THE ZD DIRAC OPERATOR AND FERMIONIC SECTOR OF S_{84}

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ABSTRACT. We construct the ZD Dirac operator $\mathcal{D} = \sum_{a \in \mathrm{ZD}^*(A_4)} \gamma_a \, \partial/\partial\phi_a$ on the 84-dimensional scalar field manifold of S_{84}, where $\gamma_a = \rho(a) \in \mathrm{Cl}(0,7)$ are the Clifford matrices of [3]. We show that $\mathcal{D}^2 = -I_8\Delta_{\mathrm{fib}}$ (the negative Fano-fibre Laplacian), with purely imaginary symbol spectrum $\pm i\|\tilde{\xi}\|$, distinct from the real integer spectrum $\{-4,-2,0,+2,+4\}$ of the adjacency matrix A. Restricting to the 4_s Majorana sector (four modes in $\ker(L_a)$ with ε-indices $\bar{\ell} = \{4,5,6,7\}$), we prove by three independent arguments including a Sylow order obstruction that the C_3 Fano-block symmetry of [13] does not induce a 3+1 decomposition of the 4_s modes *[Derived-Negative]*. Open Problem 18.8 (sterile neutrino via Z_3) is thus closed negatively for the Fano-block mechanism; the residual question of whether a different selection principle provides the $3+1$ split remains open. We introduce the fermionic extension $S_{84}^{(f)}$ via a Yukawa coupling $\sum_j \phi_{a_j} \bar{\Psi}\gamma_{s_j}\Psi$, derive the modified Dirac equation, compute the condensate shift (purely imaginary eigenvalues $\pm 2iC_0$), and identify the schematic fermion-loop correction to $\beta(\lambda)$. The precise relationship between our canonical $\mathrm{Cl}(0,7)$ basis and the Gourlay–Gresnigt Cl(8) [15] is established: $\mathbb{C}\otimes\mathrm{Cl}(0,7) \cong M_8(\mathbb{C})$ embeds as the octonionic block of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}\otimes\mathbb{S}) \cong M_{16}(\mathbb{C})$.

Epistemic key. *[Derived]* — follows from Papers 1–22 and established mathematics. *[Conjecture]* — precise statement, assumptions explicit. *[Derived-Negative]* — falsification of a stated hypothesis by algebraic proof. *[Open]* — obstacle stated precisely.

1. Introduction and Scope

This paper has two purposes: to construct a canonical differential operator on the field manifold of S_{84} from the Clifford representation of [3], and to apply it to the 4_s Majorana sector in order to address Open Problem 18.8.

Primary open problem. OP 18.8 [8] asks whether the four Majorana spinors of the 4_s sector — arising from the $\mathrm{GL}(2,3)$ spinorial structure of [7] with Frobenius–Schur indicator $\mathrm{FS}(4_s) = +1$ [8] — decompose as $3+1$ under the Z_3 cyclic symmetry of the Fano blocks B_1, B_2, B_3 of [13], with the "+1" identified as a sterile neutrino candidate.

Secondary open problems. OP 12.2 [7] asks for the construction of an A_4-valued quantum field; this paper contributes the fermionic kinetic operator while the full A_4-valued QFT remains obstructed by non-associativity (OP 12.5 [7]). OP 21.2(a) [11] asks whether the triple structure Level-3 = 3×Level-2 admits an IIB-II algebraic formulation; the result of §3 shows that the C_3 Fano action is algebraically disconnected from the fermionic 4_s sector.

Strategy. Section 2 constructs $\mathcal{D}$ from the representation $\rho : \mathrm{ZD}^*(A_4) \to \mathrm{Cl}(0,7)$ of [3], computes $\mathcal{D}^2$, and compares its spectrum with that of the adjacency matrix A. Section 3 restricts $\mathcal{D}$ to the 4_s sector, analyzes whether the C_3 Fano-block symmetry gives a $3+1$ split (negative result, three independent arguments), and positions the construction relative to external literature. Section 4 extends the action S_{84} by the fermionic Yukawa term, derives the Euler–Lagrange equations, propagator, and schematic loop contributions. Section 5 updates the open-problem register.

Date: June 12, 2026.

2020 *Mathematics Subject Classification.* 17A35, 15A66, 81T20, 20C15, 05C50.

Key words and phrases. ZD Dirac operator, sedenion algebra, Clifford algebra Cl(0,7), Majorana spinors, fermionic kinetic term, Yukawa coupling, PSL(2,7), Fano incidence, sterile neutrino hypothesis.

Relation to external literature. Gresnigt et al. [14, 15, 16] generate three fermion generations via the order-3 automorphism $\psi \in \mathrm{Aut}(\mathbb{S})$ acting on minimal left ideals of Cl(8), where ψ is chosen as an ansatz from sedenion automorphism theory. Our construction is orthogonal: $\mathcal{D}$ is derived canonically from $\mathrm{ZD}^*(A_4)$ (Paper 7 [3]), uses only the octonionic sub-algebra $\mathrm{Cl}(0,7)$, and produces a *negative* result for the Z_3 generation mechanism in the fermionic sector. The setting $(\mathrm{Cl}(0,7))$ connects to Trayling–Baylis's proposal for the sterile neutrino [19], reviewed in §3.4.

This paper does not construct a full A_4-valued QFT (OP 12.2 [7]), does not compute the fermion-loop coefficient in $\beta(\lambda)$ (OP 23.1), and does not identify a specific sterile neutrino mode among the four 4_s spinors.

2. The ZD Dirac Operator

2.1. Definition from the Clifford Representation.

The Clifford representation of [3] establishes

$$\rho : \mathrm{ZD}^*(A_4) \longrightarrow \mathrm{Cl}(0,7) \cong M_8(\mathbb{R}), \qquad \rho(e_p + \sigma e_{8+k}) := L^{\mathbb{O}}_{e_p}, \tag{2.1}$$

where $L^{\mathbb{O}}_{e_p}$ is the 8×8 real matrix of left-multiplication by e_p in $\mathbb{O} = A_3$ ([3], Definition 4.1). The map depends only on $p = \varepsilon(A)$; it is constant on each of the seven fibres $\{A \in \mathrm{ZD}^*(A_4) : \varepsilon(A) = p\}$, each of cardinality $84/7 = 12$ ([3], Remark 4.2).

Notation. Write $\gamma_p := L^{\mathbb{O}}_{e_p} \in \mathrm{Cl}(0,7)$ for $p \in \{1, \ldots, 7\}$. The seven generators satisfy

$$\{\gamma_p, \gamma_q\} = -2\,\delta_{pq}\, I_8 \qquad \forall\, p, q \in \{1, \ldots, 7\}, \tag{2.2}$$

hence $\gamma_p^2 = -I_8$ ([3], Lemma 3.1).

Remark. *(Matrix availability.)* The explicit 8×8 entries of γ_p are determined by the Baez-convention octonion multiplication table ([3], Appendix A) but are not reproduced here. Every computation in §§ 2–4 requires only (2.2).

Definition 2.1 (ZD Dirac operator, *[Derived]*)**.** Let $\{\phi_a\}_{a \in \mathrm{ZD}^*(A_4)}$ be the 84 real scalar fields of S_{84} ([6]). Define

$$\mathcal{D} \;:=\; \sum_{a \in \mathrm{ZD}^*(A_4)} \gamma_a \,\frac{\partial}{\partial \phi_a} \;=\; \sum_{p=1}^{7} \gamma_p\, \hat{\partial}_p, \tag{2.3}$$

where $\gamma_a := \rho(a)$ and $\hat{\partial}_p := \sum_{\varepsilon(a)=p} \partial/\partial\phi_a$ is the sum of the 12 partial derivatives over the ε-fibre at p. $\mathcal{D}$ acts on $C^\infty(\mathbb{R}^{84}, S)$ where $S \cong \mathbb{R}^8$ is the $\mathrm{Cl}(0,7)$ spinor module.

2.2. $\mathcal{D}^2$ and Spectral Properties.

Proposition 2.2 (Fibre Laplacian identity, *[Derived]*)**.**

$$\mathcal{D}^2 \;=\; -I_8\, \Delta_{\mathrm{fib}}, \qquad \Delta_{\mathrm{fib}} := \sum_{p=1}^{7} \hat{\partial}_p^2. \tag{2.4}$$

Proof. Squaring (2.3) and using $\partial_a \partial_b = \partial_b \partial_a$:

$$\mathcal{D}^2 = \sum_a \gamma_a^2 \partial_a^2 + \sum_{a<b} \{\gamma_a, \gamma_b\} \partial_a \partial_b.$$

From (2.2): $\gamma_a^2 = -I_8$ for all a. For $a \neq b$: if $\varepsilon(a) = \varepsilon(b) = p$ then $\{\gamma_a, \gamma_b\} = \{\gamma_p, \gamma_p\} = 2\gamma_p^2 = -2I_8$; if $\varepsilon(a) \neq \varepsilon(b)$ then $\{\gamma_a, \gamma_b\} = 0$. Collecting over fibres:

$$\mathcal{D}^2 = -I_8 \Big[\sum_a \partial_a^2 + 2 \sum_{\substack{a<b \\ \varepsilon(a)=\varepsilon(b)}} \partial_a \partial_b \Big] = -I_8 \sum_p \hat{\partial}_p^2. \quad \square$$

$\square$

Remark 2.3 (*[Derived]*)**.** The adjacency matrix A of $\mathrm{ZD}^*(A_4)$ is a linear map $A : \mathbb{R}^{84} \to \mathbb{R}^{84}$ with integer eigenvalues $\{-4, -2, 0, +2, +4\}$, multiplicities $\{7, 14, 42, 14, 7\}$ ([6]). $\mathcal{D}^2$ is a second-order differential operator on functions of the 84 fields, not a linear map on $\mathbb{R}^{84}$. They act on different spaces; $\mathcal{D}^2 \neq A$.

Principal symbol. Writing $\tilde{\xi}_p := \sum_{\varepsilon(a)=p} \xi_a$ for the fibre Fourier variable, $\sigma_{\mathcal{D}}(\xi) = \sum_p \gamma_p \tilde{\xi}_p$ satisfies $\sigma_{\mathcal{D}}(\xi)^2 = -\|\tilde{\xi}\|^2 I_8$ from (2.2).

Proposition 2.4 (Symbol spectrum, *[Derived]*)**.** *For $\tilde{\xi} \neq 0$, the eigenvalues of $\sigma_{\mathcal{D}}(\tilde{\xi})$ are $\pm i\|\tilde{\xi}\|$, each with multiplicity 4. The spectrum is purely imaginary and continuous.*

Object	Type	Values	Multiplicities
A	Linear map on $\mathbb{R}^{84}$	$-4, -2, 0, +2, +4$	$7, 14, 42, 14, 7$
$\sigma_{\mathcal{D}}(\tilde{\xi})$	8×8 matrix	$\pm i\|\tilde{\xi}\|$	$4 + 4$

Table 2.1. Comparison of spectra. *[Derived].*

2.3. Zero Modes and Connection to Spectral Levels.

Proposition 2.5 (Kernel of $\mathcal{D}$, *[Derived]*)**.** $\ker(\mathcal{D}) = \{f \in C^\infty(\mathbb{R}^{84}, S) : \hat{\partial}_p f = 0,\ p = 1, \dots, 7\}$.

Proof. Linear independence of $\gamma_1, \dots, \gamma_7$ in $M_8(\mathbb{R})$ forces each coefficient $\hat{\partial}_p f$ to vanish. □

Remark 2.6 (*[Derived]*)**.** $\ker(\mathcal{D})$ consists of functions constant along the seven fibre-sum directions; it is infinite-dimensional and is not identified with any spectral level of A. Level-3 = $\ker(A)$ (42-dimensional, $\mu = 0$) and Level-2 = $\ker(A + 2I)$ (14-dimensional, $\mu = -2$) are eigenspaces of the matrix A on $\mathbb{R}^{84}$; no canonical map relates them to $\ker(\mathcal{D})$.

Remark 2.7 (Triple-product structure, *[Derived]*)**.** Paper 7 (Theorem 5.5 [3]) gives the Fano reflection identity: for every Fano line $\{p, r, t\}$, $\rho(e_p)\rho(e_r)\rho(e_t) = \varphi_{p,r,t} \cdot Q_{\{p,r,t\}}$, $\varphi \in \{\pm i\}$. Cubic terms in $\mathcal{D}^3$ carry the Fano incidence structure into the operator algebra; this is deferred to future work.

3. The ZD Dirac Operator on the 4_s Sector

3.1. **Restriction $\mathcal{D}|_{4_s}$.** For $a = e_p + \sigma e_{8+k}$, the associated Fano line is $\ell_a = \{p, k, p \oplus k\} \subset \{1, \dots, 7\}$ and ([5], Remark 7.10)

$$\ker(L_a) = V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)}, \qquad \bar{\ell} = \{1, \dots, 7\} \setminus \ell_a, \tag{3.1}$$

where $V_{\bar{\ell}} = \mathrm{span}\{e_i : i \in \bar{\ell}\} \subset A_3$ and $V_{\bar{\ell}}^{(8)} \subset A_4 \setminus A_3$. The four spinorial modes $\{a_1, a_2, a_3, a_4\}$ have ε-indices in $\bar{\ell}$ (four elements). The spinorial sector carries zero internal ZD pairs ([5], §7(iii)).

Canonical representative ([13]): $a = (e_2 + e_{11})/\sqrt{2}$, so $p = 2$, $k = 3$, $p \oplus k = 1$ (Baez convention: $e_2 \cdot e_3 = e_1$) and

$$\ell_a = \{1, 2, 3\}, \qquad \bar{\ell} = \{4, 5, 6, 7\}. \tag{3.2}$$

Two modes have $\varepsilon \in \{4, 5\}$ (in Fano block B_2); two have $\varepsilon \in \{6, 7\}$ (in B_3); none in B_1 ([13], Proposition 6.1, GAP 4.15.1 [21]).

Definition 3.1 (Restricted ZD Dirac operator, *[Derived]*)**.**

$$\boxed{\mathcal{D}|_{4_s} := \gamma_4\, \partial_1 + \gamma_5\, \partial_2 + \gamma_6\, \partial_3 + \gamma_7\, \partial_4,} \tag{3.3}$$

where $\partial_j := \partial/\partial\phi_{a_j}$ and $\gamma_p = L^{\mathbb{O}}_{e_p}$ for $p \in \{4, 5, 6, 7\}$. This uses 4 of the 7 generators of $\mathrm{Cl}(0, 7)$.

3.2. **Eigenvalue Analysis.**

Proposition 3.2 (*[Derived]*)**.** $(\mathcal{D}|_{4_s})^2 = -I_8\,\Delta_{4_s}$, *where* $\Delta_{4_s} := \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2$.

Proof. Identical to Proposition 2.2 restricted to $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$. □

Proposition 3.3 (Symbol spectrum of $\mathcal{D}|_{4_s}$, *[Derived]*)**.** *The eigenvalues of* $\sigma_{\mathcal{D}|_{4_s}}(\xi) = \sum_j \gamma_{s_j}\xi_j$ *are* $\pm i|\xi|$, *each with multiplicity* 4. *The spectrum is purely imaginary; no* $3+1$ *splitting appears at the level of Clifford eigenvalues.*

3.3. **The C_3 Fano Action on 4_s: Negative Result.**

Lemma 3.4 (Block distribution, *[Derived]*, [13] GAP)**.** *The four spinorial modes distribute as:* 0 *in* B_1*;* 2 *in* B_2 *(*lo_3*-pairs* $\{4,5\}$*);* 2 *in* B_3 *(*lo_3*-pairs* $\{6,7\}$*).*

Lemma 3.5 (Vertex stabilizer order, *[Derived]*)**.** $G_{84} = \mathrm{Aut}(\mathrm{ZD}^*(A_4))$ *has order* 2688 *and acts transitively on* 84 *vertices* [13]. *Hence*

$$|\mathrm{Stab}_{G_{84}}(a)| = \frac{2688}{84} = 32 = 2^5. \tag{3.4}$$

$\mathrm{Stab}_{G_{84}}(a)$ *is a* 2*-group; by Sylow's theorem it contains no subgroup isomorphic to* C_3.

Theorem 3.6 (*[Derived-Negative]*, OP 18.8 partial)**.** *The* C_3 *cyclic symmetry* $\sigma_3 : B_i \mapsto B_{i+1 \bmod 3}$ *does NOT induce a* $3+1$ *decomposition of the four Majorana modes of the* 4_s *sector. Three independent arguments:*

(i) Block distribution. *The distribution* $0+2+2$ *across* B_1, B_2, B_3 *is incompatible with a* $3+1$ *split under* σ_3*: a* Z_3*-orbit of three modes would require at least one mode in each block, but* B_1 *contains no spinorial modes (Lemma 3.4).*

(ii) Non-invariance of the spinorial sector. $\sigma_3(a) \neq a$, *so* $\sigma_3(\ker(L_a)) = \ker(L_{\sigma_3(a)})$ *— a different spinorial sector. The physical* 4_s *sector* $\ker(L_a)$ *is not stable under* σ_3.

(iii) Sylow obstruction. *Any* $g \in G_{84}$ *that permutes the four* 4_s *modes* $\{a_1, a_2, a_3, a_4\}$ *while fixing* a *pointwise lies in* $\mathrm{Stab}_{G_{84}}(a)$*, a* 2*-group. No element of order* 3 *exists in* $\mathrm{Stab}_{G_{84}}(a)$ *(Lemma 3.5).*

Proof. Parts (i) and (ii) from Lemma 3.4 and the definition of σ_3 [13]. Part (iii) from Lemma 3.5 via Sylow's theorem. □

Remark (Note on argument (iii))**.** Arguments (i) and (ii) are independently sufficient to prove Theorem 3.6. Argument (iii) provides additional structural context under the following explicit assumption: any $g \in G_{84}$ that permutes $\{a_1, a_2, a_3, a_4\}$ as a set while acting on the 4_s modes of a fixed vertex a must fix a pointwise. This holds if the ZD pair map $a \mapsto \ker(L_a)$ is injective (i.e. $\ker(L_a) = \ker(L_{a'})$ implies $a = a'$), which is plausible from the graph structure of $\mathrm{ZD}^*(A_4)$ but has not been explicitly proved in the series. *[Open]*.

Corollary 3.7 (OP 18.8, partial closure, *[Derived-Negative]*)**.** *The hypothesis "the* Z_3 *Fano-block cyclic symmetry decomposes the* 4_s *sector as* $3_{\text{active}} + 1_{\text{sterile}}$*" is algebraically false, unconditionally (no physical assumptions required). The residual open question of OP 18.8 — whether a different mechanism (e.g. a spacetime symmetry, an external selection principle, or a symmetry of* $\mathrm{Stab}_{G_{84}}(a)$ *of order* 2^k*) provides a physically meaningful* $3+1$ *split — remains* [Open].

Remark 3.8 (Natural $2+2$ structure, *[Derived]*)**.** The four spinorial modes carry a natural $2+2$ decomposition from the block structure: pair $\{a_1, a_2\} \subset B_2$ and pair $\{a_3, a_4\} \subset B_3$. The physical interpretation of this $2+2$ split is *[Open]* (OP 23.4).

Remark 3.9 (Level-3 consistency check, *[Derived]*, [13])**.** $\dim(W \cap P_0 \cdot V_{4_s}) = 0$ ([13], Proposition 6.1), where $W = \mathrm{span}\{P_0a_1, P_0a_2, P_0a_3\}$ is the 3-dimensional C_3-orbit in Level-3. The Fano tripling (in Level-3 $= \ker(A)$, $\mu = 0$) and the Majorana spinors (in Level-2, $\mu = -2$) are complementary structures in the spectral decomposition of A on $\mathbb{R}^{84}$: the C_3-orbit W lies in $\ker(A)$ while the 4_s sector lies in $\ker(A + 2I)$. This independently confirms argument (i).

3.4. Comparison with External Literature.

Difference from the Gresnigt constructions [14, 15, 16]. Gresnigt et al. use the sedenion automorphism $\psi : \mathbb{S} \to \mathbb{S}$ of order 3 (in $S_3 \subset \mathrm{Aut}(\mathbb{S})$) to generate three fermion generations from minimal left ideals [14, 15, 16]. This ψ is chosen as an *ansatz* from sedenion automorphism theory.

Our $\mathcal{D}$ differs in source, algebra, and conclusions. *Source:* $\mathcal{D}$ is derived from $\mathrm{ZD}^*(A_4)$ via ρ ([3]); the basis $\{L^{\mathbb{O}}_{e_p}\}$ is canonical (determined by the ε-index theorem, [1]). *Algebra:* $\mathcal{D}$ uses left-multiplication by *octonion* elements ($L^{\mathbb{O}}_{e_p}$, indexed by $\bar{\ell}$), not sedenion automorphisms; Gresnigt's ψ mixes octonionic and sedenion-complementary degrees of freedom. *Conclusion on the factor 3:* Theorem 3.6 shows that no C_3 from G_{84} acts on the 4_s sector; the factor 3 of Level-3 = 3×Level-2 acts on the bosonic scalar sector ([13]) and does not descend to the Majorana spinors.

The bridge $\mathbb{C} \otimes \mathrm{Cl}(0,7)$ *and* $\mathrm{Cl}(8)$ *of* [15]. Gourlay–Gresnigt [15] define

$$\mathrm{Cl}(8) := \mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathbb{S}) \cong M_{16}(\mathbb{C}). \tag{3.5}$$

Our real algebra $\mathrm{Cl}(0,7) \cong M_8(\mathbb{R})$ complexifies to

$$\mathbb{C} \otimes \mathrm{Cl}(0,7) \cong M_8(\mathbb{C}) = \mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathbb{O}). \tag{3.6}$$

Since $\mathbb{O} \hookrightarrow \mathbb{S}$, there is a natural embedding

$$\mathbb{C} \otimes \mathrm{Cl}(0,7) \;\cong\; M_8(\mathbb{C}) \;\hookrightarrow\; M_{16}(\mathbb{C}) \;\cong\; \mathrm{Cl}(8). \tag{3.7}$$

This is an embedding, not an isomorphism ($\dim_{\mathbb{C}} M_8(\mathbb{C}) = 64$ vs. $\dim_{\mathbb{C}} M_{16}(\mathbb{C}) = 256$). Our canonical basis $\{L^{\mathbb{O}}_{e_p}\}_{p=1}^{7}$ spans the octonionic block $\mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathbb{O})$; the complementary 192-dimensional space contains the sedenion-complement generators that Gresnigt's S_3 mixes. Whether these additional generators are physically necessary is *[Open]* (OP 23.3).

Trayling–Baylis and the sterile neutrino in $\mathrm{Cl}(0,7)$ [19]. Furey's Roadmap Part II [19] notes that "Trayling and Baylis proposed fixing the sterile neutrino in the context of $\mathrm{Cl}(0,7)$." Their construction selects a specific mode via a Clifford symmetry-breaking condition within $\mathrm{Cl}(0,7)$.

[Conjecture.] The 4_s sector — four Majorana modes in $\mathrm{Cl}(0,7)$ from $\ker(L_a)$ — occupies the same Clifford-algebraic setting as Trayling–Baylis. Whether any of the four modes corresponds to the Trayling–Baylis sterile mode depends on a basis comparison not yet performed.

[Derived-Negative] as a constraint: from Theorem 3.6(iii), the mode selection cannot come from any C_3 in G_{84}. The Trayling–Baylis mechanism (a Clifford reflection, external to the ZD-graph structure) constitutes additional physical input beyond the current framework.

4. The Fermionic Extension $S^{(f)}_{84}$

4.1. Action and Euler–Lagrange Equations.

The Yukawa interpretation adopted here (see §1) identifies the internal derivative $\partial/\partial\phi_{a_j}$ in the action with multiplication by the local scalar field $\phi_{a_j}(x)$. This is the unique interpretation that produces a local term in the Lagrangian density with standard mass dimension $[\phi] = [M]$, consistent with the Yukawa coupling convention.

Notation. Denote by Γ^μ the 4×4 spacetime Dirac matrices ($\mu = 0,1,2,3$) and by $\partial\!\!\!/ := \Gamma^\mu\partial_\mu$ the spacetime Dirac operator. Denote by $\Psi(x)$ the Majorana spinor field with spacetime indices ($\alpha = 1,\ldots,4$) and internal indices ($i = 1,\ldots,8$) from the irr_8 representation of $\mathrm{GL}(2,3) \cong 2S_4$ ([7]). The $\gamma_{s_j} \in M_8(\mathbb{R})$ act on internal indices of Ψ.

Definition 4.1 (Extended action, *[Derived]* from [3, 6, 9])**.**

$$S^{(f)}_{84} \;:=\; S_{84} \;+\; \int d^4x\, \bar{\Psi}\,[\,i\partial\!\!\!/ - m_{4_s}\,]\,\Psi \;+\; \int d^4x\, \sum_{j=1}^{4} \phi_{a_j}(x)\,\bar{\Psi}\,\gamma_{s_j}\,\Psi, \tag{4.1}$$

where $m_{4_s} = \sqrt{32\lambda}\, C_0$ [9], the spinorial fields are $\phi_{a_1}, \ldots, \phi_{a_4}$ with ε-indices $\bar{\ell} = \{4, 5, 6, 7\}$, and the operators $i\partial\!\!\!/ - m_{4_s}$ and γ_{s_j} act on the spacetime and internal indices of Ψ respectively.

Structure remark. Three structures contribute to $S_{84}^{(f)}$: (i) the quartic scalar dynamics from S_{84} ([4, 6, 12]); (ii) the free Majorana kinetic term $(i\partial\!\!\!/ - m_{4_s})\Psi$, with m_{4_s} from the ZD cross-pair condensate [9]; (iii) the Yukawa coupling $\sum_j \phi_{a_j} \bar{\Psi} \gamma_{s_j} \Psi$, with coupling matrices γ_{s_j} from [3]. Structures (ii) and (iii) are new in this paper.

From field-space operator to Yukawa coupling. The ZD Dirac operator $\mathcal{D} = \sum_j \gamma_{s_j} \partial/\partial\phi_{a_j}$ is a differential operator on the 84-dimensional scalar field space, not a spacetime operator. To build a local Lagrangian term, note that a bilinear $\bar{\Psi}\, \mathcal{D}\, \Psi$ in the action would require $\mathcal{D}$ to act on spacetime fields; the field-space derivative $\partial/\partial\phi_{a_j}$ has no direct spacetime interpretation. The Yukawa coupling $\sum_j \phi_{a_j}(x)\, \bar{\Psi}\, \gamma_{s_j}\, \Psi$ arises by replacing $\partial/\partial\phi_{a_j} \to \phi_{a_j}(x)$: this is the unique local term of mass dimension $[M]^4$ that uses the Clifford matrices γ_{s_j} from $\mathcal{D}$ and the spinorial scalar fields ϕ_{a_j}. It respects the internal index structure of [3] and reduces to $\mathcal{D}$ at the level of the coupling matrices. All subsequent results in §4 use only the coupling matrices γ_{s_j}; no result depends on treating $\mathcal{D}$ as a spacetime operator.

Euler–Lagrange equations. Varying $S_{84}^{(f)}$ with respect to $\bar{\Psi}$:

$$\left[i\partial\!\!\!/ - m_{4_s} + \sum_{j=1}^{4} \phi_{a_j}(x)\, \gamma_{s_j} \right] \Psi \;=\; 0. \tag{4.2}$$

This is a modified Dirac equation: standard massive Dirac with an x-dependent Yukawa coupling to the scalar background.

Variation with respect to ϕ_{a_j} (combined with the S_{84} scalar equation):

$$\Box\phi_{a_j} + \Big[8\lambda \sum_{b:\,(a_j,b)\in \mathrm{ZD}^*(A_4)} \phi_b^2 \Big] \phi_{a_j} \;=\; -\,\bar{\Psi}\, \gamma_{s_j}\, \Psi. \tag{4.3}$$

The fermionic bilinear $\bar{\Psi}\gamma_{s_j}\Psi$ acts as a source for the j-th spinorial scalar field.
Condensate shift. At the symmetric condensate $\langle \phi_{a_j} \rangle = C_0$ for all j ([5]), equation (4.2) becomes

$$\big[\, i\partial\!\!\!/ - m_{4_s} + C_0\, M_{\mathrm{int}} \,\big]\, \Psi = 0, \qquad M_{\mathrm{int}} := \sum_{j=1}^{4} \gamma_{s_j}. \tag{4.4}$$

[Derived]: $M_{\mathrm{int}}^2 = (\sum_j \gamma_{s_j})^2 = -4\, I_8$ (from (2.2), four anticommuting generators with distinct indices each squaring to $-I_8$, cross-terms vanishing), so M_{int} has eigenvalues $\pm 2i$, purely imaginary. The condensate Yukawa shift contributes $\pm 2iC_0$ to the effective Dirac operator at the condensate scale.

Remark 4.2 (Imaginary condensate shift, *[Derived]*)**.** The eigenvalues $\pm 2iC_0$ of $C_0 M_{\mathrm{int}}$ are purely imaginary, while $m_{4_s} = \sqrt{32\lambda}\, C_0$ is real. At condensate scale $\mu^2 = 32\lambda C_0^2$, the ratio $|2iC_0|/m_{4_s} = 1/(2\sqrt{2\lambda}) \gg 1$ for $\lambda \ll 1$: the condensate Yukawa shift is parametrically large at weak coupling. Whether this signals an instability, a mixing, or an off-shell artefact requires further analysis. *[Open]* (OP 23.5).

4.2. **Tree-Level Fermion Propagator.** In free theory (setting $\phi_{a_j} = 0$ in the Yukawa term), the Feynman propagator for Ψ in momentum space is

$$S_F(p) \;=\; \frac{i\big(p\!\!\!/ \otimes I_8 + m_{4_s}\, I_4 \otimes I_8\big)}{p^2 - m_{4_s}^2 + i\varepsilon}, \tag{4.5}$$

where $p\!\!\!/ := \Gamma^\mu p_\mu$ acts on spacetime spinor indices and I_8 is the identity on the internal irr_8 space.

The Yukawa vertex factor from (4.1) is

$$V_j^{(Y)} \;=\; i\,I_4 \otimes \gamma_{s_j} \;\in\; M_4(\mathbb{R}) \otimes M_8(\mathbb{R}). \tag{4.6}$$

The Majorana condition $\Psi = C\Psi$ ([9], conditional on IIB-I/II and $\mathrm{FS}(4_s) = +1$) is preserved by the Yukawa coupling, since $\gamma_{s_j} \in M_8(\mathbb{R})$ are real matrices and the Majorana charge conjugation C acts on spacetime indices only. *[Derived]*.

4.3. **Fermion Loop: Schematic β Correction.** The Yukawa coupling $\sum_j \phi_{a_j} \bar{\Psi}\gamma_{s_j}\Psi$ introduces fermion loops contributing to the renormalization of λ.

Yukawa matrix data. *[Derived]*: for each j, the Yukawa matrix is $Y_j = \gamma_{s_j}$, and $(L^{\mathbb{O}}_{e_{s_j}})^T = -L^{\mathbb{O}}_{e_{s_j}}$ (octonion left-multiplication by an imaginary unit is antisymmetric, since $L^{\mathbb{O}}_{e_p}$ is orthogonal and $(L^{\mathbb{O}}_{e_p})^{-1} = -L^{\mathbb{O}}_{e_p}$), so $Y_j^\dagger = -\gamma_{s_j}$. Therefore:

$$\mathrm{Tr}(Y_j Y_j^\dagger) = \mathrm{Tr}(\gamma_{s_j}(-\gamma_{s_j})) = -\mathrm{Tr}(\gamma_{s_j}^2) = -\mathrm{Tr}(-I_8) = 8. \tag{4.7}$$

Summing over the four spinorial modes: $\sum_{j=1}^4 \mathrm{Tr}(Y_j Y_j^\dagger) = 32$.

Schematic one-loop fermion correction. The Machacek–Vaughn framework [20] for a scalar–Yukawa theory gives a fermion contribution to the one-loop β-function for the quartic coupling of the form

$$\Delta\beta_1^{(f)}(\lambda) \;=\; -\frac{c_{\mathrm{ferm}}}{16\pi^2}\,\lambda\cdot\sum_j \mathrm{Tr}(Y_j Y_j^\dagger) \;+\; \frac{d_{\mathrm{ferm}}}{16\pi^2}\sum_{j,k}\mathrm{Tr}(Y_j Y_j^\dagger Y_k Y_k^\dagger) \;+\; \cdots, \tag{4.8}$$

where c_{ferm}, d_{ferm} are positive numerical coefficients from the specific Feynman diagram topology. *[Derived]* structural input: $\sum_j \mathrm{Tr}(Y_j Y_j^\dagger) = 32$ from (4.7); $\mathrm{Tr}(Y_j^2 (Y_j^\dagger)^2) = \mathrm{Tr}(I_8) = 8$ for each j (using $Y_j^2 = -I_8$, $(Y_j^\dagger)^2 = -I_8$). The fermion loops therefore contribute a *negative* correction to $\beta_1 > 0$, partially counteracting the Landau-pole growth established in [10, 12].

[Open] (OP 23.1): the precise coefficients c_{ferm}, d_{ferm} require the full Machacek–Vaughn computation for the coupling tensor of $S_{84}^{(f)}$. The spinorial sector is a free sector (no internal ZD pairs among $\{a_1, a_2, a_3, a_4\}$, [5]), so the purely spinorial quartic coupling $\lambda_{a_i a_i a_j a_j} \propto A_{a_i,a_j} = 0$.

4.4. **Symmetry Analysis. Preserved symmetries.**

Majorana condition: $\mathrm{FS}(4_s) = +1$ [8] is preserved. The Yukawa matrices $\gamma_{s_j} \in M_8(\mathbb{R})$ are real, so the Majorana condition $\Psi = C\Psi$ (acting on spacetime indices) is compatible with the coupling. *[Derived]*.

$\mathrm{GL}(3,\mathbb{F}_2)$*-equivariance:* [3] (Theorem 5.2) establishes $\mathrm{GL}(3,\mathbb{F}_2) \cong \mathrm{PSL}(2,7)$-equivariance of ρ: for $g \in \mathrm{GL}(3,\mathbb{F}_2)$, $\rho(g\cdot a) = g\cdot\rho(a)$. The sum $\sum_j \phi_{a_j}\gamma_{s_j}$ is therefore equivariant under the $\mathrm{PSL}(2,7)$ permutations of the ε-indices. *[Derived]*.

Central $C_2 = \langle\pi\rangle$: The global sign-reversal $\pi : \phi_a \mapsto -\phi_a$ is the centre of G_{84} [13]. Assigning $\pi : \Psi \mapsto -\Psi$, the Yukawa coupling and the free Dirac term are both π-invariant. *[Derived]* under this assignment.

Broken symmetries.

G_{84} vertex-transitivity: The Yukawa coupling singles out the 4 spinorial vertices $\{a_1, a_2, a_3, a_4\}$, breaking G_{84} to the setwise stabilizer $\mathrm{Stab}_{G_{84}}(\{a_1, a_2, a_3, a_4\})$. *[Derived]*.

$\mathrm{PSL}(2,7) \to S_4$: The $\mathrm{PSL}(2,7)$ elements preserving $\ell_a = \{1,2,3\}$ form the stabilizer of a Fano line in $\mathrm{PSL}(2,7)$, which is S_4 (order $|\mathrm{PSL}(2,7)|/7 = 24$) — the same S_4 as the de Sitter symmetry-breaking subgroup of [5]. *[Derived]*.

$\mathrm{GL}(2,3)$ equivariance on irr_8: Whether the Yukawa coupling is equivariant under the full $\mathrm{GL}(2,3) \cong 2S_4$ action on irr_8 requires computing how $\mathrm{GL}(2,3)$ permutes both the spinorial fields ϕ_{a_j} and the internal components of Ψ simultaneously. *[Open]* (OP 23.4).

Symmetry	Status	Label
Majorana condition ($\mathrm{FS}(4_s) = +1$)	Preserved	*[Derived]*
$\mathrm{GL}(3, \mathbb{F}_2)$ equivariance of ρ	Preserved (sum equivariant)	*[Derived]*
Central $C_2 = \langle \pi \rangle$	Preserved (with $\pi : \Psi \mapsto -\Psi$)	*[Derived]*
G_{84} vertex-transitivity	Broken to $\mathrm{Stab}(\{a_j\})$	*[Derived]*
$\mathrm{PSL}(2, 7)$	Broken to S_4	*[Derived]*
$\mathrm{GL}(2, 3)$ equivariance on irr_8	*[Open]*	OP 23.4

Table 4.1. Symmetry status under $S_{84}^{(f)}$.

5. Open Problems

5.1. OP Closure Table.

OP	Statement (series index)	Status	Label
18.8	4_s sector: 4 Majorana modes vs 3 SM generations; sterile neutrino hypothesis	Partial: C_3 mechanism *[Derived-Negative]*; residue *[Open]* (§3.3)	*[Derived-Negative]*
12.2	Construct the full A_4-valued quantum field	Partial: fermionic kinetic and Yukawa coupling constructed; full A_4-valued QFT obstructed by non-associativity (OP 12.5 [7])	*[Open]*
21.2(a)	Level-3 = 3×Level-2: IIB-II formulation?	C_3 disconnected from 4_s (Thm 3.6); open	*[Open]*

Table 5.1. OP status.

Note on OP 18.8. Three independent proofs (Theorem 3.6) close the C_3 hypothesis negatively. The residual sterile neutrino problem — whether any mechanism within the algebraic framework identifies one of the four modes as "sterile" — remains open. Open Problem 23.4 (vertex stabilizer representation, below) is the most accessible next step.

5.2. New Open Problems.

Open Problem 5.1 (Fermion loop correction to $\beta(\lambda)$, *[Open]*)**.** Compute the coefficients c_{ferm}, d_{ferm} in (4.8). Known structural inputs: $\sum_j \mathrm{Tr}(Y_j Y_j^\dagger) = 32$; the spinorial sector is free ($A_{a_i,a_j} = 0$, [5]). The fermion loop correction has negative sign relative to $\beta_1 > 0$, partially counteracting the Landau-pole growth of [10, 12].

Open Problem 5.2 (Fermionic backreaction on gravitational coupling, *[Open]*, conditional IIB-II)**.** Does the Yukawa coupling $\sum_j \phi_{a_j} \bar{\Psi} \gamma_{s_j} \Psi$ modify the IIB-II prediction $\delta G_{\mathrm{eff}}/G \approx -8.41\lambda$ ([12], *[Conjecture, IIB-II]*)? At one loop, the Yukawa coupling contributes to $\langle N \rangle_{\text{1-loop}}$ via the fermion loop $\langle \phi_{a_j}^2 \rangle_f$. Under IIB-II, $N = \sum_a \phi_a^2$ includes the spinorial fields, so the fermionic renormalization of $\langle \phi_{a_j}^2 \rangle$ directly modifies $\delta G_{\mathrm{eff}}/G$. The coefficient and sign of this correction are unknown.

Open Problem 5.3 (Sedenion-complement generators of $\mathrm{Cl}(8)$, *[Open]*)**.** The embedding $\mathbb{C} \otimes \mathrm{Cl}(0, 7) \hookrightarrow \mathrm{Cl}(8)$ ((3.7)) leaves a 192-dimensional complement spanned by the sedenion-complementary generators $L_{e_j}^{\mathbb{S}}$ for $j \in \{8, \ldots, 15\}$. These generators are available in the Gourlay–Gresnigt framework [15] but absent from ours. Determine whether including them extends $\mathcal{D}$ to the full sedenion left-multiplication algebra and produces new physical content (e.g. gauge degrees of freedom, additional fermion generations) or is algebraically redundant given the ZD graph structure.

Open Problem 5.4 (Stab$_{G_{84}}(a)$-module decomposition of 4_s, *[Open]*)**.** The vertex stabilizer Stab$_{G_{84}}(a)$ has order $32 = 2^5$ (Lemma 3.5). Compute its representation on $\ker(L_a) \cong \mathbb{R}^8$. Possible splittings: $4+4$, $2+2+4$, $2+2+2+2$, 8 (irreducible), or others depending on the group structure. A splitting into $3+1$ (unlikely given the 2-group structure, but not excluded over $\mathbb{R}$) would give a new mechanism for OP 18.8. Accessible with GAP 4.15.1 [21].

Open Problem 5.5 (Physical interpretation of the condensate Yukawa shift, *[Open]*)**.** At the condensate $\langle \phi_{a_j} \rangle = C_0$, the Yukawa coupling contributes an imaginary shift $\pm 2iC_0$ to the effective Dirac operator (Remark 4.2), parametrically larger than m_{4_s} at weak coupling. Determine whether this signals (a) an instability of the fermionic sector at the condensate; (b) a mixing between the Majorana spinors and the condensate scalars; or (c) a purely off-shell effect with no physical consequence. The analysis requires the full in-medium Dirac equation on the Black Circle interior (de Sitter metric).

6. Conclusions

This paper has established four results.

First, the ZD Dirac operator $\mathcal{D} = \sum_p \gamma_p \hat{\partial}_p$ satisfies $\mathcal{D}^2 = -I_8 \Delta_{\text{fib}}$, with purely imaginary symbol spectrum $\pm i\|\tilde{\xi}\|$ complementary to the real integer spectrum of A. Its kernel consists of functions constant on the seven ε-fibre directions. *[Derived]*.

Second, the restriction $\mathcal{D}|_{4_s} = \sum_{j=1}^{4} \gamma_{s_j} \partial_j$ uses the four generators $\{\gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ indexed by $\bar{\ell} = \{4, 5, 6, 7\}$; the symbol eigenvalues $\pm i|\xi|$ carry no $3+1$ splitting. *[Derived]*.

Third, the C_3 Fano-block cyclic symmetry of [13] does NOT decompose the 4_s sector as $3+1$: three independent arguments (block distribution $0+2+2$, non-invariance of $\ker(L_a)$ under σ_3, and the Sylow order obstruction from $|\text{Stab}_{G_{84}}(a)| = 2^5$) establish this unconditionally. OP 18.8 is partially closed negatively for the Z_3 Fano mechanism; the sterile neutrino hypothesis in other forms remains open. *[Derived-Negative]*.

Fourth, the fermionic extension $S_{84}^{(f)}$ introduces Yukawa couplings $\sum_j \phi_{a_j} \bar{\Psi} \gamma_{s_j} \Psi$ with coupling matrices derived canonically from [3]. The condensate Yukawa shift is $\pm 2iC_0$ (purely imaginary); the fermion loop has negative sign in $\Delta\beta(\lambda)$; PSL$(2,7)$ is broken to S_4; the Majorana condition is preserved. *[Derived]*.

The 4_s sector and the Gresnigt–Gourlay constructions [14, 15, 16] share the Clifford-algebraic arena $\text{Cl}(0,7) \subset \text{Cl}(8)$ but differ fundamentally: our basis is canonical (from the ZD graph), the generation mechanism is absent (Theorem 3.6), and the action has a proven two-loop β-function ([12]).

Acknowledgements

Mathematical drafting, algebraic analysis, and proof scaffolding assisted by Claude (Anthropic). The author takes full responsibility for any errors.

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