THE FANO DICTIONARY: MODULE STRUCTURE OF V_{ZD} AND THE DARK SECTOR PARTITION

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ABSTRACT. The Fano plane $\mathrm{PG}(2,\mathbb{F}_2)$ appears twice in the Cayley-Dickson tower: in A_3 it organises multiplication; in A_4 it organises the failure of multiplication via the zero-divisor pair graph $\mathrm{ZD}^*(A_4)$. We formalise this observation as an explicit dictionary between Fano-geometric elements and their algebraic meanings in A_3 and A_4.

The central result is the $\mathrm{PSL}(2,7)$-module decomposition

$$V_{\mathrm{ZD}} := \mathrm{span}\{e_1,\dots,e_7,e_9,\dots,e_{15}\} \cong \mathbf{1}^2 \oplus \mathrm{irr}_6^2$$

[Derived], established by character theory using the Cayley-Dickson extension. Comparison with Level-2 $= \mathrm{irr}_6 \oplus \mathrm{irr}_8$ shows that the dimensional coincidence $\dim(\text{Level-2}) = \dim(V_{\mathrm{ZD}}) = 14$ is purely numerical: by Schur's lemma, no $\mathrm{PSL}(2,7)$-equivariant embedding Level-2 $\hookrightarrow V_{\mathrm{ZD}}$ exists, since irr_8 has multiplicity zero in V_{ZD} *[Derived-Negative]*. Open Problem PT1.6 is thus closed negatively. The natural algebraic home of irr_8 is identified as the Clifford spinor module of $\mathrm{Cl}(0,7)$ via the representation ρ of Paper 7 *[Derived]*. The Fano dictionary translates the bosonic/fermionic split of Level-2 into the on-line/off-line partition of $\mathrm{PG}(2,\mathbb{F}_2)$: irr_6 (bosonic, 6 modes) maps equivariantly into V_{ZD}; irr_8 (spinorial, 8 algebraic modes) lies necessarily in $\ker(L_a)$, outside V_{ZD}. Under IIB-II, this corresponds to the partition between recoverable and irrecoverable dark sector content *[Conjecture]*.

Epistemic key. *[Derived]*: follows from the series and established mathematics. *[Derived-Negative]*: precise algebraic falsification. *[Conjecture]*: precise statement, physical identification required. *[Open]*: obstacle stated precisely.

1. Introduction

The Fano plane $\mathrm{PG}(2,\mathbb{F}_2)$ — the unique projective plane over $\mathbb{F}_2$, with 7 points and 7 lines — is the combinatorial backbone of the octonion algebra A_3. Every non-trivial product $e_p \cdot e_q = \pm e_r$ in A_3 corresponds to an incidence in $\mathrm{PG}(2,\mathbb{F}_2)$; the composition norm $N(ab) = N(a)N(b)$ holds because the Fano cocycle closes on every line [3]. The Born Rule (the quantum-mechanical probability assignment via N) is valid precisely in A_3 [2].

In A_4 (sedenions) the Fano plane reappears, but in a different role. The zero-divisor pair graph $\mathrm{ZD}^*(A_4)$ — whose 84 vertices are the active zero-divisors and whose structure is organised by Fano incidence (Distinct ε-Index Theorem, [3]) — is the *broken* form of the Fano structure: the same 7 lines that governed products in A_3 now govern which products vanish in A_4. The composition norm fails (Hurwitz), while only the statistical mean $\mathbf{E}[N(av)] = N(a)$ is preserved (AMT, [2]).

This paper develops the Fano plane as a *Rosetta Stone*: the same geometric element carries one algebraic meaning in A_3 and a different meaning in A_4, with the difference encoding the physics of the IIB dark sector [2].

The central new result is the $\mathrm{PSL}(2,7)$-module structure of $V_{\mathrm{ZD}} := \mathrm{span}\{e_1,\dots,e_7,e_9,\dots,e_{15}\} \subset A_4$ (the 14-dimensional imaginary sedenion subspace introduced in Paper Tecnico 1 as the image space of the extended norm $\hat{N}$). Theorem 2.5 establishes $V_{\mathrm{ZD}} \cong \mathbf{1}^2 \oplus \mathrm{irr}_6^2$ by character theory; Theorem 2.6 shows that no $\mathrm{PSL}(2,7)$-equivariant embedding of Level-2 $= \mathrm{irr}_6 \oplus \mathrm{irr}_8$ into V_{ZD} exists,

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closing Open Problem PT1.6 negatively. Proposition 3.1 identifies irr_8 as the Clifford spinor module of $\mathrm{Cl}(0,7)$, providing its natural algebraic home.

Relation to prior work. The ε-index structure of $\mathrm{ZD}^*(A_4)$ and the role of $\mathrm{PSL}(2,7)$ as organising group were established in Papers 1–7 of this series [1, 3]. The spectral decomposition of $\mathrm{ZD}^*(A_4)$ with $\mathrm{PSL}(2,7)$-module content $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ at Level-2 ($\mu = -2$) was proved in Papers 10–11 and confirmed in Paper 21 [5, 6, 9].

2. Module Structure of V_{ZD}

2.1. **Setup and character-table conventions.** The group $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ has order 168 and six conjugacy classes, which we denote 1A, 2A, 3A, 4A, 7A, 7B with class sizes $1, 21, 56, 42, 24, 24$. Its real irreducible representations over $\mathbb{R}$ include the trivial $\mathbf{1}$, and irreducibles of dimensions 6, 7, 8; over $\mathbb{C}$ there are additionally the complex conjugate pair $\mathrm{irr}_3, \mathrm{irr}_3^*$ of dimension 3. The characters relevant here are:

Module	dim	1A	2A	3A	4A	7A	7B
$\mathbf{1}$	1	1	1	1	1	1	1
irr_6	6	6	2	0	0	-1	-1
irr_8	8	8	0	-1	0	1	1

Table 2.1. Relevant characters of $\mathrm{PSL}(2,7)$. All inner products below are computed as $\langle \chi, \psi \rangle = |G|^{-1} \sum_g \chi(g)\overline{\psi(g)}$ with $|G| = 168$.

2.2. **Decomposition of $V_1 = \mathrm{span}\{e_1, \dots, e_7\}$.**

Lemma 2.1 (*[Derived]*)**.** *The permutation character of* $\mathrm{PSL}(2,7)$ *on the* 7 *points of* $\mathrm{PG}(2,\mathbb{F}_2)$ *is*

$$\chi_{\mathrm{perm}} = (7,\ 3,\ 1,\ 1,\ 0,\ 0)$$

in the conjugacy-class order 1A, 2A, 3A, 4A, 7A, 7B.

Proof. The group $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ acts on the 7 points of $\mathrm{PG}(2,\mathbb{F}_2) = \mathrm{PG}(2,\mathbb{F}_2)$. The character value $\chi_{\mathrm{perm}}(g) = |\mathrm{Fix}(g)|$. The identity fixes all 7; the remaining values follow from the cycle-type analysis in [3], Theorem 5.2. □

Proposition 2.2 (*[Derived]*)**.** $V_1 = \mathrm{span}\{e_1, \dots, e_7\} \cong \mathbf{1} \oplus \mathrm{irr}_6$ *as* $\mathrm{PSL}(2,7)$*-modules.*

Proof. From Paper 7 (Theorem 5.2 [3]), $\mathrm{PSL}(2,7) \subset G_2 = \mathrm{Aut}(A_3)$ acts on $\{e_1, \dots, e_7\}$ by the natural permutation of imaginary octonion basis elements, which is the 7-point Fano action. Inner products of χ_{perm} against the irreducible characters give

$$\langle \chi_{\mathrm{perm}}, \mathbf{1} \rangle = \tfrac{1}{168}(7 + 63 + 56 + 42 + 0 + 0) = 1,$$
$$\langle \chi_{\mathrm{perm}}, \chi_{\mathrm{irr}_6} \rangle = \tfrac{1}{168}(42 + 126 + 0 + 0 + 0 + 0) = 1,$$

all other multiplicities vanishing. Hence $V_1 \cong \mathbf{1} \oplus \mathrm{irr}_6$. □

2.3. **Decomposition of $V_2 = \mathrm{span}\{e_9, \dots, e_{15}\}$.**

Lemma 2.3 (*[Derived]*)**.** *Every* $g \in \mathrm{PSL}(2,7)$*, extended to* A_4 *via the Cayley-Dickson formula* $\varphi(a + e_8 b) = \varphi(a) + e_8\varphi(b)$*, satisfies* $g(e_{8+k}) = e_{8+g(k)}$ *for all* $k \in \{1, \dots, 7\}$*.*

Proof. The Cayley-Dickson doubling $A_3 \to A_4$ extends every $\varphi \in G_2$ to A_4 by the formula above (see [3], §6.1). Under this extension, the index $8+k$ transforms as $8+g(k)$ since g permutes $\{1, \dots, 7\}$ and fixes e_0, e_8. □

Proposition 2.4 (*[Derived]*)**.** $V_2 = \mathrm{span}\{e_9, \dots, e_{15}\} \cong \mathbf{1} \oplus \mathrm{irr}_6$ *as* $\mathrm{PSL}(2,7)$*-modules.*

Proof. By Lemma 2.3, $g \in \mathrm{PSL}(2,7)$ acts on V_2 by $g : e_{8+k} \mapsto e_{8+g(k)}$, which is the same 7-point permutation representation as on V_1. The character and decomposition are therefore identical to those in Proposition 2.2. □

2.4. Main theorem.

Theorem 2.5 (*[Derived]*)**.**

$$V_{\mathrm{ZD}} \;=\; V_1 \oplus V_2 \;\cong\; \mathbf{1}^2 \oplus \mathrm{irr}_6^2 \quad \textit{as } \mathrm{PSL}(2,7)\textit{-modules.}$$

The character of V_{ZD} *is* $\chi_{V_{\mathrm{ZD}}} = (14,\, 6,\, 2,\, 2,\, 0,\, 0)$. *Multiplicities:* $\langle \mathbf{1}, V_{\mathrm{ZD}}\rangle = 2$, $\langle \mathrm{irr}_6, V_{\mathrm{ZD}}\rangle = 2$, *all others zero.*

Proof. Direct sum of Propositions 2.2 and 2.4 with $\chi_{V_{\mathrm{ZD}}} = 2\chi_{\mathrm{perm}}$. □

2.5. Non-existence of an equivariant embedding.

Recall from Paper 21 [9] that

$$\text{Level-2} := \ker(A+2I) \subset \mathbb{R}^{84} \;\cong\; \mathrm{irr}_6 \oplus \mathrm{irr}_8 \quad \text{as } \mathrm{PSL}(2,7)\text{-modules,}$$

with character $\chi_{\text{Level-2}} = (14,\, 2,\, -1,\, 0,\, 0,\, 0)$. Comparison with Table 2.1:

Module	dim	1A	2A	3A	4A	7A	7B
Level-2	14	14	$+2$	-1	0	0	0
V_{ZD}	14	14	$+6$	$+2$	$+2$	0	0

Table 2.2. The two 14-dimensional PSL(2,7)-modules. Decisive differences at classes 2A, 3A, 4A. *[Derived].*

The multiplicity of irr_8 in V_{ZD}:

$$\langle \chi_{\mathrm{irr}_8}, \chi_{V_{\mathrm{ZD}}}\rangle = \tfrac{1}{168}\big(8\cdot 14 + 0\cdot 6\cdot 21 + (-1)\cdot 2\cdot 56 + 0\cdot 2\cdot 42 + 0 + 0\big) = \tfrac{112-112}{168} = 0.$$

Theorem 2.6 (*[Derived-Negative]*, OP PT1.6 closed)**.** *No* $\mathrm{PSL}(2,7)$*-equivariant embedding* f : $\text{Level-2} \hookrightarrow V_{\mathrm{ZD}}$ *exists.*

Proof. By Schur's lemma, any $\mathrm{PSL}(2,7)$-equivariant linear map $f : M \to N$ decomposes along isotypic components: $f|_W = 0$ for every irreducible W that does not appear in N. Since $\langle \mathrm{irr}_8, V_{\mathrm{ZD}}\rangle = 0$, any equivariant f satisfies $f|_{\mathrm{irr}_8} = 0$, so $\ker(f) \supseteq \mathrm{irr}_8$ (an 8-dimensional subspace of Level-2). Hence f cannot be injective. □

Corollary 2.7 (*[Derived]*)**.** $\text{Level-2} \not\cong V_{\mathrm{ZD}}$ *as* $\mathrm{PSL}(2,7)$*-modules. The coincidence* $\dim(\text{Level-2}) = \dim(V_{\mathrm{ZD}}) = 14$ *is purely numerical, not a module isomorphism.*

Corollary 2.8 (*[Derived]*)**.** *A non-trivial* $\mathrm{PSL}(2,7)$*-equivariant map* f : $\text{Level-2} \to V_{\mathrm{ZD}}$ *exists if and only if* f *factors through the* irr_6 *projection:*

$$f\colon \text{Level-2} \twoheadrightarrow \mathrm{irr}_6 \xrightarrow{\iota} \mathrm{irr}_6^2 \subset V_{\mathrm{ZD}},$$

where ι *is any one of the 1-parameter family of* $\mathrm{PSL}(2,7)$*-equivariant injections* $\mathrm{irr}_6 \hookrightarrow V_{\mathrm{ZD}}$*. The* irr_8*-summand of* Level-2 *lies necessarily in* $\ker(f)$*.*

3. The Fano Dictionary and the Home of irr_8

3.1. The Fano dictionary.

The Fano plane appears in A_3 as the organiser of products and in A_4 as the organiser of zero-products. Table 3.1 makes this correspondence explicit. Every row translates one Fano-geometric element. Labels: *[Derived]* where the translation follows from the series with no additional assumptions; *[Conjecture]* where a physical identification is required.

Fano element	In A_3	In A_4	Label
Point $p \in \{1, ..., 7\}$	Imaginary unit e_p, $e_p^2 = -1$	ε-index of a ZD fibre: the 12-element family $\pi^{-1}(p) \subset \mathrm{ZD}^*(A_4)$ (from 84/7=12 per fibre)	*[Derived]*
Line $\ell = \{p, q, r\}$, $p \oplus q \oplus r = 0$	Multiplication rule $e_p \cdot e_q = \pm e_r$ (product creation)	ZD-incidence rule: $A \cdot B = 0$ iff $\varepsilon(A) = p$, $\varepsilon(B) = q$ on line ℓ (product annihilation)	*[Derived]*
Incidence $p \in \ell$	e_p participates in the product $e_p \cdot e_q = \pm e_r$	ε-index p is the A_3-foot of a ZD element; element is grounded in $V_1 \subset V_{\mathrm{ZD}}$	*[Derived]*
Non-incidence $p \notin \ell$	e_p is absent from the product defined by ℓ	$p \in \bar{\ell}$ indexes $\ker(L_a)$: $\ker(L_a) = V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)}$ [5]; the 4 off-line points generate the irrecoverable sector	*[Derived]*
Automorphism $g \in \mathrm{PSL}(2, 7)$	Algebra automorphism of A_3; preserves all products	Permutation of ε-index fibres via outer factor of $\mathrm{Aut}(A_4)$; not a subgroup of G_{84} on 84 vertices [11]	*[Derived]*
Norm N	Composition norm: $N(ab) = N(a)N(b)$ exactly	AMT: $\mathbf{E}[N(av)] = N(a)$ statistically; exact composition fails (Hurwitz)	*[Derived]*
Missing element e_8	Does not exist in A_3	Cayley-Dickson doubling generator; lies in $V_{\mathrm{ZD}}^{\perp} = \mathrm{span}\{e_0, e_8\}$, the blind spot of $\hat{N}$; IIB-I conformal boundary	*[Derived]* (alg.); *[Conjecture]* (IIB-I)

Table 3.1. Fano dictionary. Row 2 is the central entry: the same Fano-line incidence that encodes product creation in A_3 encodes product annihilation in A_4. The Fano does not disappear at the Born Rule boundary; it changes role.

3.2. **The natural home of** irr_8**.** Theorem 2.6 shows that irr_8 has no $\mathrm{PSL}(2, 7)$-equivariant image in V_{ZD}. This is a negative result about V_{ZD}, not about irr_8 itself. The following proposition identifies the correct algebraic home.

Proposition 3.1 (*[Derived]*, OP 24.3 partial)**.** *The* irr_8*-summand of* Level-2 *is the* $\mathrm{Cl}(0, 7)$*-spinor module, with the* $\mathrm{PSL}(2, 7)$*-action induced by the Clifford representation* ρ *of Paper 7.*

Proof. From Paper 7 (Definition 4.1 and Lemma 3.1 [3]), the Clifford representation $\rho : \mathrm{ZD}^*(A_4) \to \mathrm{Cl}(0, 7) \cong M_8(\mathbb{R})$ maps each ZD element to a Clifford generator $\gamma_p = L^{\mathbb{O}}_{e_p}$ acting on $\mathbb{R}^8$. This gives a group homomorphism $\mathrm{PSL}(2,7) \to \mathrm{Aut}(\mathrm{Cl}(0,7))$ whose induced action on $\mathbb{R}^8$ has character $(8,0,-1,0,1,1)$ (classes 1A,2A,3A,4A,7A,7B) — identical to χ_{irr_8} (Table 2.1), confirmed by $\langle \chi_{\mathrm{irr}_8}, \chi_{\mathrm{irr}_8} \rangle = 1$. Since $\mathrm{Cl}(0,7) \cong M_8(\mathbb{R})$ has a unique irreducible module $\mathbb{R}^8$ (the spinor module), we conclude: $\mathrm{irr}_8 =$ Clifford spinor module of $\mathrm{Cl}(0,7)$ with $\mathrm{PSL}(2,7)$-action from ρ. $\square$

Remark 3.2 (*[Derived]*)**.** irr_8 is not algebraically homeless: it lives in the Clifford spinor space $\mathbb{R}^8$ acted on by $\mathrm{Cl}(0,7)$, not in the sedenion direction space V_{ZD}. The Fano dictionary (Table 3.1, Row 4) provides the geometric explanation: the off-line Fano points $\bar{\ell}$ that index the irr_8 sector generate $\ker(L_a) = V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)}$, a subspace of A_4 that is orthogonal to V_{ZD} under the natural inner product. The spinors live where the Fano multiplication table does NOT apply.

This result connects Paper 24 to Paper 23 [12]: the ZD Dirac operator $\mathcal{D} = \sum_p \gamma_p \hat{\partial}_p$ of Paper 23 acts on the same $\mathbb{R}^8$ spinor module identified here, and the restriction $\mathcal{D}|_{4_s}$ uses precisely the generators γ_p for $p \in \bar{\ell} = \{4,5,6,7\}$ (the off-line Fano indices for the canonical representative).

4. The 14 Massless Channels

4.1. **On-line and off-line Fano partition.** The Fano dictionary (Table 3.1, Rows 3–4) partitions the 14 Level-2 modes into two sectors according to Fano-line incidence.

Proposition 4.1 (*[Derived]* (module); *[Conjecture]* (physical))**.** *The* $\mathrm{PSL}(2,7)$*-module structure of* Level-2 *mirrors the on-line/off-line partition of* $\mathrm{PG}(2,\mathbb{F}_2)$:

(i) On-line sector (irr_6*, 6 bosonic modes*)*: has a* $\mathrm{PSL}(2,7)$*-equivariant image in* V_{ZD} *(Corollary 2.8), consistent with* $\mathrm{irr}_6 \cap \ker(L_a) = 0$ *(modes not in the zero-product kernel).*

(ii) Off-line sector (irr_8*, 8 algebraic spinorial modes*)*: has no* $\mathrm{PSL}(2,7)$*-equivariant image in* V_{ZD} *(Theorem 2.6) and lies in* $\ker(L_a)$ *for every active* $a \in \mathrm{ZD}^*(A_4)$ *(Paper 10, Remark 7.10* [5]*).*

Under IIB-IV [2]*: the off-line sector is irrecoverable by Hawking radiation and crosses the CCC conformal boundary* [5]*. This physical interpretation requires identifications (a)–(c) of Paper 10, §5.6, which remain* [Open]*.*

Remark 4.2 (*[Derived]*)**.** The 8 algebraic modes of irr_8 pair under the Majorana condition of Paper 18 (Frobenius-Schur indicator $\mathrm{FS}(4_s) = +1$) to give 4 physical Majorana spinors [7, 8, 12]. The algebraic partition is $6+8$; the physical spinor count is 4.

4.2. irr_6 **as quadratic Fano geometry.**

Proposition 4.3 (*[Derived]*, Paper 21)**.** $\mathrm{irr}_6 \cong \mathrm{Sym}^2(\mathrm{irr}_3)$ *as* $\mathrm{PSL}(2,7)$*-modules, where* irr_3 *is the natural action of* $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3,\mathbb{F}_2)$ *on* $\mathbb{F}_2^3$*.*

Proof. Paper 21, Proposition 2.2 [9]. $\square$

This result has a concrete geometric meaning: the 6 bosonic massless modes transform as symmetric bilinear forms on the Fano "coordinate space" $\mathbb{F}_2^3$. They are the *quadratic Fano geometry* — the sector of the dark massless modes that knows about quadratic Fano structure. The SU(3) content $6 = \mathrm{Sym}^2(3_{\mathrm{fund}})$ (colour sextet) is consistent with this interpretation [9].

4.3. **Masslessness: no A_3 analog.** The mass formula $m_i^2 = 16\lambda C_0^2(2+\mu_i)$ gives $m^2 = 0$ at $\mu = -2$ from the exact cancellation between the condensate stabilisation $+2$ and the graph eigenvalue -2. This cancellation is algebraically forced and persists at all loop orders below the Landau pole [10].

In A_3 there is no ZD graph, no adjacency matrix, and hence no eigenvalue $\mu = -2$: the masslessness of Level-2 is a purely A_4 phenomenon with no A_3 counterpart. *[Derived-Negative].* The Fano dictionary translates it as: *the 14 modes sit exactly where the Fano multiplication fails to produce anything* — they are, in a precise algebraic sense, the "price" of the Fano's failure in A_4.

5. The Dark Sector as Broken Fano

The Fano dictionary (Table 3.1) and the module results of §§ 2–4 combine into the following structural picture.

In A_3: The Fano plane is complete. Every line encodes a product; the norm N satisfies $N(ab) = N(a)N(b)$ exactly; the Born Rule holds. $\mathrm{PSL}(2,7)$ is a subgroup of the full automorphism group $G_2 = \mathrm{Aut}(A_3)$.

At the $A_3 \to A_4$ boundary (IIB-I): The composition norm fails. $N(ab) = N(a)N(b)$ cannot hold for $n \geq 4$ (Hurwitz); only the AMT $\mathbf{E}[N(av)] = N(a)$ survives. The Fano multiplication table produces zero-divisors instead of invertible products.

In A_4: The Fano plane is *broken* but not destroyed. Three structures survive the transition *[Derived]*:

(i) *Symmetry:* $\mathrm{PSL}(2,7)$ acts on the ε-index fibres of $\mathrm{ZD}^*(A_4)$, organising the 84 ZD elements into 7 families of 12.
(ii) *Sign:* The parity constraint $\pi_A \cdot \pi_B \cdot \pi_C = -1$ for every ZD triple (Paper 1 [1]) carries the Fano sign convention into A_4.
(iii) *Norm:* N persists as a well-defined quadratic form (AMT); V_{ZD} is the subspace of A_4 that $\hat{N}$ sees without ambiguity.

What is *new* in A_4 relative to A_3:

(i) *Kernel sector:* The off-line Fano complement $\bar{\ell}$ generates $\ker(L_a)$ — a 4-dimensional irrecoverable subspace per ZD element, accounting for exactly $1/4$ of A_4.
(ii) *Spinor homelessness in V_{ZD}:* The irr_8 sector of Level-2, indexed by $\bar{\ell}$, has no $\mathrm{PSL}(2,7)$-equivariant image in V_{ZD} (Theorem 2.6). It lives instead in the Clifford spinor module of $\mathrm{Cl}(0,7)$ (Proposition 3.1).

Under IIB-II (gravity $= N$) and IIB-III ($A_4 \setminus A_3$ is the algebraic dark sector [2]), the bosonic irr_6 modes are "$\hat{N}$-visible" and the spinorial irr_8 modes are "$\hat{N}$-invisible" — the on-line/off-line Fano partition determines what gravity can and cannot see in the dark sector. This physical reading is *[Conjecture, IIB-II+III]*; the algebraic structure is *[Derived]*.

6. Open Problems

Open Problem 6.1 (Preferred equivariant embedding, *[Open]*)**.** The two irr_6 summands of V_{ZD} — one in $V_1 = \mathrm{span}\{e_1, \dots, e_7\}$ and one in $V_2 = \mathrm{span}\{e_9, \dots, e_{15}\}$ — give a 1-parameter family of equivariant embeddings of the bosonic irr_6 sector of Level-2 into V_{ZD}. Does the dynamics of S_{84} [4, 10] select a canonical embedding (into V_1, V_2, or a diagonal)? Under IIB-II, the V_1 embedding would identify the 6 massless bosons with A_3-visible directions; the V_2 embedding with $A_4 \setminus A_3$ (dark) directions.

Open Problem 6.2 (Spectral projection of V_{ZD} trivial summands, *[Open]*)**.** V_{ZD} contains two $\mathrm{PSL}(2,7)$-trivial directions $\Sigma_1 = e_1 + \cdots + e_7 \in V_1$ and $\Sigma_2 = e_9 + \cdots + e_{15} \in V_2$. The trivial $\mathrm{PSL}(2,7)$-representation does not appear in Level-2 ($\mathrm{irr}_6 \oplus \mathrm{irr}_8$ contains no trivial summand). For a 4-regular graph, the uniform vector in $\mathbb{R}^{84}$ is an eigenvector with eigenvalue 4 (Level-5, $\mu = +4$). Determine the precise spectral projection of Σ_1, Σ_2 via a natural map $V_{\mathrm{ZD}} \to \mathbb{R}^{84}$ and compute which spectral levels they contribute to.

Open Problem 6.3 (Natural home of irr_8: full equivariant map, *[Open]*)**.** Proposition 3.1 identifies irr_8 as the spinor module of $\mathrm{Cl}(0,7)$ with $\mathrm{PSL}(2,7)$-action from ρ. Determine whether there exists a $\mathrm{PSL}(2,7)$-equivariant map $\mathrm{irr}_8 \to \mathrm{span}\{e_8\} \oplus V_{\mathrm{ZD}}$ (i.e. into the full imaginary A_4 rather than V_{ZD} alone) that extends ρ naturally. The candidate is $\mathrm{Im}^*(A_4) = V_{\mathrm{ZD}} \oplus \mathrm{span}\{e_8\}$ of dimension 15.

Open Problem 6.4 (AMT-minimality of $\mathrm{ZD}^*(A_4)$, *[Open]*)**.** Does there exist a proper algebraic sub-structure $S \subsetneq \mathrm{ZD}^*(A_4)$ such that removing S from A_4 still allows the AMT $\mathbf{E}[N(av)] = N(a)$ to hold? If no such S exists, $\mathrm{ZD}^*(A_4)$ is AMT-minimal — the "minimal residual Fano" compatible with norm preservation in A_4. This would convert the structural observation of §5 into a theorem.

Open Problem 6.5 (Fano derivation of $\mu = -2$, *[Open]*)**.** Is there a purely Fano-combinatorial proof that the ZD graph adjacency matrix has eigenvalue -2 with the $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ content of Level-2? Currently the eigenvalue $\mu = -2$ is established by direct graph-spectral computation (Papers 10–11 [5, 6]); a Fano-incidence derivation would provide a deeper explanation of why the cancellation $2 + \mu_2 = 0$ occurs precisely in the Fano-permutation sector.

Open Problem 6.6 (Bosonic embedding and dark-sector observability, *[Open]*)**.** The equivariant irr_6 embedding into $V_1 \subset V_{\mathrm{ZD}}$ (the A_3-visible octonionic directions) versus $V_2 \subset V_{\mathrm{ZD}}$ (the $A_4 \setminus A_3$ directions) has a direct implication under IIB-III: V_1 modes could couple to the Standard Model sector under assumption (A2) of Paper 21 [9], while V_2 modes are strictly dark. Does the irr_6 embedding preferred by Open Problem 6.1 fall in V_1, V_2, or a combination?

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