FANO INCIDENCE AND THE LEVEL-3 TRIPLING

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ABSTRACT. Four distinct occurrences of the integer 3 have been observed in the series of papers on zero-divisors in Cayley–Dickson algebras: (1) Furey's three Standard Model generations from Cl(6) at $n = 3$; (2) the factor 3 in $|G_{ef}^{(n)}| = 3 \times 2^{2^{n-1}+(n-4)}$, arising from three orbits of $C_3 \le G_{class}$ on C_{ef} axis-class blocks; (3) the tension between the four Majorana spinors of the 4_s-sector and the three SM generations; (4) the exact identity Level-3 $= 3 \times$ Level-2 as $\mathrm{PSL}(2,7)$-modules, with the factor 3 fixed by the characteristic polynomial $x^3(x+2)$ of the ZD pair graph adjacency matrix.

This paper proves that Occurrences 2 and 4 share the same algebraic origin: the Fano incidence parameter 3 = (lines through a point in $PG(2, \mathbb{F}_2)$). The ZD axis class for each Fano axis direction v decomposes into three Fano-line blocks $\{B_1, B_2, B_3\}$, one per Fano line through v. The antisymmetric block indicators a_1, a_2, a_3 project to three linearly independent directions in Level-3 *[Derived, GAP 4.15.1]*.

Occurrences 1 and 3 are shown to be algebraically independent of this Fano parameter: Furey's 3 is an $n = 3$ phenomenon with no direct map to the $n = 4$ Fano-block structure, and the tensor 4 of the 4_s-sector comes from 2^{n-2} with $n = 4$, not from the incidence number 3.

The π-antisymmetry of Level-3 is established analytically: the global sign-reversal $\pi\colon (p, s, q) \mapsto (p, -s, q)$ is a graph automorphism of $\mathrm{ZD}^*(\mathcal{A}_4)$ *[Derived]*, and every zero-eigenvector of the adjacency matrix is π-antisymmetric *[Derived, GAP 4.15.1]*. Three open problems are stated.

Epistemic key. [Derived] — follows from Papers 1–21 and established mathematics. *[Conjecture]* — precise statement, assumptions explicit. *[Speculation]* — structurally motivated, not yet falsifiable. *[Open]* — obstacle stated precisely.

CONTENTS

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1. Introduction

1.1. **Background and motivation.** The series of papers [1, 3] on zero-divisors in Cayley–Dickson algebras has established a precise algebraic framework for the sedenion zero-divisor pair graph $\mathrm{ZD}^*(\mathcal{A}_4)$ (84 vertices, 168 edges, $\mathrm{PSL}(2,7)$ symmetry). Four occurrences of the integer 3 appear in this framework, all derived or observed independently.

- **Occurrence 1** (external, $n = 3$): Furey [8] derives three Standard Model generations from minimal left ideals of $\mathrm{Cl}(6) \cong M_8(\mathbb{C})$ at the octonionic level $n = 3$.
- **Occurrence 2** ($n \geq 4$): The effective automorphism group $G_{ef}^{(n)}$ satisfies $|G_{ef}^{(n)}| = 3 \times 2^{2^{n-1}+(n-4)}$, where the factor 3 is the number of orbits of $C_3 \leq G_{class}$ on the C_{ef} axis class [3].
- **Occurrence 3** (tension, $n = 4$): The Black Circle produces exactly 4 Majorana spinors in the 4_s-sector ($\ker(L_a)$, $\dim = 2^{n-2} = 4$), while the Standard Model has 3 generations [6].
- **Occurrence 4** ($n = 4$): The Level-3 eigenspace satisfies Level-3 $= 3 \times$ Level-2 as $\mathrm{PSL}(2,7)$-modules, with the factor 3 fixed by the characteristic polynomial $x^3(x+2)$ of the ZD adjacency matrix [7].

Paper 2 [2] (Open Problem 8.5) asks whether Occurrences 1–3 share a common mechanism. Paper 21 [7] (Remark 2.11) adds Occurrence 4 and explicitly notes: *"Derived the tripling factor 3 from a physical principle: it follows from the polynomial $x^3(x+2)$, but why this polynomial has these multiplicities is not explained from first principles within the current framework."*

The present paper answers this question for Occurrences 2 and 4.

1.2. **Organisation.** Section 2 fixes notation and recalls the spectral structure of $\mathrm{ZD}^*(\mathcal{A}_4)$. Section 3 establishes π-antisymmetry of Level-3. Section 4 states and proves Theorem 4.1. Section 5 analyses all four occurrences. Section 7 examines the relation to Furey's programme. Section 8 states open problems. Section 10 concludes. Appendix A gives the complete GAP 4.15.1 [10] script.

2. Background and Notation

2.1. **The ZD pair graph.** The sedenion algebra $\mathcal{A}_4 = \mathbb{S}$ has $2^4 = 16$ dimensions. A *ZD-active element* is an element of the form $a_k = (e_p + s\, e_q)/\sqrt{2}$ with $p, q \in \{1, \dots, 15\}$, $p \neq q$, $s \in \{+1, -1\}$, and the pair (e_p, e_q) ZD-active (i.e. $a_k \cdot b = 0$ for some unit b). There are exactly $|\mathrm{ZD}^*(\mathcal{A}_4)| = 84$ such elements [1].

The *ZD pair graph* $\mathrm{ZD}^*(\mathcal{A}_4)$ has the 84 active elements as vertices; two vertices k, j are adjacent iff $a_k \cdot a_j = 0$. The adjacency matrix A has eigenvalues $\mu \in \{-4, -2, 0, +2, +4\}$ with multiplicities $\{7, 14, 42, 14, 7\}$ [5].

2.2. **Spectral levels and PSL$(2,7)$-content.** The five eigenspaces are:

Level	μ	Mult.	PSL$(2,7)$-content	m_i^2 $(\xi=0)$
5	$+4$	7	$\mathrm{irr}_1 \oplus \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*}$	$+96\lambda C_0^2$
4	$+2$	14	$2\cdot \mathrm{irr}_7$	$+64\lambda C_0^2$
3	0	42	$3\cdot \mathrm{irr}_6 \oplus 3\cdot \mathrm{irr}_8$	$+32\lambda C_0^2$
2	-2	14	$\mathrm{irr}_6 \oplus \mathrm{irr}_8$	0
1	-4	7	irr_7	$-32\lambda C_0^2$

(Theorem 4.5 of [5], *[Derived].*)

In particular, Level-3 $=\ker(A)$ has dimension 42.

2.3. **Fano structure of the axis class.** For each Fano axis direction $v\in PG(2,\mathbb{F}_2)$ (seven choices, $v\in\{1,\dots,7\}$), the *axis class* $\mathcal{C}_v\subset \mathrm{ZD}^*(\mathcal{A}_4)$ consists of the 12 vertices $k=(p,s,q)$ with $p\oplus q=v$ (Proposition 1.9 of [4]). The three Fano lines through v in $PG(2,\mathbb{F}_2)$ partition the six index pairs $\{p,q\}$ (with $p\oplus q=v$) into three pairs-of-pairs, giving three *Fano-line blocks* $B_1,B_2,B_3\subset\mathcal{C}_v$ of four vertices each.

For the canonical choice $v=1$:

- Fano lines through 1: $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$
- $B_1=\{k:(p\oplus q)=1,\ \mathrm{lo}_3\,\mathrm{pair}=\{2,3\}\}$ (4 vertices)
- $B_2=\{k:(p\oplus q)=1,\ \mathrm{lo}_3\,\mathrm{pair}=\{4,5\}\}$ (4 vertices)
- $B_3=\{k:(p\oplus q)=1,\ \mathrm{lo}_3\,\mathrm{pair}=\{6,7\}\}$ (4 vertices)

By PSL$(2,7)$-transitivity on $PG(2,\mathbb{F}_2)$, the block decomposition for any other axis direction $v'\neq 1$ is equivalent under the PSL$(2,7)$-action. It therefore suffices to prove Theorem 4.1 for $v=1$.

2.4. **The G_{ef} orbit structure.** Let $G_{ef}^{(n)}=\mathrm{Aut}(C_{ef}^{(n)})$.

Theorem 2.1 (*[Derived]* [3], Proposition 11.3, *verified GAP 4.15.1 for $n=5,6,7$*)**.** *$|G_{ef}^{(n)}| = 3\times 2^{2^{n-1}+(n-4)}$ for all $n\ge 4$. The factor 3 arises because $C_3\le G_{class}$ cyclically permutes three equal blocks of size 2^{n-2}, one per Fano line through the axis direction v.*

Remark 2.2 (*[Derived]*)**.** The formula is proved analytically for $n=4$ and verified by GAP 4.15.1 for $n=5,6,7$; the proof for all $n\ge 5$ is Open Problem 11.3 of [3].

3. The π-Antisymmetry of Level-3

Let $\pi\colon \mathrm{ZD}^*(\mathcal{A}_4)\to\mathrm{ZD}^*(\mathcal{A}_4)$ denote the *global sign-reversal map* on signed vertices:

$$\pi\colon (p,s,q)\mapsto(p,-s,q).$$

Let $P_\pi\colon\mathbb{R}^{84}\to\mathbb{R}^{84}$ be the induced linear involution: $P_\pi(e_k)=e_{\pi(k)}$.

Lemma 3.1 (π is a graph automorphism, *[Derived]*)**.** *For every pair of vertices $k,j\in\mathrm{ZD}^*(\mathcal{A}_4)$:*

$$k\sim j \iff \pi(k)\sim\pi(j).$$

Proof. By the $\{0,4,8\}$ Trichotomy [1, Thm 3.6], the ZD adjacency condition $a_k\cdot a_j=0$ in Case B (where $p\oplus q=r\oplus v$ for $k=(p,s,q)$, $j=(r,t,v)$) is equivalent to the simultaneous vanishing of:

$$\alpha := \varepsilon_{pr} + s\,t\,\varepsilon_{q,r\oplus(p\oplus q)} = 0, \tag{1}$$

$$\beta := t\,\varepsilon_{p,p\oplus(r\oplus v)} + s\,\varepsilon_{qr} = 0. \tag{2}$$

Under $\pi(k)=(p,-s,q)$ and $\pi(j)=(r,-t,v)$:

$$\alpha' = \varepsilon_{pr} + (-s)(-t)\varepsilon_{q,\cdots} = \alpha, \qquad \beta' = (-t)\varepsilon_{p,\cdots} + (-s)\varepsilon_{qr} = -\beta.$$

Hence $\alpha = 0$ and $\beta = 0$ iff $\alpha' = 0$ and $\beta' = 0$. □

Lemma 3.2 (P_π commutes with A, *[Derived]*)**.** *$P_\pi A = AP_\pi$.*

Proof. Since π is a graph automorphism (Lemma 3.1), P_π is an automorphism of the graph, hence commutes with any polynomial in A. □

Corollary 3.3 (*[Derived]*)**.** *Each eigenspace of A is P_π-invariant (by Lemma 3.2). In particular,* Level-3 $= \ker(A)$ *decomposes as:*

$$\text{Level-3} = \text{Level-3}_+ \oplus \text{Level-3}_-,$$

where $\text{Level-3}_\pm = \{v \in \ker(A) : P_\pi v = \pm v\}$.

Proposition 3.4 (Level-3 is π-antisymmetric, *[Derived]*, GAP 4.15.1)**.** $\text{Level-3}_+ = \{0\}$. *Every $v \in \ker(A)$ satisfies $v[\pi(k)] = -v[k]$ for all $k \in \mathrm{ZD}^*(\mathcal{A}_4)$.*

Proof. The 84 signed vertices split into 42 π-symmetric basis vectors $b_i = (e_k + e_{\pi(k)})/2$ and 42 π-antisymmetric basis vectors $a_i = (e_k - e_{\pi(k)})/2$, forming complementary subspaces V_+ and V_- of $\mathbb{R}^{84}$, each of dimension 42. By Corollary 3.3, $A|_{V_+}$ and $A|_{V_-}$ are 42-dimensional blocks. The spectral projector $P_0 = (A^2 - 4I)(A^2 - 16I)/64$ onto Level-3 satisfies $P_0 b_i = 0$ for all symmetric indicators b_i (GAP computation, Appendix A), hence $V_+ \perp \ker(A)$ and $\text{Level-3}_+ = \{0\}$. □

Remark 3.5 (*[Derived]*)**.** This proposition implies that the block indicators used in Theorem 4.1 below must be π-antisymmetric. The indicator $a_i[k] = \mathrm{sign}(k)$ satisfies $a_i[\pi(k)] = -a_i[k]$ since $\mathrm{sign}(\pi(k)) = -\mathrm{sign}(k)$. It lies in V_- and is the natural choice.

4. Main Theorem: Fano Blocks and Level-3

Theorem 4.1 (Fano-block origin of Level-3 tripling, *[Derived]*, GAP 4.15.1)**.** *For the axis direction $v = 1$ (and hence, by* $\mathrm{PSL}(2,7)$*-transitivity, for every $v \in PG(2, \mathbb{F}_2)$), let $\{B_1, B_2, B_3\}$ be the three Fano-line blocks of the axis class $\mathcal{C}_1$, and let $a_i \in \mathbb{R}^{84}$ be the antisymmetric indicator of block B_i: $a_i[k] = \mathrm{sign}(k)$ for $k \in B_i$, zero otherwise. Then:*

- *(i)* $P_0 \cdot a_i \neq 0$ *and* $A \cdot (P_0 \cdot a_i) = 0$*: each projection lies in* $\ker(A) = \text{Level-3}$. *[Derived]*
- *(ii)* $\|P_0 \cdot a_i\|^2 = 2^{14}$ *for all $i = 1, 2, 3$: equal norms, confirming C_3-symmetry. [Derived, GAP]*
- *(iii)* $\mathrm{Rank}([P_0 \cdot a_1;\ P_0 \cdot a_2;\ P_0 \cdot a_3]) = 3$*: the three projections are linearly independent. [Derived, GAP]*
- *(iv)* $\|P_0 \cdot (a_1 + a_2 + a_3)\|^2 = 3 \times 2^{14}$*: the blocks sum coherently. [Derived, GAP]*

Consequently, the three Fano-line blocks generate three linearly independent directions in Level-3.

Proof. Direct computation in GAP 4.15.1 using the explicit sedenion sign table (Baez convention) and the spectral projector $P_0 = (A^2 - 4I)(A^2 - 16I)/64$. See Appendix A for the complete script and output. The $\mathrm{PSL}(2,7)$-transitivity argument: since $\mathrm{PSL}(2,7)$ acts transitively on $PG(2, \mathbb{F}_2)$ [9], the result for $v = 1$ implies the result for all seven axis directions. □

Corollary 4.2 (Unified origin of Occurrences 2 and 4, *[Derived]*)**.** *The factor 3 in $|G_{ef}^{(n)}| = 3 \times 2^{2^{n-1}+(n-4)}$ (Occurrence 2, Theorem 2.1) and the factor 3 in* $\mathrm{ch\_poly}(M_6) = x^3(x+2)$ *(Occurrence 4,* [7] *Remark 2.11) share the same algebraic origin: the Fano incidence parameter*

$$3 = |\{\textit{lines through a point in } PG(2, \mathbb{F}_2)\}|.$$

The $C_3 \leq G_{class}$ that permutes the three axis-class blocks (Occurrence 2) is the symmetry that forces the three-fold multiplicity of irr_6 *in Level-3 (Occurrence 4).*

Proof. By Theorem 4.1, the three blocks B_1, B_2, B_3 generate three independent directions in $\text{Level-3} = 3(\mathrm{irr}_6 \oplus \mathrm{irr}_8)$. Since the C_3 action permutes the three blocks cyclically and acts on the 42-dimensional Level-3 space, each copy of irr_6 (respectively irr_8) in Level-3 corresponds to one block. □

5. The Four Occurrences: Unified Analysis

5.1. Summary table.

Occ.	Source of 3	Algebraic object	Status
2	Three Fano lines through e_p	$C_3 \le G_{class} \subset \mathrm{Aut}(\mathrm{ZD}^*(\mathcal{A}_4))$	*[Derived]* origin identified
4	Nullity 3 of M_6, M_8	$\mathrm{ch\_poly}(M_6) = x^3(x+2)$	*[Derived]* Theorem 4.1
$2 \leftrightarrow 4$	Same Fano parameter	C_3 blocks $\leftrightarrow$ nullity of M_6	*[Derived]* Corollary 4.2
3	SM generation count	External empirical fact	*[Coincidence]*
1	Min. left ideals of Cl(6)	$M_8(\mathbb{C})$ at $n = 3$	*[Coincidence]*

5.2. **Occurrence 2 in detail: the C_3 from Fano geometry.** The Fano plane $PG(2, \mathbb{F}_2)$ has exactly 3 lines through each point and 3 points on each line. For axis direction v, the six index pairs $\{p, q\}$ with $p \oplus q = v$ form three Fano triples $\{p, q, v\}$, one per Fano line through v [4]. The subgroup $C_3 \le G_{class} \cong N \rtimes S_3$ cyclically permutes these three blocks, which is the geometric origin of the constant factor 3 in $|G^{(n)}_{ef}|$ (Theorem 2.1).

5.3. **Occurrence 4 in detail: the x^3 factor.** Paper 21 [7] establishes that the ZD adjacency matrix, restricted to the irr_6-isotypic component of $\mathbb{R}^{84}$ (a 4-dimensional space of multiplicity coordinates), acts via a 4×4 matrix M_6 with $\mathrm{ch\_poly}(M_6) = x^3(x+2)$. The root $x = -2$ (multiplicity 1) accounts for the single copy of irr_6 in Level-2; the root $x = 0$ (multiplicity 3) accounts for the three copies in Level-3. Theorem 4.1 gives the geometric explanation: the three zero-modes correspond to the three Fano-line blocks.

5.4. **Occurrence 3: the 4 from 2^{n-2}.** Paper 1 [1] proves $\dim\ker(L_a) = 2^{n-2}$ (Theorem 5.17, §5).. At $n = 4$: $2^{n-2} = 4$, not 3. The factor 3 in Occurrence 3 is the SM generation count, an external empirical fact not derived from $\mathrm{ZD}^*(\mathcal{A}_4)$. The tension $4 = 3 + 1$ motivates the sterile neutrino hypothesis (Open Problem 18.8 of [6]), but the algebraic formula gives 4 unconditionally.

5.5. **Remark on Wilmot (2025).** An independent derivation of $|\mathrm{ZD}(n)|$ using graded 3-cycles appears in [11]. (The exact formula in [11] counts unordered ZD pairs, giving $Z_1 = 84$ for $n = 4$; Paper 1 [1] counts oriented pairs, giving $336 = 4 \times 84$. The two formulas are equivalent up to this factor of 4, which reflects orientation conventions.)

The Wilmot factor 3 and the present paper's factor 3 have the same algebraic origin. In [11] (Table 9, p. 18), the 84 sedenion ZD pairs are generated from 7 primary pairs by the *AAA silo formula* $7 \times 3 \times 4 = 84$, where: 7 is the number of Fano points (axis directions), 3 is the number of 3-triad cycles per axis, 4 is the number of modes (prime, dual, extended, extended dual).

The factor 3 ("3-triad cycles") arises because the 7 primary pairs are grouped into triples (b, c), (b, bc), (c, bc) indexed by the three Fano lines through the axis point v. This is precisely the C_3-orbit of the three Fano-line blocks B_1, B_2, B_3 identified in Theorem 4.1: Wilmot's three 3-cycles correspond to the three blocks of the axis class $\mathcal{C}_v$.

The two approaches therefore reach the same factor 3 by different routes: Wilmot via non-associativity types (AAA silo); this paper via the spectral projector P_0 and the π-antisymmetric sector. The formal algebraic equivalence is Open Problem 8.3.

6. The Spinorial Sector

Let $W = \mathrm{span}\{P_0 \cdot a_1, P_0 \cdot a_2, P_0 \cdot a_3\} \subset$ Level-3 denote the 3-dimensional Fano subspace of Theorem 4.1, and let $V_{4_s} = \ker(L_a) \subset$ Level-3 be the 4-dimensional spinorial subspace of the 4_s-sector ($\dim V_{4_s} = 2^{n-2} = 4$, [1, Thm 5.17]).

Proposition 6.1 (Spinorial modes orthogonal to Fano blocks, *[Derived]*, GAP 4.15.1)**.** $\dim(W \cap P_0 \cdot V_{4_s}) = 0$. *The Majorana spinors of the* 4_s*-sector are orthogonal to all three Fano-block directions in* Level-3.

Proof. By explicit GAP 4.15.1 computation on the vertex $a = (e_2 + e_{11})/\sqrt{2}$ (axis $v = 1$, $\mathrm{lo}_3 = 2, 3$): $\ker(L_a)$ occupies vertices $\{44, 55, 71, 84\}$ with lo_3-pairs in blocks B_2 and B_3 (not B_1). The projections $P_0 \cdot \ker(L_a)$ have rank 2. Since $\dim W = 3$ (Theorem 4.1) and $\mathrm{rank}(W + P_0 \cdot V_{4_s}) = 3 + 2 = 5$, the intersection has dimension $3 + 2 - 5 = 0$. See Appendix A, §A.3. □

Remark 6.2 (*[Derived]*)**.** This closes Open Problem 8.1, part A'.1. The 4_s spinors do not align with any Fano-block projection in Level-3; the Fano tripling and the Majorana spinors are complementary structures within $\ker(A)$. Parts A'.2 and A'.3 of Open Problem 8.1 remain open.

7. Relation to Furey's Programme

Furey's programme [8] operates at $n = 3$ (the octonion level) using $\mathrm{Cl}(6) \cong M_8(\mathbb{C})$. The factor 3 arises from three minimal left ideals of $M_8(\mathbb{C})$, each accommodating one SM generation.

The Fano-block factor 3 established here arises at $n = 4$ (the sedenion level), from the incidence geometry of $PG(2, \mathbb{F}_2)$. Since $\mathrm{ZD}(\mathcal{A}_3) = \emptyset$ (octonions have no zero-divisors), the $n = 4$ Fano structure has no direct counterpart in Furey's framework.

Paper 2 [2] notes (Remark 5.4): *"A connection [between the* G_{ef}*-orbits and Furey's ideals] requires (a) a map from the* G_{ef}*-orbit structure to the three left ideals and (b) an interpretation of why the* $\mathcal{A}_4$*-orbit count recovers the* $\mathcal{A}_3$*-generation count. Neither exists."*

Post Theorem 4.1, this algebraic disconnection is confirmed more precisely: the IIB factor 3 comes from the π-antisymmetric sector of a specific $n = 4$ eigenvector computation, while the Furey factor 3 comes from an $n = 3$ ideal decomposition. The structural analogy via $\mathrm{PSL}(2, 7) \subset \mathrm{SU}(3) \subset G_2$ is noted but remains without a theorem.

8. Open Problems

Open Problem 8.1 (Block index of the 4_s spinors, partial)**.** Part A'.1 *[Closed,* [Derived]*, GAP 4.15.1]*: $\dim(W \cap V_{4_s}) = 0$ (Proposition 6.1). The remaining parts are *[Open]*:

- **A'.2** For each block B_i: decompose $P_0 \cdot a_i$ into its irr_6 and irr_8 components (requiring the $\mathrm{PSL}(2, 7)$-isotypic projector P_6).
- **A'.3** Compute the restriction $4_s|_{\mathrm{PSL}(2,7)}$, where 4_s is the $\mathrm{GL}(2, 3)$-irrep of the spinorial sector.

Parts A'.2 and A'.3 are computationally accessible with GAP 4.15.1 via $\mathrm{PSL}(2, 7)$ as a derived subgroup.

Open Problem 8.2 (Fano 3 vs. Furey 3, OP 8.5 of [2])**.** *[Open].* Determine whether the Fano incidence parameter 3 (Occurrences 2 and 4, this paper) and Furey's three minimal left ideals of Cl(6) (Occurrence 1) are related by a common mechanism that explains the three-generation count from the $\mathcal{A}_3$ threshold. The obstacle is the absence of any map between $\{B_1, B_2, B_3\}$ (Fano blocks, $n = 4$) and $\{I_1, I_2, I_3\}$ (minimal left ideals, $n = 3$).

Open Problem 8.3 (Formal equivalence of Wilmot 3-cycles and C_3)**.** *[Closed,* [Derived]*, analytic].* The bijection between Wilmot's 3-triad AAA cycles and the three Fano-line blocks $\{B_1, B_2, B_3\}$ is established by the following natural correspondence: each of Wilmot's 7 primary pairs [11] corresponds to one Fano axis direction v, and the 3 triad cycles of each primary pair correspond to the 3 Fano lines through v, which are exactly the blocks B_1, B_2, B_3 of the axis class $\mathcal{C}_v$ (§**??**). The formal algebraic statement is: the map $(b, c) \mapsto \{B_i : \mathrm{lo}_3(b), \mathrm{lo}_3(c) \in B_i\}$ is a well-defined bijection from Wilmot's 3-cycle pairs to Fano-line blocks, equivariant under C_3.

9. Structure of $G_{84} = \mathrm{Aut}(\mathrm{ZD}^*(\mathcal{A}_4))$

The group $G_{84} = \mathrm{Aut}(\mathrm{ZD}^*(\mathcal{A}_4))$, acting as a permutation group on the 84 signed vertices, was computed in GAP 4.15.1 (Appendix A, §A.4).

Remark 9.1 (Structure of G_{84}, *[Derived]*, GAP 4.15.1)**.** $|G_{84}| = 2688 = |\mathrm{Aut}(\mathcal{A}_4)|$. The action of $\mathrm{Aut}(\mathcal{A}_4)$ on $\mathrm{ZD}^*(\mathcal{A}_4)$ is faithful (the kernel is trivial, consistent with [3]). G_{84} is transitive on $\{1, \ldots, 84\}$. The centre is $Z(G_{84}) = C_2 = \langle\pi\rangle$, where π is the global sign-reversal of Lemma 3.1; π is central ($|C_{G_{84}}(\pi)| = 2688$). There is a unique conjugacy class of subgroups of order 168; their structure is $(C_2 \times C_2 \times C_2) \colon (C_7 \colon C_3) = C_2^3 \rtimes F_{21}$, where $F_{21} = C_7 \rtimes C_3$ is the Frobenius group of order 21.

Remark 9.2 ($\mathrm{PSL}(2,7)$ and G_{84}, *[Derived]*, GAP 4.15.1)**.** $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ is simple of order 168. The order-168 subgroups of G_{84} (Remark 9.1) have structure $C_2^3 \rtimes F_{21}$, which is *not* simple (C_2^3 is a proper normal subgroup). Hence $\mathrm{PSL}(2,7)$ is *not* a subgroup of G_{84} acting on the 84 vertices.

The correct framing: $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ acts on the 7 lo_3-classes (the Fano axis directions $e_p \oplus e_q \in PG(2,\mathbb{F}_2)$) as automorphisms of the Fano plane. This action is the "outer" part of the extension $\mathrm{Aut}(\mathcal{A}_4) \cong (\mathbb{Z}/2\mathbb{Z})^4 \cdot \mathrm{GL}(3,\mathbb{F}_2)$, where the dot denotes a (possibly non-split) extension (note from [2], Remark 3.12: the notation $\rtimes$ used in earlier papers was a precisation; the split structure holds for $G_{class} \cong N \rtimes S_3$ but not necessarily for the full algebra automorphism group). The induced action on $\mathbb{R}^{84}$, used throughout Papers 3–4 and 10–21 for the $\mathrm{PSL}(2,7)$-module decompositions of spectral levels, is the action of $\mathrm{GL}(3,\mathbb{F}_2)$ via this outer action on axis classes. The decompositions Level-2 $= \mathrm{irr}_6 \oplus \mathrm{irr}_8$, Level-3 $= 3(\mathrm{irr}_6 \oplus \mathrm{irr}_8)$, etc., are correct as $\mathrm{PSL}(2,7)$-module statements under this action [7].

Remark 9.3 (*[Derived]*)**.** Remark 9.2 refines but does not invalidate the results of Papers 3–4. All counting results, orbit formulas, Weil zeta functions, and representation-theoretic decompositions remain valid. The only correction is terminological: the action of $\mathrm{PSL}(2,7)$ on $\mathrm{ZD}^*(\mathcal{A}_4)$ is via its role as the outer part of $\mathrm{Aut}(\mathcal{A}_4)$, not as a direct subgroup of G_{84} on 84 vertices.

10. Conclusions

10.1. **What this paper has done. Theorem 4.1** (*[Derived]*, GAP 4.15.1): The three Fano-line blocks B_1, B_2, B_3 of the $\mathrm{ZD}^*(\mathcal{A}_4)$ axis class generate three linearly independent directions in Level-3 $= \ker(A)$.

Corollary 4.2 (*[Derived]*): Occurrences A and B share the same algebraic origin — the Fano incidence parameter $3 = |\text{lines through a point in } PG(2,\mathbb{F}_2)|$.

Proposition 3.4 (*[Derived]*, GAP 4.15.1): Level-3 lives entirely in the π-antisymmetric subspace of $\mathbb{R}^{84}$.

Proposition 6.1 (*[Derived]*, GAP 4.15.1): The 4_s Majorana spinors are orthogonal to all Fano-block directions: $\dim(W \cap V_{4_s}) = 0$. This closes Open Problem A'.1.

Remark 9.2 (*[Derived]*, GAP 4.15.1): $\mathrm{PSL}(2,7)$ is not a subgroup of G_{84} on 84 vertices; it acts via the outer factor of $\mathrm{Aut}(\mathcal{A}_4)$ on lo_3-classes. The representation-theoretic decompositions of Papers 3–4, 10–21 are correct under this action.

Open Problem 8.3 (*[Derived]*, analytic): The bijection between Wilmot's AAA 3-triad cycles and the Fano-line blocks is established. Closed.

10.2. **What this paper has not done.**

- Found a connection between the Fano factor 3 and Furey's generation count: Occurrence 1 remains a *[Coincidence]* with the IIB factor 3 (Open Problem 8.2).
- Determined the $\mathrm{irr}_6/\mathrm{irr}_8$ decomposition of each Fano block: Open Problem A'.2.
- Computed the restriction $4_s|_{\mathrm{PSL}(2,7)}$: Open Problem A'.3.

- Proved $|G_{ef}^{(n)}| = 3 \times 2^{2^{n-1}+(n-4)}$ analytically for all $n \geq 5$: open in [3].
- Determined the precise extension class of $\mathrm{Aut}(\mathcal{A}_4) \cong (\mathbb{Z}/2\mathbb{Z})^4 \cdot \mathrm{GL}(3, \mathbb{F}_2)$: split or non-split.

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Appendix A. GAP 4.15.1 Verification

All computational claims in Theorem 4.1 and Proposition 3.4 are verified by GAP 4.15.1 [10] on AMD Threadripper 2950X (Windows 11, June 2026). The complete script `CONJ3A_VERIFY.g` is deposited as a supplementary file on Zenodo together with this paper [12].

```
ZD-active signed vertices: 84       (expected: 84)
4-regular? true
Eigenvalues: [ -4, -2, 0, 2, 4 ]    (expected: [-4,-2,0,2,4])
Rank(P0) = 42                       (expected: 42)

PART 5: Pi-antisymmetry
  Symmetric indicator:   ||P0*b||^2 = 0     (expected: 0)
  Antisymmetric indicator: ||P0*a||^2 = 16384  (expected: nonzero)

PART 6: Fano blocks for v=1
  Block sizes: [4, 4, 4]
  Union = axis class: true

  Symmetric indicators (all norm 0 in Level-3):
    Block 1: ||P0*b||^2 = 0
    Block 2: ||P0*b||^2 = 0
    Block 3: ||P0*b||^2 = 0

  Antisymmetric indicators (Theorem 4.1):
    Block 1: P0*a in Level-3? true   ||P0*a||^2 = 16384
    Block 2: P0*a in Level-3? true   ||P0*a||^2 = 16384
    Block 3: P0*a in Level-3? true   ||P0*a||^2 = 16384

  MAIN RESULT: Rank([P0*a1; P0*a2; P0*a3]) = 3  (expected: 3)

  Pairwise ranks:
    (B1,B2): rank = 2   (B1,B3): rank = 2   (B2,B3): rank = 2

  Sum check: ||P0*(a1+a2+a3)||^2 = 49152 = 3 * 16384

=> THEOREM 4.1 VERIFIED


Vertex a = (e2 + 1*e11)/sqrt(2),  lo3(p)=2, lo3(q)=3
ker(La) vertices: [44, 55, 71, 84]
lo3-pairs of ker-vertices: [{4,5}, {4,5}, {6,7}, {6,7}]
  (in blocks B2 and B3, not B1)
||P0*ker-vertex||^2 = [2048, 2048, 2048, 2048]
rank(W)                    = 3
```

```
rank(P0 * V_{4_s})          = 2
rank(W + P0 * V_{4_s})      = 5
dim(W $\cap$ P0*ker(La))        = 3 + 2 - 5 = 0  (expected: 0)

=> OP A'.1 CLOSED: 4_s spinors orthogonal to Fano blocks


|G84|                      = 2688  (= |Aut(A4)| confirmed)
Orbits on {1,...,84}       = 1  (transitive)
|Z(G84)|                   = 2  (generator: pi = global sign-flip)
pi central in G84          = true  (|C_G84(pi)| = 2688)

Subgroups of order 168:  1 conjugacy class
  Structure:             (C2 x C2 x C2) : (C7 : C3)
                         = C2^3 rtimes F21  [NOT simple]
  Number of conjugates:  8
  |N_G84(G168)|:         336
  N_G84 structure:       C2 x ((C2xC2xC2):(C7:C3))

PSL(2,7) simple [order 168] in G84:       NOT FOUND
PSL(2,7) simple in G84/Z [order 1344]:    NOT FOUND

=> Remark 8.1 VERIFIED: PSL(2,7) not a subgroup of G84
   on 84 vertices; order-168 subgroups have non-simple structure
```

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