elements). The modules are genuinely inequivalent: no inner automorphism of $\mathrm{PSL}(2,7)$ maps one to the other.

2.4. Refutation of the $\mathbf{G_2}$-Goldstone hypothesis.

Theorem 2.4 (*[Derived]*)**.** *The 14 Level-2 modes of* $\mathrm{ZD}^*(\mathcal{A}_4)$ *are not the Goldstone bosons of any spontaneous breaking of* G_2 *triggered by* irr_7 *condensation. Three independent obstructions:*

(i) Representational. *If a VEV in the* irr_7 *direction broke* $G_2 \to H$*, the massless modes would transform as* $\mathfrak{g}_2/\mathfrak{h}$*, hence as* $\mathrm{adj}(G_2)|_H$ *modulo* $\mathrm{adj}(H)$*. For the reduction to* $\mathrm{PSL}(2,7)$*, this requires* $\mathrm{irr}_3 \oplus \mathrm{irr}_{3^*} \oplus \mathrm{irr}_8$ *(Proposition 2.1), but the actual Level-2 content is* $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ *([4], Theorem 4.5). These are inequivalent* $\mathrm{PSL}(2,7)$*-modules (Proposition 2.3 and the character table above).*

(ii) Combinatorial (Goldstone count). *For a continuous symmetry* G_2 *(dim 14) broken to a continuous subgroup* H*, Goldstone's theorem* [12] *gives* $\dim G_2 - \dim H$ *massless modes. A VEV in the 7-dimensional representation of* G_2 *has generic continuous stabiliser* $\mathrm{SU}(3)$ *(dim 8)* [9]*, giving* $14 - 8 = 6$ *Goldstone bosons, not 14.*

(iii) Structural. $G_2 = \mathrm{Aut}(\mathbb{O})$ *acts on* $\mathrm{Im}(\mathbb{O}) = \mathcal{A}_3$*, not on* $\mathcal{A}_4$*. The Cayley–Dickson construction* $\mathcal{A}_3 \to \mathcal{A}_4$ *breaks* G_2 *to a discrete subgroup;* G_2 *is not a symmetry of the action* S_{84} *and admits no Noether current in* S_{84} *that could be spontaneously broken.*

Remark 2.5 (*[Derived]*)**.** Corollary 2.2 provides a fourth corroborating datum: irr_7 has zero overlap with $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)}$. The algebraic channel through which a Level-1 condensate could seed G_2 Goldstone bosons does not exist at the representation level.

2.5. Positive residue: SU(3) quantum numbers of Level-2.

Proposition 2.6 (*[Derived]*, [3])**.** *Under the canonical* $\mathrm{SU}(3)$*-path embedding of* $\mathrm{PSL}(2,7)$ *into* G_2 *(Paper 10* [3]*, Theorem 7.5 and Remark 7.6(ii)), the Level-2 eigenspace decomposes under* $\mathrm{SU}(3)$ *as:*

$$\text{Level-2}\big|_{\mathrm{SU}(3)} = \underbrace{8}_{\mathrm{adj}(\mathrm{SU}(3))} \oplus \underbrace{6}_{\mathrm{Sym}^2(3_{\mathrm{fund}})}. \tag{3}$$

Proof. Under the SU(3)-path embedding: the 8-dimensional adjoint of SU(3) restricts to irr_8 of $\mathrm{PSL}(2,7)$; the 6-dimensional $\mathrm{Sym}^2(3_{\mathrm{fund}})$ restricts to $\mathrm{irr}_6 = \mathrm{Sym}^2(\mathrm{irr}_3)$ (Proposition 2.3). □

Remark 2.7 (*[Derived]*, cond. (A2))**.** The Standard Model Higgs boson is a colour singlet ($\mathbf{1}$ under $\mathrm{SU}(3)_{\mathrm{color}}$). Level-2 carries $\mathbf{8}\oplus\mathbf{6}$ (non-singlets). No renormalisable Higgs–Level-2 coupling invariant under $\mathrm{SU}(3)_{\mathrm{color}}$ exists. The SM Higgs mechanism cannot generate a mass gap for Level-2. This exclusion is conditional on assumption (A2) of the IIB programme (identification $\mathcal{A}_3 \leftrightarrow \mathrm{SU}(3)_{\mathrm{color}}$).

Remark 2.8 (*[Derived]*, structural)**.** The SU(3)-decomposition table (Proposition 2.10 below) shows that the two non-conjugate $\mathrm{PSL}(2,7) \hookrightarrow G_2$ embeddings of [3] give *identical* SU(3) content on Levels 1 and 5 (both $1\oplus 3\oplus \bar{3}$). Assumption (A2) is therefore insufficient to select which embedding is physically realised; a G_2-level identification is required. This is a strengthened obstacle to OP 7.5 of [2].

2.6. Open Problem 21.1.

Open Problem 2.9 (SU(3) quantum numbers and IIB-III, new)**.** Determine whether the SU(3) quantum numbers $8\oplus 6$ of Level-2 are consistent with the IIB-III charge assignments ([2], Conjecture 3.11), and whether any BSM scalar in a non-singlet SU(3) representation (e.g. a colour-octet scalar $\Phi_8 \sim (\mathbf{8},\mathbf{1},0)$) can couple to Level-2 at tree level. The primary obstacle: assumption (A2) does not specify an explicit assignment of hypercharge and isospin quantum numbers to the 7 imaginary units $\{e_1,\dots,e_7\}$ of $\mathcal{A}_3$.

2.7. SU(3) decomposition of all five spectral levels.

Proposition 2.10 (*[Derived]*)**.** *Under the* SU(3)*-path embedding of* [3]*, the complete* SU(3)*-decomposition of the five spectral levels is:*

Level	μ	PSL(2,7) *content*	SU(3) *content*	*Source*
5	$+4$	$\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*}$	$1\oplus 3\oplus \bar{3}$	[3]
4	$+2$	$2\cdot \mathrm{irr}_7$	$2\,(1\oplus 3\oplus \bar{3})$	[3]
3	0	$3\cdot \mathrm{irr}_6 \oplus 3\cdot \mathrm{irr}_8$	$3\,(6\oplus 8)$	[4]
2	-2	$\mathrm{irr}_6 \oplus \mathrm{irr}_8$	$6\oplus 8$	*this paper*
1	-4	irr_7	$1\oplus 3\oplus \bar{3}$	[3], *Rmk 7.6(ii)*

The SU(3)*-mirror symmetry*

$$\text{Level-1}\big|_{\mathrm{SU}(3)} = \text{Level-5}\big|_{\mathrm{SU}(3)} = 1\oplus 3\oplus \bar{3} \tag{4}$$

is an exact algebraic identity: $\mathrm{irr}_7|_{\mathrm{SU}(3)} = 1\oplus 3\oplus \bar{3}$ *follows from the* $G_2 \supset \mathrm{SU}(3)$ *branching rule* $7_{G_2}|_{\mathrm{SU}(3)} = 1\oplus 3\oplus \bar{3}$.

Remark 2.11 (*[Derived]*; *[Speculation]* for generational interpretation)**.** From the table:

$$\text{Level-3} = 3\,(\mathrm{irr}_6 \oplus \mathrm{irr}_8) = 3\times \text{Level-2} \tag{5}$$

as PSL(2,7)-modules, and under SU(3): Level-3 $= 3\,(8\oplus 6)$.

The algebraic origin: the 4×4 restrictions M_6 and M_8 of the ZD pair graph adjacency matrix to the irr_6- and irr_8-isotypic components of $\mathbb{R}^{84}$ share the same characteristic polynomial $x^3(x+2)$. This polynomial assigns exactly 3 copies of each representation to Level-3 ($\mu = 0$, roots 0 with multiplicity 3) and exactly 1 copy to Level-2 ($\mu = -2$, root -2 with multiplicity 1). The 3:1 ratio is algebraically fixed by $\mathrm{ZD}^*(\mathcal{A}_4)$ and is independent of λ and C_0.

Structural precision: Level-3 contains 3 copies of the *pair* $(\mathrm{irr}_6, \mathrm{irr}_8)$, not of one factor alone, since M_6 and M_8 have the same polynomial. This is a structural constraint of $\mathrm{ZD}^*(\mathcal{A}_4)$, not a numerical coincidence.

Generational observation. Under assumption (A2), Level-2 carries $\mathrm{SU}(3)_{\mathrm{color}}$ quantum numbers $8\oplus 6$ and Level-3 carries $3\,(8\oplus 6)$. The Standard Model has exactly 3 generations of fermions.

The factor 3 is structurally consistent with the SM generation count under (A2). This is *[Speculation]* because: (i) the identification of Level-3 modes with SM generations is not derived; (ii) Level-3 modes are bosonic (φ_k scalars), not SM fermions; (iii) the 4 Majorana spinors of Level-3 (Paper 19 [7]) constitute a separate fermionic sector not involved in this boson tripling.

A possible interpretation: the 3 copies of Level-2 inside Level-3 represent 3 internal degrees of freedom of the massive sector that decouple when $m^2 \to 0$. Whether this admits a precise formulation within IIB-II is *[Open]* (OP 21.2 below).

Open Problem 2.12 (SU(3)-mirror symmetry and triple structure, new)**.** **(a)** Determine the physical significance of the SU(3)-mirror symmetry Level-$1|_{\mathrm{SU}(3)} =$ Level-$5|_{\mathrm{SU}(3)} = 1 \oplus 3 \oplus \bar{3}$ ((4)): does this identity imply a dynamical relation between tachyon condensation and condensate structure under IIB-II?

(b) Determine whether the triple structure Level-3 $= 3 \times$ Level-2 ((5)) admits an algebraic formulation in IIB-II analogous to the mirror symmetry: do the 3 internal copies of Level-2 inside Level-3 represent 3 distinct dynamical roles, and does the 3:1 ratio persist at higher loop order?

3. Level-2 under the de Sitter Symmetry-Breaking Subgroup S_4

3.1. **Setup and motivation.** Paper 10 [3] established (Theorem 7.3 and Remark 7.4) that the de Sitter inner metric of the Black Circle breaks the Fano symmetry $\mathrm{PSL}(2,7) \to S_4$ (order 24, the point stabiliser of $\mathrm{PG}(2,\mathbb{F}_2)$). There are 14 conjugate copies of S_4 in $\mathrm{PSL}(2,7)$, corresponding to 7 point stabilisers and 7 line stabilisers. Choosing the de Sitter inner metric is algebraically equivalent to choosing a gauge in the CSS code structure: the 14 S_4-copies correspond to the 14 stabilisers of the Steane $[[7,1,3]]$ code ([3], Remark 7.4(iii)).

The Section 2 results are stated at the $\mathrm{PSL}(2,7)$ level. The physical symmetry of the Black Circle interior is S_4. This section computes the restriction Level-$2|_{S_4} = \mathrm{irr}_6|_{S_4} \oplus \mathrm{irr}_8|_{S_4}$ via the class fusion map $S_4 \hookrightarrow \mathrm{PSL}(2,7)$ and GAP 4.15.1, and derives its physical implications.

This computation closes OP 21.3 as [Derived]*, GAP 4.15.1.*

3.2. **The S_4 character table and fusion map.** S_4 has 5 conjugacy classes with representative orders $(1,2,2,3,4)$ and sizes $(1,6,3,8,6)$ [13]:

Irrep	dim	1A	2A	2B	3A	4A	Description
irr_1	1	1	1	1	1	1	trivial
$\mathrm{irr}_{1'}$	1	1	-1	1	1	-1	sign
irr_2	2	2	0	2	-1	0	via $S_4/V_4 \cong S_3$
irr_3	3	3	1	-1	0	-1	standard
$\mathrm{irr}_{3'}$	3	3	-1	-1	0	1	standard$\otimes$sign

Fusion map. $\mathrm{PSL}(2,7)$ has exactly one conjugacy class of involutions (class 2A, 21 elements). Both S_4-classes of involutions — transpositions 2A (order 2, 6 elements) and double transpositions 2B (order 2, 3 elements) — fuse into $\mathrm{PSL}(2,7)$-class 2A. The remaining classes fuse unambiguously: S_4-3A $\to$ $\mathrm{PSL}(2,7)$-3A, S_4-4A $\to$ $\mathrm{PSL}(2,7)$-4A.

Remark 3.1 (*[Derived]*)**.** Both conjugacy classes of $S_4 \subset \mathrm{PSL}(2,7)$ (point-stabilisers and line-stabilisers, related by the outer automorphism of $\mathrm{PSL}(2,7)$) give identical restrictions for irr_6 and irr_8, since the outer automorphism fixes all real representations of $\mathrm{PSL}(2,7)$. The choice of representative S_4-class in the computation does not affect the result.

3.3. **Decomposition theorem.**

Proposition 3.2 (*[Derived]*, GAP 4.15.1 [10])**.**

$$\mathrm{irr}_6\big|_{S_4} = \mathrm{irr}_1 \oplus \mathrm{irr}_2 \oplus \mathrm{irr}_3, \tag{6}$$

$$\mathrm{irr}_8\big|_{S_4} = \mathrm{irr}_2 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_{3'}, \tag{7}$$

$$\text{Level-2}\big|_{S_4} = \mathrm{irr}_1 \oplus 2\,\mathrm{irr}_2 \oplus 2\,\mathrm{irr}_3 \oplus \mathrm{irr}_{3'}. \tag{8}$$

Dimension check: $1 + 4 + 6 + 3 = 14$. ✓

Proof. Using the S_4-character table above and the fusion map, restricted characters are obtained by pulling back along the class fusion: $\chi(\mathrm{irr}_6|_{S_4}) = (6, 2, 2, 0, 0)$ (in S_4-order 1A, 2A, 2B, 3A, 4A); $\chi(\mathrm{irr}_8|_{S_4}) = (8, 0, 0, -1, 0)$. Inner products against the five S_4-irreducibles give multiplicity vectors $[1, 0, 1, 1, 0]$ and $[0, 0, 1, 1, 1]$ respectively. Summing: Level-2 has multiplicity vector $[1, 0, 2, 2, 1]$. All values verified by GAP 4.15.1 (Appendix A). □

Corollary 3.3 (*[Derived]*)**.** *The sign representation* $\mathrm{irr}_{1'}$ *of* S_4 *does not appear in* Level-2$|_{S_4}$. *No Level-2 mode is odd under all odd permutations of* S_4.

Corollary 3.4 (*[Derived]*)**.** *The representation* $\mathrm{irr}_{3'} = \mathrm{irr}_3 \otimes \mathrm{irr}_{1'}$ *appears exclusively in* $\mathrm{irr}_8|_{S_4}$*, not in* $\mathrm{irr}_6|_{S_4}$. *Under assumption (A2), the three modes transforming as* $\mathrm{irr}_{3'}$ *in the de Sitter geometry are exclusively of colour-octet origin (***8***), with no colour-sextet component (***6***).*

3.4. **Physical interpretation.** The de Sitter geometry resolves the 14 massless Level-2 modes into **five S_4-multiplets**:

Multiplet	Modes	SU(3) origin (cond. (A2))	Physical character
irr_1 (singlet)	1	from irr_6	Fully S_4-symmetric; the unique sub-Planck d.o.f. at condition (C)
irr_2, copy 1	2	from irr_6	S_3-sensitive; colour-sextet
irr_2, copy 2	2	from irr_8	S_3-sensitive; colour-octet
irr_3, copy 1	3	from irr_6	Standard triplet; colour-sextet
irr_3, copy 2	3	from irr_8	Standard triplet; colour-octet
$\mathrm{irr}_{3'}$ (odd triplet)	3	from irr_8 only	Sign-sensitive; exclusively colour-octet

The two irr_2 copies and the two irr_3 copies are S_4-indistinguishable by their transformation properties but are distinguished by their SU(3) quantum numbers under (A2). The decomposition reveals a two-layer structure: a geometric stratum (S_4-multiplet) and an algebraic stratum (SU(3) content within that multiplet).

Remark 3.5 (*[Derived]*)**.** The curvature coupling ξ in S_{84} enters as $(\xi/2)R\sum_k N(\varphi_k) = (12\xi/r_s^2)\sum_k \varphi_k^2$. This is a PSL(2,7)-invariant quadratic form, hence in particular S_4-invariant, and assigns the *same mass* $m^2 = 12\xi/r_s^2$ to all 14 Level-2 modes regardless of their S_4-multiplet. The S_4 decomposition therefore provides a **second negative result** for ξ as a symmetry-breaking mass mechanism: not only does ξ fail to split the Level-2 14-fold degeneracy at the PSL(2,7) level, it also fails at the de Sitter symmetry-breaking level S_4. The maximum S_4-compatible mass pattern from any S_4-invariant mechanism is $(1, 4, 6, 3)$ — one mass per irrep type. With SU(3) labels under (A2), the maximum distinguishable pattern is $(1, 2, 2, 3, 3, 3)$. Any finer splitting requires breaking S_4.

Remark 3.6 (*[Derived]* for mass formula; *[Speculation]* for conditions (B)–(D))**.** From [4] Theorem 5.4 and Proposition 5.11, in Planck units ($G = l_{\mathrm{Pl}} = 1$):

$$m_i^2 = \sqrt{\frac{3\lambda}{2\pi}} \cdot \frac{2 + \mu_i}{r_s}. \tag{9}$$

The masses scale as r_s^{-1} and as $\lambda^{1/2}$. The masslessness of Level-2 ($2 + \mu_2 = 0$) is exact at $\xi = 0$ for all $r_s > 0$ and all $\lambda > 0$.

The minimum Black Circle is determined by which geometric condition on r_s is physically correct. Four candidates are identified (OP 4.2):

Cond.	r_s	M_{BH}	Planck scale	Origin
(A)	$2l_{\mathrm{Pl}}$	M_{Pl}	Length	Standard semiclassical GR
(B)	l_{Pl}	$M_{\mathrm{Pl}}/2$	Length	Min. de Sitter radius
(D)	$l_{\mathrm{Pl}}/1.361$	$M_{\mathrm{Pl}}/2.72$	Time	$\tau_{\mathrm{grow}} = t_{\mathrm{Pl}}$ ([6])
(C)	$l_{\mathrm{Pl}}/(2\pi)$	$M_{\mathrm{Pl}}/(4\pi)$	Length	Black Circle as 1D circle ([3], Remark 4.3)

Spectral masses at each condition ($\lambda = 0.1$, all in M_{Pl}; from (9)):

Level	(A) $r_s = 2l_{\mathrm{Pl}}$	(B) $r_s = l_{\mathrm{Pl}}$	(D) $r_s = l_{\mathrm{Pl}}/1.361$	(C) $r_s = l_{\mathrm{Pl}}/(2\pi)$
5 ($\mu = +4$)	0.810	1.145	1.336	2.870
4 ($\mu = +2$)	0.661	0.935	1.090	2.343
3 ($\mu = 0$)	0.467	0.661	0.771	1.657
2 ($\mu = -2$)	0	0	0	0
M_{BH}	1.000	0.500	0.367	0.080
m_3/M_{BH}	0.47	1.32	2.10	20.7

Self-consistency *[Derived]*: Conditions (B), (D), (C) all require $r_s \le l_{\mathrm{Pl}}$, violating the semiclassical GR approximation underlying Papers 10–18. The formula $\tau_{\mathrm{grow}} = 1.361\, r_s/c$ used to define condition (D) is itself derived under $r_s \gg l_{\mathrm{Pl}}$ ([6]); applying it at $r_s = l_{\mathrm{Pl}}/1.361$ extrapolates beyond its validity. Only condition (A) is self-consistent with the current IIB framework. Conditions (B)–(D) require an extension of IIB beyond semiclassical GR.

Crossover *[Derived]*: The transition $m_3 = M_{\mathrm{BH}}$ occurs at

$$r_s^{\mathrm{cross}} = \left(8\sqrt{\frac{3\lambda}{2\pi}}\right)^{1/3} \approx 1.203\, l_{\mathrm{Pl}} \quad (\lambda = 0.1).$$

For all $\lambda > 0$: $r_s^{\mathrm{cross}} < 2l_{\mathrm{Pl}}$ (sub-Planck for all coupling values). The "purely Level-2" regime ($m_3 > M_{\mathrm{BH}}$) is universally outside the current IIB validity.

At condition (C), $m_3/M_{\mathrm{BH}} \ge 6.6$ for all $\lambda \gtrsim 10^{-3}$ (λ-independent for practical purposes). Condition (D) ($\tau_{\mathrm{grow}} = t_{\mathrm{Pl}}$, the most IIB-internally motivated candidate) gives $m_3/M_{\mathrm{BH}} \approx 2.10$: Level-3 modes are within reach at $\approx 2\, M_{\mathrm{BH}}$.

Remark 3.7 (*[Derived]*, combining [3] and Proposition 3.2)**.** The 14 S_4-copies of [3], Remark 7.4(iii) correspond to the 14 stabilisers of the Steane [[7, 1, 3]] code. The S_4 decomposition of Level-2 gives the mode content of the "gauge-fixed" sector: $1 \oplus 2 \cdot 2 \oplus 2 \cdot 3 \oplus 3$. The irr_1 singlet is the unique mode invariant under the de Sitter geometry. Under IIB-IV *[Conjecture]*: if any single Level-2 mode is a CCC-crossing candidate, the irr_1 singlet is the natural choice — it is simultaneously (i) massless at $\xi = 0$ *[Derived]*, (ii) at the exact massless boundary of the $\mathrm{SO}(1, 4)$ complementary series ([4], Corollary 5.9) *[Derived]*, and (iii) fully S_4-symmetric *[Derived]*. The interpretation as a CCC candidate is *[Speculation]*, cond. (A2) and IIB-IV.

4. Open Problems

Open Problems 21.1 and 21.2 are stated in full in §2 (Open Problems 2.9 and 2.12 respectively). They concern the consistency of the $8 \oplus 6$ quantum numbers with IIB-III charges, and the physical significance of the SU(3)-mirror symmetry Level-$1|_{\mathrm{SU}(3)}$ = Level-$5|_{\mathrm{SU}(3)}$ together with the triple structure Level-3 $= 3 \times$ Level-2.

Open Problem 4.1 (de Sitter S_4 decomposition of Level-2 — **CLOSED**)**.** *Closed by Proposition 3.2 (§3),* [Derived]*, GAP 4.15.1.* Level-$2|_{S_4} = \mathrm{irr}_1 \oplus 2\,\mathrm{irr}_2 \oplus 2\,\mathrm{irr}_3 \oplus \mathrm{irr}_{3'}$. Maximum S_4-compatible mass pattern: $(1, 4, 6, 3)$. With additional SU(3) resolution: $(1, 2, 2, 3, 3, 3)$. OP 8.6 of [2] is updated: three mechanisms excluded (G_2-Goldstone, SM Higgs, ξ-splitting). The curvature coupling ξ remains the only unexcluded mechanism but produces uniform mass (Remark 3.5).

Open Problem 4.2 (Minimum Black Circle condition, new)**.** Determine which of four minimum conditions for the Black Circle — (A) $r_s = 2l_{\mathrm{Pl}}$ (standard semiclassical GR, *[Derived]*), (B) $r_s = l_{\mathrm{Pl}}$ (min. de Sitter radius, *[Speculation]*), (D) $r_s = l_{\mathrm{Pl}}/1.361$ ($\tau_{\mathrm{grow}} = t_{\mathrm{Pl}}$, *[Speculation]*), (C) $2\pi r_s = l_{\mathrm{Pl}}$ (Planck circumference, *[Speculation]*) — emerges from IIB first principles. The spectral masses and m_3/M_{BH} ratios at each condition are given in Remark 3.6. The "purely Level-2" regime ($m_3 > M_{\mathrm{BH}}$) requires $r_s < 1.20\,l_{\mathrm{Pl}}$ for all $\lambda > 0$, hence is universally sub-Planckian and outside the current IIB framework. *[Derived]*

The cavity bound $\lambda G M^2 < 2.14 \times 10^{-3}$ [5] is an upper bound on M for fixed λ; it provides no lower bound on r_s. *[Derived]*

Closing OP 21.4 requires an extension of IIB beyond semiclassical GR. *[Open]*

Three sub-questions: (a) Can the IIB framework sustain Black Circles with $M_{\mathrm{BH}} < M_{\mathrm{Pl}}$ without invoking external GR? (b) Is there a topological argument privileging condition (C) for an object with S^1 topology? (c) Does the S_4-singlet irr_1 of Level-2 play a distinguished role at the minimum Black Circle?

5. Conclusions

This paper has investigated whether the 14 Level-2 modes ($\mu = -2$) of $\mathrm{ZD}^*(\mathcal{A}_4)$ are the Goldstone bosons of a spontaneous breaking $G_2 \to \mathrm{PSL}(2,7)$ triggered by condensation of the Level-1 tachyon (irr_7).

The central result is negative. Three independent proofs: the adjoint of G_2 restricted to $\mathrm{PSL}(2,7)$ decomposes as $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*} \oplus \mathrm{irr}_8$, while Level-2 $= \mathrm{irr}_6 \oplus \mathrm{irr}_8$. Characters disagree at classes 2A and 4A. *[Derived]*, GAP 4.15.1

Positive SU(3) structure. Level-2 carries SU(3) quantum numbers $8 \oplus 6$ (Proposition 2.6). The SM Higgs mechanism is excluded as a mass source under (A2) (Remark 2.7). *[Derived]*, cond. (A2)

de Sitter S_4 decomposition. Level-$2|_{S_4} = \mathrm{irr}_1 \oplus 2\,\mathrm{irr}_2 \oplus 2\,\mathrm{irr}_3 \oplus \mathrm{irr}_{3'}$. The ξ-coupling assigns uniform mass to all 14 modes (third negative result for OP 8.6). *[Derived]*, GAP 4.15.1

Tripling. Level-3 $= 3\times$Level-2 as $\mathrm{PSL}(2,7)$-modules, with the factor 3 algebraically fixed by the characteristic polynomial $x^3(x+2)$. Under SU(3): Level-3 $= 3\,(8 \oplus 6)$. Generational interpretation: *[Speculation]*, cond. (A2). *[Derived]* for the algebraic fact.

What this paper has not done.

- Identified a positive mass mechanism for Level-2: G_2-Goldstone, SM Higgs, and ξ-splitting are all excluded; no positive mechanism has been found.
- Closed OP 7.5 of [2]: Remark 2.8 strengthens the obstacle — (A2) is insufficient to select the physically realised G_2 embedding.
- Resolved OP 21.4: the minimum Black Circle condition and the physical significance of the "purely Level-2" regime require a sub-Planck extension of IIB.
- Derived the tripling factor 3 from a physical principle: it follows from the polynomial $x^3(x+2)$, but why this polynomial has these multiplicities is not explained from first principles within the current framework.

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Appendix A. GAP 4.15.1 Verification

All representation-theoretic results are verified with GAP 4.15.1 [10] on AMD Threadripper 2950X (Windows 11, June 2026).

```
G  := PSL(2,7);
ct := CharacterTable(G);
irr := Irr(ct); nc := NrConjugacyClasses(ct);
pm2 := PowerMap(ct,2);

# chi_{adj(G2)} = Lambda^2(irr7) - irr7
chi7  := First(irr, chi -> chi[1] = 7);
chiL2 := List([1..nc],
          i -> (chi7[i]^2 - chi7[pm2[i]])/2);
chi14 := List([1..nc], i -> chiL2[i] - chi7[i]);
Print("chi_14: ", chi14, "\n");
decomp := MatScalarProducts(ct, irr,
            [ClassFunction(ct, chi14)]);
Print("Mults:  ", decomp[1], "\n");

# Sym^2(irr_3) = irr_6 ?
chi3   := First(irr, chi -> chi[1] = 3);
chi3s2 := List([1..nc],
           i -> (chi3[i]^2 + chi3[pm2[i]])/2);
Print("Sym^2:  ",
  MatScalarProducts(ct,irr,
    [ClassFunction(ct,chi3s2)])[1], "\n");
```

```
# GAP class order: (1A,3A,7A,7B,2A,4A)
# Standard order: (1A,2A,3A,4A,7A,7B)
chi_14: [ 14, -1, 0, 0, -2, 2 ]   # = (14,-2,-1,+2,0,0)
Mults:  [ 0, 1, 1, 0, 0, 1 ]      # irr3+irr3*+irr8
Sym^2:  [ 0, 0, 0, 1, 0, 0 ]      # pure irr6
```

```
G   := PSL(2,7);
ctG := CharacterTable(G);
irrG := Irr(ctG);
irr6G := First(irrG, chi -> chi[1] = 6);
irr8G := First(irrG, chi -> chi[1] = 8);

# S4 as maximal subgroup of order 24
H    := First(MaximalSubgroupClassReps(G),
              H -> Order(H) = 24);
ctH  := CharacterTable(H);
irrH := Irr(ctH);
fusion := FusionConjugacyClasses(ctH, ctG);
Print("Fusion: ", fusion, "\n");
```

```
# Restrict and decompose
nc_H    := NrConjugacyClasses(ctH);
irr6res := List([1..nc_H], i->irr6G[fusion[i]]);
irr8res := List([1..nc_H], i->irr8G[fusion[i]]);
lev2res := List([1..nc_H],
             i->irr6res[i]+irr8res[i]);
f := x -> MatScalarProducts(ctH,irrH,
            [ClassFunction(ctH,x)])[1];
Print("irr6|S4:    ", f(irr6res), "\n");
Print("irr8|S4:    ", f(irr8res), "\n");
Print("Level-2|S4: ", f(lev2res), "\n");
Print("S4 dims:    ",
      List(irrH,chi->chi[1]), "\n");
```

```
# S4 class order: (1A,3A,2A,4A,2B), sizes [1,8,6,6,3]
# Fusion: S4-2A and S4-2B both -> PSL-2A (one inv. class)
Fusion:     [ 1, 2, 5, 6, 5 ]
irr6|S4:    [ 1, 0, 1, 1, 0 ]  # irr1+irr2+irr3
irr8|S4:    [ 0, 0, 1, 1, 1 ]  # irr2+irr3+irr3'
Level-2|S4: [ 1, 0, 2, 2, 1 ]  # irr1+2irr2+2irr3+irr3'
S4 dims:    [ 1, 1, 2, 3, 3 ]
# Dim check: 1+0+2*2+2*3+3 = 14  OK
```

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Independent Researcher, Modena, Italy
Email address: giovanni.levratti.research@gmail.com