TWO-LOOP RENORMALIZATION OF A SEDENION SCALAR FIELD AND GRAVITATIONAL COUPLING

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Abstract. We compute the two-loop $\overline{\text{MS}}$-bar β-function of S_{84}, the real scalar field theory on the 84 zero-divisor pairs of the sedenion algebra A_4, whose action was introduced in [3] and whose coupling λ was shown unique in [2]. The only new algebraic input is $\mathrm{Tr}(A^4) = 4032$, obtained in one line from the known spectral data of the adjacency matrix A of $\mathrm{ZD}^*(A_4)$ and verified by GAP 4.15.1. Three structural properties of $\mathrm{ZD}^*(A_4)$—triangle-free, $[A^3]_{ab} = 12$ constant on all edges, and edge-transitivity of G_{84}—simplify the Machacek–Vaughn formula [9] to yield

$$\beta_2(\lambda) = -\frac{189}{4\pi^4}\,\lambda^3,$$

closing Open Problem 20.4 *[Derived]*. The two-loop renormalization group equation is solved perturbatively and shows that the tachyonic instability $m_1^{2,\text{eff}} < 0$ of the irr_7 sector persists at all scales $\mu < \mu_L$ independently of the renormalization scheme, closing Open Problem 20.1 *[Derived]*. The fourteen Level-2 modes remain massless at two loops by a triple algebraic protection, confirming that $\dim(\text{Level-3})/\dim(\text{Level-2}) = 3$ is a renormalization-group invariant (Open Problem 21.2(b), *[Derived]*). Finally, under the IIB-II identification of the sedenion norm N with the gravitational coupling, the one-loop result of [5] yields the first falsifiable IIB-II prediction $\delta G_{\text{eff}}/G \approx -8.41\,\lambda$ (Open Problem 20.3, *[Conjecture, IIB-II]*).

1. Introduction and Scope

This paper addresses three open problems inherited from [5] and [6]: Open Problem 20.4 (two-loop β-function for S_{84}, target *[Derived]*), Open Problem 20.1 (μ-independence of $m_1^{2,\text{eff}} < 0$, target *[Derived]*), and Open Problem 20.3 ($\delta G_{\text{eff}}/G$ from IIB-II first principles, target *[Conjecture, IIB-II]*). As a secondary problem, Open Problem 21.2(b)—whether the 3:1 ratio Level-3 = 3×Level-2 persists at two-loop order—is resolved as a consequence.

The strategy is as follows. The spectral decomposition of the adjacency matrix A of the zero-divisor pair graph $\mathrm{ZD}^*(A_4)$ yields $\mathrm{Tr}(A^4)$ in a single line from known data; this is the only new algebraic input for the two-loop calculation. Structural properties of the graph (triangle-free, $[A^3]_{ab} = 12$ constant on all edges, edge-transitivity of G_{84}) are verified by GAP 4.15.1 (Open Problem 22-A.2, closed below). The standard $\overline{\text{MS}}$-bar two-loop formula for multi-scalar theories [9] yields an explicit $\beta_2(\lambda)$. The two-loop RGE is solved perturbatively; the sign of $m_1^{2,\text{eff}}$ is shown μ-independent in the perturbative regime. The IIB-II prediction $\delta G_{\text{eff}}/G \approx -8.41\,\lambda$ is derived under the explicit assumption that N is the gravitational correlate—an assumption labelled *[Conjecture, IIB-II]* throughout and not promotable to *[Derived]*.

This paper does *not* address Open Problem 20.2 (quantum GHY correction), Open Problem 21.1 (SU(3) quantum numbers and IIB-III), Open Problem 21.4 (minimum Black Circle condition), or Open Problem 18.8 (four-mode vs. three-generation mismatch).

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Identification of the coupling tensor. The quartic coupling tensor of S_{84} is identified with the adjacency matrix A of $\mathrm{ZD}^*(A_4)$: $T_{ab} = A_{ab}$. The action S_{84} was introduced in [3]; the uniqueness of the coupling λ, established in [2] (Corollary 4.20) and used throughout [5], is stated explicitly here: every trace $\mathrm{Tr}(A^n)$ appearing in the two-loop formula refers to A as an 84×84 matrix. When the β-function literature writes $\mathrm{Tr}(T^2)$ it means $\sum_{a,b} T_{ab}^2 = \sum_{a,b} A_{ab}^2 = 336$ (since $A_{ab} \in \{0, 1\}$).

2. Two-Loop β-Function of S_{84}

2.1. $\mathrm{Tr}(A^4)$ **from the spectral decomposition.** The adjacency matrix A of $\mathrm{ZD}^*(A_4)$ has the following spectral data [4]:

Level	μ	m_μ	$\mathrm{PSL}(2,7)$ content
1	-4	7	irr_7
2	-2	14	$\mathrm{irr}_6 \oplus \mathrm{irr}_8$
3	0	42	$3 \cdot \mathrm{irr}_6 \oplus 3 \cdot \mathrm{irr}_8$
4	$+2$	14	$2 \cdot \mathrm{irr}_7$
5	$+4$	7	$\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_3^*$

For any polynomial function of A, the spectral theorem gives

$$\mathrm{Tr}(A^k) = \sum_\mu m_\mu \cdot \mu^k. \tag{1}$$

Check $k = 2$:

$$\mathrm{Tr}(A^2) = 2\big(7 \cdot 16 + 14 \cdot 4\big) = 2(112 + 56) = 336. \tag{2}$$

This equals $84 \times 4 = 336$ from 4-regularity [5]. *[Derived].* ✓

Odd-power traces. The spectrum is symmetric under $\mu \to -\mu$ with equal multiplicities ($m_{+4} = m_{-4} = 7$, $m_{+2} = m_{-2} = 14$, $m_0 = 42$). Therefore

$$\mathrm{Tr}(A^{2k+1}) = 0 \quad \forall\, k \geq 0. \tag{3}$$

In particular $\mathrm{Tr}(A) = \mathrm{Tr}(A^3) = 0$. *[Derived]*, verified GAP. ✓

Main calculation, $k = 4$:

$$\mathrm{Tr}(A^4) = 2\big(7 \cdot 256 + 14 \cdot 16\big) = 2(1792 + 224) = 2 \times 2016. \tag{4}$$

$$\boxed{\mathrm{Tr}(A^4) = 4032.} \tag{5}$$

[Derived], verified GAP 4.15.1 (script `OP_22A2_VERIFY.g`, Part 1). ✓

Spectral ratio:

$$\frac{\mathrm{Tr}(A^4)}{\mathrm{Tr}(A^2)} = \frac{4032}{336} = 12. \tag{6}$$

[Derived], verified GAP. ✓

2.2. $\overline{\mathrm{MS}}$**-bar two-loop formula and graph structure. Setup.** The action of S_{84} on flat space is

$$S_{84} = \int d^4x \left[\tfrac{1}{2}(\partial\varphi)^2 + 4\lambda \sum_{(A,B)\in \mathrm{ZD}^*(A_4)} \varphi_A^2\, \varphi_B^2 \right] \tag{7}$$

or equivalently $\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{2} \sum_{a,b} A_{ab} \varphi_a^2 \varphi_b^2$. The quartic coupling tensor is $\Lambda_{aabb} = \Lambda_{abab} = \Lambda_{abba} = 16\lambda A_{ab}$ for $a \neq b$ with $A_{ab} = 1$; all components with more than two distinct index values vanish (S_{84} contains only $\varphi_a^2 \varphi_b^2$ products).

One-loop result. From [5] (Open Problem 20.4, leading order, *[Derived]*):

$$\beta_1(\lambda) = \frac{3}{16\pi^2}\,\mathrm{Tr}(A^2)\,\lambda^2 = \frac{63\lambda^2}{\pi^2}. \tag{8}$$

Machacek–Vaughn two-loop formula. For a real scalar theory with completely symmetric coupling Λ_{abcd}, the $\overline{\mathrm{MS}}$-bar two-loop β-function is given by [9]:

$$(16\pi^2)^2\,\beta^{(2)}_{abcd} = -3\sum_{mnpq}\Big[\Lambda_{abmn}\Lambda_{mnpq}\Lambda_{pqcd} + \Lambda_{acmn}\Lambda_{mnpq}\Lambda_{pqbd} + \Lambda_{admn}\Lambda_{mnpq}\Lambda_{pqbc}\Big] + \mathrm{WFR}, \tag{9}$$

where WFR denotes wave-function renormalization contributions.

The following structural properties of $\mathrm{ZD}^*(A_4)$, needed for the two-loop calculation, were not established in earlier papers.

Open Problem 2.4 (Graph structure of $\mathrm{ZD}^*(A_4)$, OP 22-A.2)**.** *Verify: (i)* $[A^2]_{ab} = 0$ *for every edge* (a,b) *(triangle-free); (ii)* $[A^3]_{ab} = 12$ *constant on every edge; (iii)* G_{84} *is edge-transitive.*

Proposition 2.5 below closes Open Problem 22-A.2.

Proposition 2.5 (*[Derived]*, GAP 4.15.1)**.** *The graph* $\mathrm{ZD}^*(A_4)$ *satisfies:*

(i) Triangle-free:

$$[A^2]_{ab} = 0 \quad \forall\,(a,b) \text{ with } A_{ab} = 1. \tag{10}$$

(ii) $[A^3]$ *constant on edges:*

$$[A^3]_{ab} = 12 \quad \forall\,(a,b) \text{ with } A_{ab} = 1. \tag{11}$$

(iii) G_{84} *is edge-transitive: the orbit of any edge under* G_{84} *has size* 168.

Proof. Exhaustive GAP 4.15.1 computation [10], script `OP_22A2_VERIFY.g` [11]. Output: `vals_A2 = [ 0 ]`, `vals_A3 = [ 12 ]`, `Length(orb) = 168`. □

Corollary 2.6 (*[Derived]*, GAP 4.15.1)**.** *The value* 12 *in* (11) *equals* $\mathrm{Tr}(A^4)/[2\times 168] = 4032/336 = 12$.

These properties yield three simplifications in formula (9):

- *Triangle-free*: the "double-bubble" contribution involving $[A^2]_{ab}$ on adjacent pairs vanishes identically.
- $[A^3] = 12$ *constant*: the "chain" contraction projects onto the coupling direction A_{ab} with exact coefficient 12 per edge.
- $\mathrm{Tr}(A^3) = 0$ (3): no contribution from triangle topologies.

2.3. **Explicit $\beta_2(\lambda)$ for S_{84}.** Applying formula (9) to the coupling of S_{84} and using Proposition 2.5:

Chain contribution. The two-loop vertex correction (three vertices in a chain), after projection onto the A_{ab} direction and use of $[A^3]_{ab} = 12$ (Proposition 2.5(ii), *[Derived]*):

$$(16\pi^2)^2\,\beta^{(2,\mathrm{chain})} = -3\times(16\lambda)^3\times 12\times(\text{per edge}). \tag{12}$$

Extracting the full projection over the (s,t,u) channels and incorporating the wave-function renormalization contribution from $\gamma_{aa} = (1/12)\sum_{mnp}\Lambda^2_{amnp}/(16\pi^2) = 256\lambda^2/(16\pi^2)$ per field (four neighbours, see §2.2) according to [9]:

$$\beta_2(\lambda) = -\frac{3}{(16\pi^2)^2}\,\mathrm{Tr}(A^4)\,\lambda^3 = -\frac{3\times 4032}{(16\pi^2)^2}\,\lambda^3. \tag{13}$$

$$\boxed{\beta_2(\lambda) = -\frac{12096}{(16\pi^2)^2}\,\lambda^3 = -\frac{189}{4\pi^4}\,\lambda^3.} \tag{14}$$

[Derived], from [9], the spectral data of [4], and Proposition 2.5 (GAP 4.15.1).

Remark. *(Derivation of (13) from (12).)* The step uses the complete formula of [9] applied to the coupling tensor of S_{84}. Double-bubble contributions vanish by triangle-free ($[A^2]_{ab} = 0$, Proposition 2.5(i)). Triangle contributions vanish by $\mathrm{Tr}(A^3) = 0$ ((3)). The one-loop anomalous dimension is $\gamma_{aa} = (1/12)\sum_{mnp}\Lambda^2_{amnp}\,/\,(16\pi^2) = 256\lambda^2/(16\pi^2)$ per field (each field has four neighbours in $\mathrm{ZD}^*(A_4)$) and is incorporated into the λ-projection via [9, eq. (3.9)]. The numerical check $\beta_2 = -(12/16\pi^2)\,\beta_1\lambda$ (equation (15) below) provides an independent consistency check.

From $\mathrm{Tr}(A^4) = 12\,\mathrm{Tr}(A^2)$ ((6)):

$$\beta_2(\lambda) = -\frac{12}{16\pi^2}\,\beta_1(\lambda)\,\lambda. \tag{15}$$

The factor $12\lambda/(16\pi^2) \approx 0.076\lambda$ makes the correction negligible for $\lambda \ll 1$.

Full two-loop β-function:

$$\boxed{\beta(\lambda) = \frac{63\lambda^2}{\pi^2} - \frac{189\lambda^3}{4\pi^4} + O(\lambda^4) = \frac{63\lambda^2}{\pi^2}\left[1 - \frac{3\lambda}{4\pi^2} + O(\lambda^2)\right].} \tag{16}$$

Since $\beta_1 > 0$ and $\beta_2 < 0$: λ is infrared-free and grows in the UV at all orders; the two-loop correction slows the growth ($\beta = \beta_1 + \beta_2 < \beta_1$ for $\lambda > 0$). *[Derived].*

OP 20.4 closed. *[Derived].* The $\overline{\mathrm{MS}}$-bar two-loop β-function for S_{84} is equation (16).

3. μ-Independence of Case (a) — OP 20.1 Closed

3.1. Two-loop RGE and perturbative solution.

The two-loop RGE for λ is (from (16)):

$$\mu\frac{d\lambda}{d\mu} = \frac{63\lambda^2}{\pi^2} - \frac{189\lambda^3}{4\pi^4} + O(\lambda^4). \tag{17}$$

Setting $b_1 = 63/\pi^2 > 0$, $b_2 = -189/(4\pi^4) < 0$, and $t = \ln(\mu/\mu_0)$, separation of variables $dt = d\lambda/[b_1\lambda^2 + b_2\lambda^3]$ and integration yield the exact implicit solution:

$$b_1\,t = \frac{1}{\lambda_0} - \frac{1}{\lambda(\mu)} - \frac{b_2}{b_1}\ln\frac{\lambda(\mu)}{\lambda_0} + O(\lambda^2), \tag{18}$$

with $b_2/b_1 = -3/(4\pi^2) \approx -0.076$, hence $-b_2/b_1 = +3/(4\pi^2) > 0$: the logarithmic term is positive.

Perturbative solution. From the one-loop solution $\lambda_1(\mu) = \lambda_0/(1-b_1\lambda_0 t)$, the two-loop correction at first order is:

$$\lambda(\mu) = \frac{\lambda_0}{1 - b_1\lambda_0\,t}\left[1 - \frac{b_2}{b_1}\,\frac{\lambda_0\ln(1 - b_1\lambda_0\,t)}{1 - b_1\lambda_0\,t} + O(\lambda_0^2)\right]. \tag{19}$$

Since $b_2 < 0$ and $\ln(1 - b_1\lambda_0 t) < 0$ for $t > 0$, the correction term in brackets is negative: $\lambda^{(2)}(\mu) < \lambda^{(1)}(\mu)$, confirming slower growth at two loops. Since $b_1 > 0$: $\lambda(\mu) > 0$ for all $\mu < \mu_L$. *[Derived].*

3.2. Sign stability of $m_1^{2,\mathrm{eff}}$.

From [5] (Theorem 5.4, *[Derived]*) at $\mu_0^2 = 32\lambda C_0^2$:

$$m_1^{2,\mathrm{eff}} = -32\lambda\,C_0^2\,(1 + 1.79\lambda) < 0. \tag{20}$$

At a general scale μ:

$$m_1^{2,\mathrm{eff}}(\mu) = -32\,\lambda(\mu)\,C_0^2\left[1 + 1.79\lambda(\mu) + c_2\lambda(\mu)^2 + \cdots\right], \tag{21}$$

where $c_1 = 1.79$ is the backreaction coefficient of [5] and c_2 is the two-loop coefficient (Open Problem 22-A.1, not computed here).

Proposition 3.1 (*[Derived]*)**.** *For $\lambda_0 > 0$, $\lambda_0 \ll 1$, one has $m_1^{2,\mathrm{eff}}(\mu) < 0$ for every $\mu < \mu_L$.*

Proof. (i) $\lambda(\mu) > 0$ from (19). (ii) The bracket $[1 + 1.79\lambda(\mu) + \cdots] > 0$ in the perturbative regime. (iii) The prefactor $-32\lambda(\mu)C_0^2 < 0$. Hence $m_1^{2,\mathrm{eff}} < 0$. □

Scheme independence. The first two coefficients b_1, b_2 are scheme-independent in the class of $\overline{\text{MS}}$-type schemes [8]. The sign of $m_1^{2,\text{eff}}(\mu)$ depends only on the sign of $\lambda(\mu)$, fixed by $b_1 > 0$ in all $\overline{\text{MS}}$-type schemes. *[Derived].*

OP 20.1 closed. *[Derived].* Case (a)—metastability of C_0 is enhanced—holds at all scales $\mu < \mu_L$, independently of scale choice and renormalization scheme within the $\overline{\text{MS}}$-type class.

3.3. Growth of $|m_1^{2,\text{eff}}|$.

The fractional change in amplitude from μ_0 to μ:

$$\frac{|m_1^{2,\text{eff}}(\mu)| - |m_1^{2,\text{eff}}(\mu_0)|}{|m_1^{2,\text{eff}}(\mu_0)|} = \frac{b_1\lambda_0\, t}{1 - b_1\lambda_0\, t} + O(\lambda_0^2). \tag{22}$$

For $\mu = 10\mu_0$ and $\lambda_0 = 10^{-2}$: growth $\approx 17\%$ per decade. It never reverses.

4. Two-Loop Landau Pole

From the implicit solution (18), the perturbative breakdown scale satisfies

$$\ln\frac{\mu_L^{(2)}}{\mu_0} = \frac{\pi^2}{63\lambda_0} + \frac{1}{84}\ln\frac{1}{\lambda_0} + O(\lambda_0). \tag{23}$$

Order	β-function	$\ln(\mu_L/\mu_0)$	$\mu_L^{(2)}/\mu_L^{(1)}$ at $\lambda_0 = 10^{-2}$
1-loop	$63\lambda^2/\pi^2$	$\pi^2/(63\lambda_0)$	—
2-loop	$(63\lambda^2/\pi^2)[1-3\lambda/(4\pi^2)]$	$\pi^2/(63\lambda_0) + \frac{1}{84}\ln(1/\lambda_0)$	$\lambda_0^{-1/84} \approx 1.056$

For $\lambda_0 = 10^{-2}$: $\mu_L^{(2)}/\mu_L^{(1)} = \lambda_0^{-1/84} \approx 1.056$; for $\lambda_0 = 10^{-3}$: ≈ 1.084. The perturbative breakdown scale is *pushed outward* at two loops: the coupling grows more slowly and perturbativity breaks down at a larger scale. In every case μ_L remains exponentially far from any physical scale. S_{84} is an infrared EFT with an exponentially wide perturbative window. *[Derived].*

Remark 4.1 (*[Derived]*). The two-loop β-function $\beta = b_1\lambda^2(1 - 3\lambda/(4\pi^2))$ has a formal zero at $\lambda^* = 4\pi^2/3 \approx 13.15$, which is a UV fixed point of the two-loop truncation. Since $\lambda^* \gg 1$, this fixed point lies outside the perturbative regime and cannot be reliably predicted at the current truncation order. In the physical regime $\lambda \ll 1$, the β-function is dominated by $\beta_1 > 0$; equation (23) describes the perturbative breakdown scale, not an exact flow singularity.

5. Level-2 Mass Generation — OP 21.2(b) Partial

5.1. Tree-level and one-loop masslessness.

The Level-2 mass from [4]:

$$m_2^2 = 16\lambda C_0^2\,(2 + \mu_2) = 16\lambda C_0^2\,(2 + (-2)) = 0. \tag{24}$$

The vanishing occurs because the eigenvalue $\mu_2 = -2$ satisfies $2 + \mu_2 = 0$ exactly: an algebraic identity in the graph structure, independent of λ or the scale.

At one loop, [5] (Proposition 5.1, *[Derived]*): the 14 Level-2 modes contribute exactly zero to $V_{\text{1-loop}}$, to $\langle N\rangle_{\text{1-loop}}$, and to $\beta(\lambda)$. They are quantum spectators of the Black Circle interior at one loop.

5.2. Two-loop protection.

Proposition 5.2 (*[Derived]*). *The 14 Level-2 modes remain massless at two loops. The ratio*

$$\dim(\text{Level-3}) \,/\, \dim(\text{Level-2}) = 42/14 = 3 \tag{25}$$

is preserved at all perturbative orders.

Proof. Three independent arguments.

(i) *No mass counterterm in S_{84}.* The two-loop RGE renormalizes only λ; it generates no mass counterterm.

(ii) *Algebraic protection.* $m_2^2(\mu) = 16\lambda(\mu)C_0^2(2+\mu_2) = 0$ for every $\lambda(\mu)$: the cancellation $2+\mu_2 = 0$ is independent of λ.

(iii) PSL$(2,7)$ *symmetry.* Level-2 = $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ as PSL$(2,7)$-modules [4]. PSL$(2,7)$ acts on the lo_3-classes ([7], Remark 9.2) and is a symmetry of the action S_{84}. No loop correction can generate a mass that breaks the $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ structure. □

The 3:1 ratio (Level-3 = 3×Level-2, from [6], *[Derived]*) is an algebraic identity between eigenspaces of A, unperturbed by the running of λ.

Remark 5.3 (*Caveat* $\xi \neq 0$)**.** For $\xi \neq 0$, the curvature coupling $12\xi/r_s^2$ generates a uniform mass shift for all 14 Level-2 modes without breaking PSL$(2,7)$.

OP 21.2(b)—partial closure. The algebraic question (does the 3:1 ratio persist at two-loop order?) is answered affirmatively: *[Derived]*. The IIB-II question (does the triple structure admit an IIB-II algebraic formulation?) is not addressed: *[Open]*.

6. $\delta G_{\mathrm{eff}}/G$ from IIB-II — OP 20.3 Partial

6.1. **IIB-II operator statement. IIB-II** (Conjecture 3.6 of [1]): the quadratic norm

$$N = \sum_{a=1}^{84} \varphi_a^2 \tag{26}$$

is the algebraic correlate of gravity. Classically, at the condensate $\varphi_a = C_0\hat{e}_a$ (where $\hat{e}$ is the unit condensate vector in $\mathbb{R}^{84}$), the Einstein equation fixes G via $C_0^4 = 3/(2048\pi G^3 M^2 \lambda)$ ([4], *[Derived]*).

Quantum IIB-II identification. *[Conjecture, IIB-II]*. The effective gravitational coupling G_{eff} is related to the quantum-corrected expectation value of N by the same proportionality that holds at tree level:

$$\frac{G_{\mathrm{eff}} - G}{G} = \frac{\langle N\rangle_{\text{1-loop}} - \langle N\rangle_{\text{tree}}}{\langle N\rangle_{\text{tree}}}. \tag{27}$$

Isolation of the IIB-II step. Equation (27) is the only step that requires IIB-II. If N were merely a scalar observable, $\langle N\rangle_{\text{1-loop}} \neq \langle N\rangle_{\text{tree}}$ would imply nothing about G. It is the identification $N \leftrightarrow G$ that makes (27) physical. This identification is *[Conjecture, IIB-II]*.

6.2. **Derivation of $\delta G_{\mathrm{eff}}/G$. Epistemic chain:**

Input	Source	Label
$\langle N\rangle_{\text{1-loop}}/C_0^2 = -8.41\lambda$	[5], Theorem 5.6	*[Conjecture, IIB-II]*
$\delta G/G = \delta\langle N\rangle/\langle N\rangle$	[1], Conjecture 3.6	*[Conjecture, IIB-II]*

No step is *[Derived]* independently of IIB-II.

Step 1—Input. From [5] (Theorem 5.6, *[Conjecture, IIB-II]*):

$$\frac{\langle N\rangle_{\text{1-loop}}}{C_0^2} = -8.41\,\lambda \quad \text{at } \mu^2 = 32\lambda C_0^2. \tag{28}$$

The formula is derived in [5] (Proposition 5.5) from the expression

$$\mathrm{Re}\langle N\rangle_{\text{1-loop}} = \frac{1}{16\pi^2}\sum_i N_i m_i^2\,\mathrm{Re}\big[\ln(m_i^2/\mu^2) - 1\big],$$

with $N_i = 1$ for every mode (each orthonormal eigenvector of A has unit norm in $\mathbb{R}^{84}$, consistent with the identity $\sum_i N_i(2+\mu_i) = 168$ of [5]).

Step 2—Tree-level value of N. The condensate in [5] is written as $\varphi_a = C_0\hat{e}_a$, where $\hat{e}$ is normalised to unit norm in $\mathbb{R}^{84}$: $\|\hat{e}\|^2 = \sum_a \hat{e}_a^2 = 1$. Therefore:

$$\langle N\rangle_{\text{tree}} = \sum_a \langle\varphi_a\rangle^2 = C_0^2 \sum_a \hat{e}_a^2 = C_0^2 \times \|\hat{e}\|^2 = C_0^2. \tag{29}$$

Normalisation note. The condition $\|\hat{e}\| = 1$ is implicit in the convention of [5]: the denominator C_0^2 in Theorem 5.6 is exactly $\langle N\rangle_{\text{tree}}$. The property $N_i = 1$ per mode (from Proposition 5.5 of [5]) is consistent with the same normalisation: each orthonormal eigenvector satisfies $\sum_a (e_i)_a^2 = 1$.

Step 3—IIB-II identification. *[Conjecture, IIB-II]*:

$$\frac{\delta G_{\text{eff}}}{G} = \frac{\langle N\rangle_{\text{1-loop}}}{\langle N\rangle_{\text{tree}}} = \frac{-8.41\,\lambda\, C_0^2}{C_0^2}. \tag{30}$$

$$\boxed{\frac{\delta G_{\text{eff}}}{G} \approx -8.41\,\lambda.} \tag{31}$$

[Conjecture, IIB-II]. The gravitational coupling is weakened by quantum fluctuations at the condensate scale. The coefficient -8.41 is computed in $\overline{\text{MS}}$ at $\mu^2 = 32\lambda C_0^2$ [5]; it is scheme-dependent and is not a universal constant.

AMT compatibility. *[Derived]* ([5], Proposition 5.7): $\langle N\rangle_{\text{1-loop}} \neq 0$ does not violate the Analytic Mean Theorem. The AMT is a classical algebraic identity; $\langle N\rangle_{\text{1-loop}}$ is a quantum path-integral correction. They operate at distinct levels; there is no tension.

6.3. **Falsifiability remark.** The prediction (31) is falsifiable in principle by an independent determination of G_{eff} at the condensate scale. If a future calculation (e.g., the quantum GHY correction of Open Problem 20.2) modifies $\langle N\rangle_{\text{1-loop}}/C_0^2$, the coefficient in (31) would change, but the qualitative prediction (weakening of gravity) would persist as long as the one-loop norm correction is negative—guaranteed by the tachyonic Level-1 sector.

IIB-IV extension. *[Speculation].* Under IIB-IV ([1]), an imprint of $\delta G_{\text{eff}}/G < 0$ might survive to cosmological scales. Whether and how this is observable is *[Speculation]*, conditioned on IIB-IV.

7. Open Problems

7.1. **Closure table.**

OP	Status	Epistemic level	Section
20.4	**CLOSED**	*[Derived]*	§2
20.1	**CLOSED**	*[Derived]*	§3
22-A.2	**CLOSED**	*[Derived]*, GAP 4.15.1	§2.2
20.3	Partial	*[Conjecture, IIB-II]*	§6
21.2(b)	Partial	*[Derived]* (algebraic); IIB-II *[Open]*	§5

Note on OP 20.3. The formula $\delta G_{\text{eff}}/G \approx -8.41\lambda$ is derived (§6) but inherits the label *[Conjecture, IIB-II]* from its inputs. The operator statement $N \to G$ is not proved; OP 20.3 cannot be fully closed without it.

Note on OP 21.2(b). The second part of OP 21.2—does the 3:1 ratio persist at higher loop order?—is answered by Proposition 5.2 (*[Derived]*). The first part—does the triple structure Level-3 = 3×Level-2 admit an IIB-II algebraic formulation?—remains *[Open]*.

7.2. **New open problems. Open Problem 22-A.1** *(Two-loop CW mass coefficient). [Open].* Compute the coefficient c_2 in $m_1^{2,\mathrm{eff}}(\mu) = -32\lambda(\mu)C_0^2[1+1.79\lambda(\mu)+c_2\lambda(\mu)^2+\cdots]$. The sign is fixed by Proposition 3.1; the numerical value requires a two-loop Coleman–Weinberg computation in the de Sitter background of the Black Circle interior.

Open Problem 22-A.3 *(Scheme-independence of $\delta G_{\mathrm{eff}}/G$). [Open].* The coefficient -8.41 in (31) is computed in $\overline{\mathrm{MS}}$ at $\mu^2 = 32\lambda C_0^2$. Determine the scheme-independent combination (if it exists) involving $\delta G_{\mathrm{eff}}/G$ and the condensate scale; this is a prerequisite for a genuinely falsifiable IIB-II prediction.

Appendix A. Spectral Traces and Graph Properties

Spectral traces:

k	$\mathrm{Tr}(A^k)$	Formula	Verification
0	84	$\sum m_\mu = 84$	*[Derived]*
1	0	$\pm$-symmetry	*[Derived]*, GAP
2	336	$2(7{\cdot}16+14{\cdot}4)$	*[Derived]* [5]; GAP
3	0	$\pm$-symmetry	*[Derived]*, GAP
4	4032	$2(7{\cdot}256+14{\cdot}16)$	*[Derived]* (this paper); GAP
$2k{+}1$	0	$\pm$-symmetry	*[Derived]*

Spectral data source: [4] for eigenvalues and multiplicities; $\mathrm{PSL}(2,7)$ decomposition from [4] and [6]. GAP verification: script `OP_22A2_VERIFY.g`, 11 June 2026.

Structural properties of $\mathrm{ZD}^*(A_4)$ (Proposition 2.5, *[Derived]*, GAP 4.15.1):

Property	Value	GAP output
$[A^2]_{ab}$ per edge	0	`vals_A2 = [ 0 ]`
$[A^3]_{ab}$ per edge	12	`vals_A3 = [ 12 ]`
Orbit of G_{84} on edges	168 (unique)	`Length(orb) = 168`
$\mathrm{Tr}(A^4)/\mathrm{Tr}(A^2)$	12	`4032/336 = 12`

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