FANO BLOCK STRUCTURE, THE COMPLETE FERMIONIC CHAIN AND ONE-LOOP YUKAWA CORRECTIONS IN S_{84}

GIOVANNI LEVRATTI

ABSTRACT. This paper is Paper 25 of the Cayley–Dickson zero-divisor series, a *consolidation paper*: every result is either derived from the cumulative algebra of Papers 1–24 without new assumptions, or computed by direct GAP 4.15.1/Python calculation on objects already constructed. Three open problems are closed. *OP A′.2* (Paper 22-B [10]): the $\mathrm{PSL}(2,7)$-isotypic split of Fano block indicators in Level-3 is determined analytically and by GAP Burnside projection [13, 14], giving $c_6 = 8/7$, $c_8 = 20/7$, $c_6 : c_8 = 2 : 5$, C_3-uniform; the hypothesised 0+4+4 algebraic distribution of Level-3 irr_8 across Fano blocks is *[Derived-Negative]*. *OP 23.1* (Paper 23 [11]): the one-loop fermion-loop correction to the S_{84} beta function (Majorana convention, $\mathrm{FS}(4_s)$=+1) is $\Delta\beta^{(f)}(\lambda) = 4(\lambda-4)/\pi^2$, with c_{ferm}=2, d_{ferm}=−2; the complete one-loop result is $\beta^{(\mathrm{total})} = (63\lambda^2+4\lambda-16)/\pi^2$. *OP 24.2* (Paper 24 [12]): $\Sigma_1 = e_1+\cdots+e_7$ projects to Level-5 (μ=+4) and $\Sigma_2 = e_9+\cdots+e_{15}$ to Level-3 (μ=0, purely irr_8) [15]. The complete fermionic chain $\rho \to \mathrm{irr}_8$=$\mathrm{Cl}(0,7)$-spinor $\to \mathrm{FS}(4_s)$=+1 $\to D|_{4_s}$ is collected as a corollary for the first time. OP 23.3 is partially closed: the physical content of the $\mathrm{Cl}(0,7)$ block is exactly irr_8=4_s. Section 6 provides a systematic 2.5-page acknowledgement of sixteen prior and concurrent works, with sixteen new bibliography entries.

CONTENTS

Date: June 2026.

2020 *Mathematics Subject Classification.* 17A35, 15A66, 81T20, 20C15, 05C50.

Key words and phrases. Cayley–Dickson algebras, sedenions, zero-divisor graph, Fano block decomposition, $\mathrm{PSL}(2,7)$-module, isotypic split, Clifford algebra $\mathrm{Cl}(0,7)$, Majorana spinors, fermionic chain, Yukawa coupling, one-loop beta function, Machacek–Vaughn.

1. Introduction

This paper is Paper 25 of the Cayley–Dickson zero-divisor series. It is a *consolidation paper*: every result is either derived from the cumulative algebra of Papers 1–24 without new assumptions, or computed by direct GAP [13]/Python [15] calculation on objects already constructed. The paper has three explicit goals.

Goal A: Close derivable open problems. Three open problems from earlier papers can now be settled using only results already in the series.

OP A′.2 (Paper 22-B [10]) asked for the PSL$(2,7)$-isotypic split of the Fano block indicators in Level-3 $= \ker(A)$. We prove that the block indicator a_i lies in $\ker(A)$ analytically (Lemma 2.1), establish C_3-uniformity by a group-equivariance argument (Theorem 2.3), and compute the exact split by GAP Burnside projection [13, 14] (Theorem 2.7):

$$c_6 = \|P_6\, a_i\|^2 = \tfrac{8}{7}, \qquad c_8 = \|P_8\, a_i\|^2 = \tfrac{20}{7}, \qquad c_6 : c_8 = 2 : 5.$$

As a consequence, the hypothesised 0+4+4 algebraic distribution of Level-3 irr_8 across Fano blocks is refuted by $c_8 \neq 0$ together with the C_3-equivariance of P_8 *[Derived-Negative]*.

OP 23.3 (Paper 23 [11]) asked for the physical content of the $\mathrm{Cl}(0,7)$ block in $\mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathcal{S})$. We show it equals $\mathrm{irr}_8 = 4_s$ exactly *[Derived]* (Proposition 3.3), partially closing OP 23.3; the 192-dimensional sedenion-complement remains open.

OP 24.2 (Paper 24 [12]) asked for the spectral projections of $\Sigma_1 = e_1 + \cdots + e_7$ and $\Sigma_2 = e_9 + \cdots + e_{15}$ onto eigenspaces of A. We prove $\Sigma_1 \to$ Level-5 ($\mu = +4$) analytically and $\Sigma_2 \to$ Level-3 ($\mu = 0$) via a sign-cancellation argument *[Derived]* (Theorem 5.3). A Python computation [15] further shows that $v_2 = \varphi_2(\Sigma_2)$ is *purely* irr_8 within Level-3 ($\|P_6 v_2\|^2 = 0$, $\|P_8 v_2\|^2 = 84$) *[Computed, Python, Theorem 5.5]*.

We also collect, for the first time in the series, the complete four-link chain connecting the ZD graph to the physical Majorana sector: $\rho \to \mathrm{irr}_8 = \mathrm{Cl}(0,7)$-spinor $\to \mathrm{FS}(4_s) = +1 \to D|_{4_s}$ (Corollary 3.1, *[Derived]*), with each link attributed to its source paper.

Goal B: The one-loop Yukawa correction to $\beta(\lambda)$. This is the only genuinely new computation in Paper 25. Using the Machacek–Vaughn formula [16, 17] with the Yukawa coupling matrices $Y_j = \gamma_{s_j}$ of Paper 23 [11], and the Majorana convention $\mathrm{FS}(4_s) = +1$ of Paper 19 [6], we derive:

$$\Delta\beta^{(f)}(\lambda) = \frac{4(\lambda - 4)}{\pi^2}, \qquad c_{\mathrm{ferm}} = 2, \quad d_{\mathrm{ferm}} = -2,$$

closing OP 23.1 *[Derived]* (Theorem 4.4). The complete one-loop beta function of $S_{84}^{(f)}$ is:

$$\beta^{(\mathrm{total})}(\lambda) = \frac{63\lambda^2 + 4\lambda - 16}{\pi^2}.$$

Goal C: Note on Related Literature. Section 6 provides a systematic 2.5-page acknowledgement of prior and concurrent work, distinguishing the series' contributions from de Marrais, Bronoff, Potton, Biss–Christensen–Dugger–Isaksen, Gillard–Gresnigt, Wilmot, Tang, Dorofeev, Masi, Dzhunushaliev, Aliferis–Leontaris–Vlachos, Okubo, and Penrose–Tod. Sixteen new `\bibitem` entries are provided.

Paper structure. Section 2 closes OP A′.2. Section 3 closes OP 23.3 partially and collects the fermionic chain corollary. Section 4 closes OP 23.1. Section 5 closes OP 24.2. Section 6 is the literature note. Section 7 is the updated open problem register.

Epistemic conventions. The series uses *[Derived]* for results that follow from prior papers without new input; *[Derived-Negative]* for algebraic falsifications; [Computed, GAP 4.15.1] or [Computed, Python] for machine-verified calculations; *[Conjecture]* for precise statements requiring further identification; *[Open]* for precisely stated obstacles. All results in this paper are *[Derived]*, *[Derived-Negative]*, or computed; no new *[Conjecture]* is introduced.

2. Fano Block Decomposition of Level-3

This section closes Open Problem A′.2 from Paper 22-B [10].

2.1. **Setup and notation.** Let $\mathbb{A}_4$ denote the sedenion algebra, $\mathrm{ZD}^*(A_4)$ its set of 84 signed zero-divisors, and A the 84×84 adjacency matrix of the ZD graph. The five eigenspaces (spectral levels) of A are:

$$\text{Level-}k, \quad \mu \in \{-4, -2, 0, +2, +4\}, \quad \text{multiplicities } \{7, 14, 42, 14, 7\}.$$

Level-3 $= \ker(A)$, dimension 42, carries the $\mathrm{PSL}(2,7)$-module structure $3 \cdot (\mathrm{irr}_6 \oplus \mathrm{irr}_8)$ *[Derived]* [10].

For each Fano axis direction $v \in \mathrm{PG}(2, \mathbb{F}_2)$, the **axis class** $\mathcal{C}_v \subset \mathrm{ZD}^*(A_4)$ consists of the 12 vertices with $p \oplus q = v$. The three Fano lines through v partition the six lo_3-pairs $\{p, q\}$ into three **Fano-line blocks** $B_1, B_2, B_3 \subset \mathcal{C}_v$, each of four vertices.

For the canonical choice $v = 1$:

- B_1: lo_3-pair $\{2, 3\}$ (Fano line $\{1, 2, 3\}$ — **on-line block**)
- B_2: lo_3-pair $\{4, 5\}$ (Fano line $\{1, 4, 5\}$ — **off-line block**)
- B_3: lo_3-pair $\{6, 7\}$ (Fano line $\{1, 6, 7\}$ — **off-line block**)

Let $a_i \in \mathbb{R}^{84}$ denote the sign-weighted indicator of block B_i:

$$a_i[(p, s, q)] = s \quad \text{for } (p, s, q) \in B_i, \qquad a_i[(p, s, q)] = 0 \text{ otherwise},$$

so $\|a_i\|^2 = 4$. Let P_0 denote the orthogonal projector onto $\ker(A)$. Let P_6, P_8 be the $\mathrm{PSL}(2,7)$-isotypic projectors onto the $3 \cdot \mathrm{irr}_6$ and $3 \cdot \mathrm{irr}_8$ components of Level-3, built via the Burnside formula

$$P_6 = \frac{\dim \mathrm{irr}_6}{|\mathrm{PSL}(2,7)|} \sum_{g \in \mathrm{PSL}(2,7)} \chi_6(g)\, \rho(g), \qquad \chi_6(g) = |\mathrm{Fix}(g)| - 1,$$

where ρ is the $\mathrm{PSL}(2,7)$ outer action on $\mathbb{R}^{84}$ [10] and $|\mathrm{Fix}(g)|$ counts the fixed Fano points of g.

The cyclic group $C_3 \leq \mathrm{PSL}(2,7)$ that stabilizes the Fano point v and permutes the three Fano lines through v acts on $\mathbb{R}^{84}$ via the outer action of $\mathrm{Aut}(\mathbb{A}_4)$ [10]:

$$\sigma_3 : B_i \mapsto B_{i+1 \bmod 3},$$

cyclically permuting the three Fano-line blocks and their indicators $a_i \mapsto a_{i+1}$.

Note. This C_3 is the stabilizer subgroup $C_3 \leq \mathrm{Stab}_{\mathrm{PSL}(2,7)}(v) \cong S_3$ inside the point-stabilizer of order 24, and it coincides with the $C_3 \leq F_{21} \leq G_{84}$ that permutes the three Fano lines through v [10]; the two copies are identified by the outer action embedding.

2.2. **Block indicators lie in Level-3.**

Lemma 2.1 (*[Derived]*)**.** *$a_i \in \ker(A)$ for each $i \in \{1, 2, 3\}$.*

Proof. For $i = 1$ ($v = 1$, lo_3-pair $\{2, 3\}$). The cases $i = 2, 3$ follow by C_3-symmetry since $a_{i+1} = \sigma_3(a_i)$ and A commutes with σ_3 (the $\mathrm{PSL}(2,7)$ outer action preserves A; see Theorem 2.7, check 4 below [13, 14]).

Since eigenvalue $+4$ has multiplicity 7 in the spectrum of A [10], the ZD graph has exactly 7 connected components, one per axis class; vertices of distinct axis classes share no edges.

We must show $(A \cdot a_1)[k] = 0$ for every vertex $k = (p, s, q)$.

Case 1: k has axis $p \oplus q \neq 1$. By the 7-component structure, no neighbor of k has axis 1, so $(A \cdot a_1)[k] = 0$.

Case 2: $k = (p, s, q) \in \mathrm{supp}(a_1)$, i.e. $\{p, q\} = \{2, 3\}$, axis 1. The ZD adjacency conditions require distinct lo_3-pairs [1], so no neighbor of k has lo_3-pair $\{2, 3\}$. Hence no neighbor of k lies in $\mathrm{supp}(a_1)$, giving $(A \cdot a_1)[k] = 0$.

Case 3: $k = (r, t, l)$ with axis 1 and lo_3-pair $\{r, l\} \neq \{2, 3\}$.

For $\{r, l\} = \{4, 5\}$: the alpha-condition of [1] gives, for $(4, t, 5)$ adjacent to $(2, \sigma, 3)$:

$$\alpha = \sigma[4][2] - t\sigma\, \sigma[5][3] = (-1) - t\sigma \cdot (+1) = 0 \quad \Rightarrow \quad \sigma = -t.$$

Here $\sigma[4][2] = -1$ and $\sigma[5][3] = +1$ are Baez–CD sign-table entries [3]. The unique neighbor in $\{2, 3\}$ of this type is $(2, -t, 3)$, with a_1-value $-t$. For $(4, t, 5)$ adjacent to $(3, \sigma', 2)$:

$$\alpha = \sigma[4][3] - t\sigma'\, \sigma[5][2] = (-1) - t\sigma' \cdot (-1) = -1 + t\sigma' = 0 \quad \Rightarrow \quad \sigma' = t.$$

The unique neighbor is $(3, t, 2)$, with a_1-value t. Total contribution: $(-t) + t = 0$.

For $\{r, l\} = \{6, 7\}$: for $(6, t, 7)$ adjacent to $(2, \sigma, 3)$, $\sigma[6][2] - t\sigma\, \sigma[7][3] = (+1) - t\sigma \cdot (+1) = 0$, giving $\sigma = t$, neighbor $(2, t, 3)$ with value t; for $(6, t, 7)$ adjacent to $(3, \sigma', 2)$, $\sigma[6][3] - t\sigma'\, \sigma[7][2] = (-1) - t\sigma' \cdot (+1) = 0$, giving $\sigma' = -t$, neighbor $(3, -t, 2)$ with value $-t$. Sum: $t + (-t) = 0$.

Hence $(A \cdot a_1)[k] = 0$ in all cases. □

Corollary 2.2. *$P_0\, a_i = a_i$ and $c_6 + c_8 = \|a_i\|^2 = 4$.*

2.3. The $\mathbf{C_3}$-uniformity theorem.

Theorem 2.3 (C$_3$ uniformity, *[Derived]*)**.** *Define $c_6 := \|P_6\, a_i\|^2$ and $c_8 := \|P_8\, a_i\|^2$. Then c_6 and c_8 are independent of $i \in \{1, 2, 3\}$.*

Proof. The projectors P_6 and P_8 are $\mathrm{PSL}(2, 7)$-equivariant by construction (the Burnside projector commutes with every $\rho(g)$), hence in particular C_3-equivariant. Since σ_3 is an isometry and $\sigma_3(a_i) = a_{i+1}$:

$$\|P_6\, a_{i+1}\|^2 = \|P_6(\sigma_3 a_i)\|^2 = \|\sigma_3(P_6 a_i)\|^2 = \|P_6 a_i\|^2. \qquad \square$$

Corollary 2.4 (*[Derived]*)**.** *The $\mathrm{irr}_6 : \mathrm{irr}_8$ decomposition of a_i within Level-3 is the same for all three Fano blocks B_1, B_2, B_3.*

2.4. The $\mathbf{C_3}$-restriction of $\mathbf{irr_6}$.

Proposition 2.5 (*[Derived]*)**.** $\mathrm{irr}_6\big|_{C_3} = 2 \cdot \mathbf{1} \oplus 2\chi_1 \oplus 2\chi_2$*, where $\mathbf{1}, \chi_1, \chi_2$ are the three irreducible characters of C_3.*

Proof. Since $\mathrm{irr}_6 = \mathrm{Sym}^2(\mathrm{irr}_3)$ [8], and σ_3 acts on irr_3 with eigenvalues $1, \omega, \omega^2$ ($\omega = e^{2\pi i/3}$) each of multiplicity 1, the symmetric-square eigenvalues are the six pairwise products $\{1, \omega, \omega^2, \omega^2, 1, \omega\} = 2 \cdot \{1, \omega, \omega^2\}$. □

Remark 2.6. The permutation representation of C_3 on $\{B_1, B_2, B_3\}$ is $\mathbf{1} \oplus \chi_1 \oplus \chi_2$. Since $\mathrm{irr}_6|_{C_3}$ is exactly twice this representation, the irr_6 content distributes symmetrically across the three blocks in a C_3-equivariant sense. The precise fraction per block is $c_6/\|a_i\|^2 = 2/7$ (from Theorem 2.7 below).

2.5. Exact isotypic split: GAP computation.

Theorem 2.7 (Exact isotypic split, *[Computed, GAP 4.15.1, v6, 14 June 2026]*)**.** *With the $\mathrm{PSL}(2, 7)$ outer action on $\mathrm{ZD}^*(A_4)$ given by*

$$\rho(g) : (p, s, q) \longmapsto \big(g(p),\ s \cdot \varepsilon_g(p, q),\ g(q)\big), \quad \varepsilon_g(p, q) := \sigma[g(p)][g(q)] \cdot \sigma[p][q] \in \{+1, -1\},$$

where $\sigma[\cdot][\cdot]$ is the Baez–CD octonion sign table [3], the Burnside projector P_6 gives:

$$\boxed{c_6 = \frac{8}{7}, \qquad c_8 = \frac{20}{7}, \qquad c_6 : c_8 = 2 : 5.}$$

The value is the same for all three Fano blocks by Theorem 2.3.

Proof. Direct GAP 4.15.1 computation [13], script `OP_A_PRIME_2_ISOTYPIC_v6.g` [14] (14 June 2026). Verification checks:

- ZD graph: 84 vertices, 4-regular, 168 edges. ✓
- $\mathrm{Tr}(A^2) = 336$; $\dim\ker A = 42$. ✓
- $a_1 \in \ker A$ $(A \cdot a_1 = 0)$. ✓
- $\rho(g)$ preserves A for all 168 elements of $\mathrm{GL}(3, \mathbb{F}_2)$. ✓
- $\sum_g \chi_6(g) = 0$; $\frac{1}{168}\sum_g \chi_6(g)^2 = 1$ (irr_6 irreducible). ✓
- $c_6 + c_8 = 4 = \|a_1\|^2$; B2 and B3 yield identical c_6, c_8. ✓

□

Remark 2.8 (Sign correction for the PSL(2,7) outer action). The twist $\varepsilon_g(p,q) \in \{+1,-1\}$ compensates the change of orientation of the Fano line $\{p, q, p \oplus q\}$ under g. Two of the seven Fano lines have reversed orientation in the Baez–CD sign table versus the XOR-canonical choice: $\{1,6,7\}$ (Baez cycle $7 \to 6 \to 1$) and $\{3,5,6\}$ (Baez cycle $6 \to 5 \to 3$). The 21-element Frobenius subgroup $F_{21} \le \mathrm{GL}(3,\mathbb{F}_2)$ preserves the Baez sign table without correction; the remaining 147 elements require the twist. With ε_g included, all 168 elements are verified graph automorphisms [13].

2.6. Physical irr_8 distribution: Level-2 versus Level-3.

Lemma 2.9 (*[Derived]*, Paper 23 [11] Lemma 3.4)**.** *The four Majorana spinors of the 4_s-sector* $\ker(A + 2I) \cap \ker(L_a)$ *in **Level-2** (eigenvalue $\mu = -2$) occupy vertices with* lo_3*-pairs* $\{4,5\}$ *and* $\{6,7\}$ *(blocks B_2, B_3). The distribution across (B_1, B_2, B_3) is* 0+2+2*.*

Theorem 2.10 (*[Derived-Negative]*, Paper 23 [11] Theorem 3.6)**.** *C_3 does NOT decompose the four Level-2 Majorana modes as* 3+1*; the* 0+2+2 *distribution is permanently asymmetric.*

Remark 2.11 (Spectral-sector distinction). The 0+2+2 pattern of Lemma 2.9 belongs to **Level-2** ($\ker(A+2I)$, eigenvalue $\mu = -2$, spinorial sector 4_s). This is a different spectral sector from **Level-3** ($\ker A$, eigenvalue $\mu = 0$, module structure $3 \cdot (\mathrm{irr}_6 \oplus \mathrm{irr}_8)$, dimension 42). There is no reason to expect Level-3 irr_8 to mirror the Level-2 pattern.

Indeed, $c_8 = 20/7 \neq 0$ (Theorem 2.7) shows that the block indicator a_1, supported entirely on B_1, has nonzero projection onto the irr_8 subspace of Level-3. Therefore at least one Level-3 irr_8 eigenmode has nonzero support on B_1.

Moreover, since P_8 is PSL(2,7)-equivariant (hence C_3-equivariant), the irr_8 subspace of Level-3 is C_3-invariant, and the same isometry argument as in Theorem 2.3 shows the total irr_8 weight on B_1 equals that on B_2 and on B_3. The hypothesized 0+4+4 algebraic distribution is therefore ***[Derived-Negative]***: excluded by $c_8 \neq 0$ and by C_3-equivariance of P_8. The correct statement is that the Level-3 irr_8 eigenmodes have **C_3-uniform** support across B_1, B_2, B_3.

2.7. Summary: status of OP A′.2.

Claim	Status
$a_i \in \ker(A)$ for all i — proved analytically	*[Derived]* — Lemma 2.1
C_3-uniform split: c_6, c_8 independent of i	*[Derived]* — Theorem 2.3
$\mathrm{irr}_6\vert_{C_3} = 2 \cdot (\mathbf{1} \oplus \chi_1 \oplus \chi_2)$	*[Derived]* — Proposition 2.5
Exact values $c_6 = 8/7$, $c_8 = 20/7$, ratio $2:5$	*[Computed, GAP 4.15.1]* [13, 14] — Theorem 2.7
Level-2 irr_8 (4_s): distribution 0+2+2	*[Derived]* [11] — Lemma 3.4
Level-3 irr_8: distribution 0+4+4 — **excluded**	*[Derived-Negative]* — Theorem 2.7 + Remark 2.11
Level-3 irr_8: C_3-uniform support across blocks	*[Derived]* — Remark 2.11

OP A′.2 status: CLOSED.

3. The Complete Fermionic Chain

This section partially closes OP 23.3 [11] *and states explicitly the fermionic chain corollary implicit across Papers 7, 19, 23, 24* [3, 6, 11, 12].

3.1. The fermionic chain.

Corollary 3.1 (Complete fermionic chain, *[Derived]*)**.** *The fermionic sector of $S_{84}^{(f)}$ is completely characterised by the following four-step chain.*

Step (i): $\rho : \mathrm{ZD}^*(A_4) \to \mathrm{Cl}(0,7) \cong M_8(\mathbb{R})$ [3]. *Every ZD element maps to a Clifford generator:*

$$\rho(e_p + \sigma e_{8+k}) := L^{\mathcal{O}}_{e_p} =: \gamma_p,$$

where $L^{\mathcal{O}}_{e_p}$ is the 8×8 real matrix of left-multiplication by e_p in $\mathcal{O} = A_3$. The map depends only on the ε-index p; it is constant on each of the seven fibres, each of size 12. The generators satisfy $\{\gamma_p, \gamma_q\} = -2\delta_{pq} I_8$, so $\gamma_p^2 = -I_8$ [3].

Step (ii): $\mathrm{irr}_8 = \mathrm{Cl}(0,7)$*-spinor module* [12]. *The representation ρ induces $\mathrm{PSL}(2,7) \to \mathrm{Aut}(\mathrm{Cl}(0,7))$ whose action on $\mathbb{R}^8$ has character $(8, 0, -1, 0, 1, 1)$ on the six conjugacy classes of $\mathrm{PSL}(2,7)$, matching χ_{irr_8} exactly. Since $\mathrm{Cl}(0,7) \cong M_8(\mathbb{R})$ has a unique irreducible module $\mathbb{R}^8$ (the standard spinor module):*

$$\mathrm{irr}_8 \;=\; \mathrm{Cl}(0,7)\text{-spinor module with } PSL(2,7)\text{-action from } \rho.$$

Step (iii): $\mathrm{FS}(4_s) = +1$ [6]. *The Frobenius–Schur indicator of the 4_s spinorial irreducible of $\mathrm{GL}(2,3) \cong 2S_4$ equals $+1$. Since $\mathrm{FS} = +1$ is the criterion for the existence of a G-invariant symmetric bilinear form, the 4_s sector admits a Majorana condition: there exist four linearly independent Majorana spinors $\Psi^{(j)}$, $j = 1, \ldots, 4$.*

Step (iv): $D|_{4_s} = \sum_{p \in \bar{\ell}} \gamma_p \, \partial_p$ [11]. *For the canonical ZD element $a = (e_2 + e_{11})/\sqrt{2}$ with Fano line $\ell_a = \{1, 2, 3\}$ and off-line complement $\bar{\ell} = \{4, 5, 6, 7\}$:*

$$D\big|_{4_s} = \gamma_4 \partial_1 + \gamma_5 \partial_2 + \gamma_6 \partial_3 + \gamma_7 \partial_4,$$

where $\partial_j = \partial / \partial \phi_{a_j}$ and $\gamma_p = L^{\mathcal{O}}_{e_p}$ for $p \in \bar{\ell}$.

The chain is complete: every link is [Derived] *from the respective paper without additional hypotheses.*

Remark 3.2 (Yukawa coupling and coupling matrix). The Yukawa extension $S_{84}^{(f)}$ [11] adds

$$\mathcal{L}_{\mathrm{Yuk}} = \sum_{j=1}^{4} \phi_{a_j}(x) \, \bar{\Psi} \, \gamma_{s_j} \, \Psi,$$

where $s_j \in \bar{\ell} = \{4,5,6,7\}$ and $Y_j := \gamma_{s_j} \in M_8(\mathbb{R})$ are the Yukawa coupling matrices. These satisfy:

$$\mathrm{Tr}(Y_j Y_j^\dagger) = \mathrm{Tr}(-\gamma_{s_j}^2) = \mathrm{Tr}(I_8) = 8, \quad \Sigma := \sum_{j=1}^{4} \mathrm{Tr}(Y_j Y_j^\dagger) = 32,$$

and $M_{\mathrm{int}} := \sum_j \gamma_{s_j}$ satisfies $M_{\mathrm{int}}^2 = -4I_8$ [11].

3.2. **Physical content of Cl(0,7) and partial closure of OP 23.3.** Paper 23 [11] establishes the Clifford algebra embedding:

$$\mathbb{C} \otimes \mathrm{Cl}(0,7) \cong M_8(\mathbb{C}) \hookrightarrow M_{16}(\mathbb{C}) \cong \mathrm{Cl}(8) := \mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathcal{S}),$$

where $\mathcal{S} \cong \mathbb{R}^{16}$ is the sedenion algebra. The embedding is as the octonionic block $\mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathcal{O})$; the complementary $256 - 64 = 192$-dimensional space consists of sedenion-complement generators.

Proposition 3.3 ($\mathrm{Cl}(0,7)$ physical content, *[Derived]*; OP 23.3 partially closed)**.** *The physical content of the* $\mathrm{Cl}(0,7)$ *block in* $\mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathcal{S})$ *is exactly* $\mathrm{irr}_8 = 4_s$ *— the Majorana sector — with the* $PSL(2,7)$*-action of Corollary 3.1, Step (ii). The 192-dimensional sedenion-complement space has no identified physical content within the current framework.*

Proof. From Step (ii) of Corollary 3.1, irr_8 is the unique irreducible $\mathbb{R}^8$-module of $\mathrm{Cl}(0,7) \cong M_8(\mathbb{R})$, with the $\mathrm{PSL}(2,7)$-action induced by ρ. The $\mathrm{Cl}(0,7)$ block therefore carries irr_8 (algebraically) and the four Majorana spinors (physically, by Step (iii)). No other representation of $\mathrm{PSL}(2,7)$ appears in the $\mathrm{Cl}(0,7)$ block. The 192-dimensional complement contains sedenion-complement generators not reached by ρ [11]; whether they have physical content is *[Open]* (OP 23.3 residue). ☐

OP 23.3 status: PARTIALLY CLOSED *[Derived]*. The $\mathrm{Cl}(0,7)$ content is $\mathrm{irr}_8 = 4_s$ (Proposition 3.3). The 192-dimensional sedenion-complement space remains *[Open]*.

4. One-Loop Yukawa Correction to $\beta(\lambda)$

This section closes OP 23.1. It is the only genuinely new numerical result in Paper 25. All ingredients are in Papers 7, 19, 20, 23 [3, 6, 7, 11]*. The two-loop scalar correction is in Paper 22-A* [9]*; here we compute only the one-loop fermion correction.*

4.1. **Setup and Machacek–Vaughn framework.** The Yukawa sector of $S_{84}^{(f)}$ is [11]:

$$\mathcal{L}_{\mathrm{Yuk}} = \sum_{j=1}^{4} \phi_{a_j}(x)\, \bar{\Psi}\, \gamma_{s_j}\, \Psi,$$

where $s_j \in \bar{\ell} = \{4,5,6,7\}$, Ψ is the 8-component Majorana spinor, and $Y_j := \gamma_{s_j} \in M_8(\mathbb{R})$ are the Yukawa coupling matrices. The quartic coupling λ is the unique S_{84} quartic [4], with one-loop pure-scalar beta function [7]:

$$\beta^{(1)}(\lambda) = \frac{63\lambda^2}{\pi^2} = \frac{1008\lambda^2}{16\pi^2}.$$

To compute the fermion-loop correction $\Delta\beta^{(f)}(\lambda)$, we apply the Machacek–Vaughn formula [16, 17] in the $\overline{\mathrm{MS}}$ scheme with

$$Y^{(2)} := \sum_{j=1}^{4} Y_j Y_j^\dagger, \qquad Y^{(4)} := \Big(\sum_{j=1}^{4} Y_j Y_j^\dagger\Big)^2.$$

The formula for Dirac fermions gives $16\pi^2 \Delta\beta|_{\mathrm{Dirac}} = 4\lambda\, \mathrm{Tr}[Y^{(2)}] - 4\, \mathrm{Tr}[Y^{(4)}]$. Since Ψ is Majorana ($\mathrm{FS}(4_s) = +1$, Paper 19 [6] Prop. 4.2), we apply a factor of $\frac{1}{2}$:

$$\boxed{16\pi^2 \Delta\beta^{(f)}\big|_{\mathrm{Majorana}} = 2\lambda\, \mathrm{Tr}[Y^{(2)}] - 2\, \mathrm{Tr}[Y^{(4)}].}$$

4.2. Evaluation of the trace invariants.

Lemma 4.1 (*[Derived]*)**.** *The Yukawa coupling matrices* $Y_j = \gamma_{s_j}$ *satisfy* $Y_j Y_j^\dagger = I_8$ *for each* $j = 1, \ldots, 4$.

Proof. The generators $\gamma_p = L^{\mathcal{O}}_{e_p}$ are 8×8 real matrices. Left-multiplication is an isometry on $\mathbb{R}^8$, so $\gamma_p^\top = \gamma_p^{-1}$; and $\gamma_p^{-1} = L^{\mathcal{O}}_{-e_p} = -\gamma_p$ [3]. Hence $\gamma_p^\dagger = \gamma_p^\top = -\gamma_p$, giving $Y_j Y_j^\dagger = \gamma_{s_j}(-\gamma_{s_j}) = -\gamma_{s_j}^2 = -(-I_8) = I_8$. □

Corollary 4.2 (*[Derived]*)**.** $\mathrm{Tr}(Y_j Y_j^\dagger) = 8$ *and* $\Sigma := \sum_{j=1}^4 \mathrm{Tr}(Y_j Y_j^\dagger) = 32$*. These reproduce* [11] *exactly.*

Proposition 4.3 (*[Derived]*)**.** $Y^{(2)} = 4I_8$, $Y^{(4)} = 16I_8$, *and consequently* $\mathrm{Tr}[Y^{(2)}] = 32$, $\mathrm{Tr}[Y^{(4)}] = 128$.

Proof. $Y^{(2)} = \sum_j I_8 = 4I_8$ by Lemma 4.1. Squaring gives $Y^{(4)} = 16I_8$. Traces follow immediately. □

4.3. The one-loop Yukawa correction.

Theorem 4.4 (OP 23.1, *[Derived]*)**.** *The one-loop fermion-loop correction to the* S_{84} *beta function, in the Majorana convention with* $\mathrm{FS}(4_s) = +1$*, is:*

$$\Delta\beta^{(f)}(\lambda) = \frac{1}{16\pi^2}\left[2\lambda\,\mathrm{Tr}[Y^{(2)}] - 2\,\mathrm{Tr}[Y^{(4)}]\right] = \frac{64\lambda - 256}{16\pi^2} = \frac{4(\lambda-4)}{\pi^2}.$$

The Machacek–Vaughn coefficients are $c_{\mathrm{ferm}} = 2$, $d_{\mathrm{ferm}} = -2$*. (Dirac values:* $c = 4$, $d = -4$*; the Majorana factor* $\frac{1}{2}$ *halves both.)*

Proof. Substituting Proposition 4.3 into the Majorana MV formula: $16\pi^2 \Delta\beta^{(f)} = 2\lambda \cdot 32 - 2 \cdot 128 = 64\lambda - 256$. □

Sign verification. At the BF stability bound $\lambda G M^2 < 2.14 \times 10^{-3}$ [5] (i.e. $\lambda \ll 1$), the dominant term is $-256/16\pi^2 \approx -1.62 < 0$, confirming the prediction of Paper 23 [11] ("fermion loop correction has negative sign" *[Derived]*). The zero of $\Delta\beta^{(f)}$ occurs at $\lambda^* = 4$, far above the physically relevant regime, so $\Delta\beta^{(f)} < 0$ throughout.

Corollary 4.5 (Complete one-loop β-function, *[Derived]*)**.**

$$\beta^{(\mathrm{total})}(\lambda) = \frac{63\lambda^2 + 4\lambda - 16}{\pi^2}, \qquad 16\pi^2 \frac{d\lambda}{dt}\bigg|_{\textit{one loop}} = 1008\lambda^2 + 64\lambda - 256.$$

Remark 4.6 (Physical implications). (i) *Sign competition.* At $\lambda \sim 10^{-3}$, the fermion constant term $-16/\pi^2 \approx -1.62$ dominates by $\sim 10^5$ over the scalar quadratic term $63\lambda^2/\pi^2 \approx 6.4 \times 10^{-6}$. The fermion loop is not a small perturbation at small λ.

(ii) *Infrared instability.* $\beta^{(\mathrm{total})} < 0$ for $\lambda < \lambda_{\mathrm{IR}} \approx 0.473$ (zero from $63\lambda^2 + 4\lambda - 16 = 0$). The two-loop Yukawa correction is *[Open]* (OP 25.2).

(iii) *IIB-II relevance.* Under $N =$ gravity *[Conjecture]* (IIB-II, [2]), the negative fermion contribution $\Delta\beta^{(f)} < 0$ corresponds to a gravitational coupling that weakens in the infrared, potentially consistent with asymptotic safety. This remains *[Conjecture]* (IIB-II).

OP 23.1 status: CLOSED *[Derived].*

5. Spectral Projections of Σ_1 and Σ_2

This section closes OP 24.2 [12].

5.1. **Setup.** We define:

$$\Sigma_1 := e_1 + \cdots + e_7 \in V_1 \subset V_{\mathrm{ZD}}, \qquad \Sigma_2 := e_9 + \cdots + e_{15} \in V_2 \subset V_{\mathrm{ZD}},$$

where $V_1 = \mathrm{span}\{e_1, \ldots, e_7\}$ is the octonion imaginary subspace and $V_2 = \mathrm{span}\{e_9, \ldots, e_{15}\}$ is the sedenion-complement subspace [12]. Each vertex $a = (p, \sigma, q) \in \mathrm{ZD}^*(A_4)$ represents $(e_p + \sigma e_{8+q})/\sqrt{2}$. Define natural maps:

$$v_1 := \varphi_1(\Sigma_1) = (1, \ldots, 1) \in \mathbb{R}^{84}, \qquad v_2 := \varphi_2(\Sigma_2) = (\sigma(a))_a \in \{+1, -1\}^{84}.$$

5.2. **Projection of Σ_1.**

Proposition 5.1 (*[Derived]*)**.** *$v_1 \in$ Level-5, i.e. $A \cdot v_1 = +4 \cdot v_1$.*

Proof. For any 4-regular graph, $A \cdot \mathbf{1} = 4 \cdot \mathbf{1}$. The ZD graph is 4-regular [1]. Level-5 is the eigenspace of A with eigenvalue $+4$, of dimension 7 [10]. □

Remark 5.2. v_1 is the unique (up to scale) $\mathrm{PSL}(2, 7)$-invariant vector in Level-5.

5.3. **Projection of Σ_2.**

Theorem 5.3 (*[Derived]*)**.** *$v_2 \in$ Level-3, i.e. $A \cdot v_2 = 0$.*

Proof. $(A \cdot v_2)[a] = \sum_{b\sim a} \sigma(b)$. From the alpha-conditions of Lemma 2.1 Case 3 [1], for any vertex $a = (r, t, l)$, the 4 neighbors in the two off-diagonal lo$_3$-pairs carry signs that cancel pairwise: the two neighbors in $\{p_1, q_1\}$ have signs $-t, +t$; the two in $\{p_2, q_2\}$ have signs $+t, -t$. Hence $(A \cdot v_2)[a] = (-t + t) + (-t + t) = 0$.

Computationally verified: Python [15] (14 June 2026): $A \cdot v_2 = 0$ for all 84 vertices. □

Remark 5.4 (Sign structure of edges). Among the 168 edges, exactly 84 connect vertices of equal sign and 84 connect vertices of opposite sign — the combinatorial expression of $A \cdot v_2 = 0$.

5.4. **Isotypic content of v_2 in Level-3.**

Theorem 5.5 (*[Computed, Python [15], 14 June 2026]*)**.** *The sedenion-sign vector v_2 is* purely irr_8 *within Level-3:*

$$\|P_6\, v_2\|^2 = 0, \qquad \|P_8\, v_2\|^2 = 84 = \|v_2\|^2.$$

Proof. The Burnside sum $w_6 = \sum_g \chi_6(g)\rho(g)v_2$ vanishes identically [15], giving $\|P_6\, v_2\|^2 = 0$. Since $v_2 \in$ Level-3 (Theorem 5.3), $\|P_8\, v_2\|^2 = \|v_2\|^2 = 84$. □

Remark 5.6. Structural separation: the octonion imaginary sum Σ_1 is stable (Level-5, $\mu = +4$), while the sedenion-complement imaginary sum Σ_2 is massless-spinorial (Level-3, $\mu = 0$, purely irr_8).

OP 24.2 status: CLOSED.

Open Problem 5.7 (OP 25.3)**.** The vanishing $\|P_6\, v_2\|^2 = 0$ means $v_2 \perp$ (irr_6 subspace of Level-3). Determine whether this has a representation-theoretic explanation: which specific irr_8 vector in the 24-dimensional $3 \cdot \mathrm{irr}_8$ subspace does v_2 correspond to? *[Open].* Accessible via the explicit irr_8 Burnside projector P_8.

6. Note on Related Literature

6.1. **Prior and concurrent work: what is new in this series. de Marrais (2000–2007).** The discovery that the 84 signed zero-divisors of $\mathbb{A}_4$ organise into 42 "Assessors" via 7 "Box-Kites" indexed by the Fano plane, and that $\mathrm{PSL}(2, 7)$ appears as the organising group, is due to de Marrais [18, 19]. This is prior work the series builds on. What is new relative to de Marrais: the $\mathrm{PSL}(2, 7)$-equivariant module decomposition $\mathrm{irr}_6 \oplus \mathrm{irr}_8$ of Level-2; the adjacency spectrum

$\{-4,-2,0,+2,+4\}$ with multiplicities $\{7,14,42,14,7\}$; and the automorphism group G_{84} of order 2688. De Marrais does not construct the ZD graph, compute its spectrum, or decompose $\mathbb{R}^{84}$ as a $\mathrm{PSL}(2,7)$-module.

Bronoff (2016). A ResearchGate preprint (non-peer-reviewed) identifies the 1-eigenspace of L_a on $\mathbb{A}_4$ with SU(3) and derives three generations [20]. This constitutes prior art on the "eigentheory of L_a implies SM symmetries / 42 pairs" bridge. The series differs in object: we study the adjacency matrix A on $\mathbb{R}^{84}$, not the operator $M_a = |a|^{-2}L_{a^*}L_a$ studied by Bronoff and by Biss–Christensen–Dugger–Isaksen [23].

Potton (2023–24). Two ResearchGate preprints (non-peer-reviewed) classify the 84 ZD by Clifford grade parity [21, 22]. The series' sign-parity constraint $\pi_A\pi_B\pi_C = -1$ [1] is a product-of-signs condition on ZD triples, not a grade-parity superselection rule. The two frameworks are conceptually adjacent but algebraically distinct; Potton does not obtain the group-theoretic structure or the ZD graph.

Biss–Christensen–Dugger–Isaksen (2009). This peer-reviewed paper establishes the rigorous eigentheory of $M_a = |a|^{-2}L_{a^*}L_a$ on A_n [23]: M_a is diagonalizable with non-negative eigenvalues, and a is a zero-divisor iff $0 \in \mathrm{spec}(M_a)$. The series' adjacency spectrum is for the graph adjacency matrix A on $\mathbb{R}^{84}$, a different object.

Gillard–Gresnigt (2019). Three fermion generations with gauge symmetry $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{em}$ arise from $\mathbb{C}\otimes\mathbb{S}$ via Clifford-algebra left ideals of Cl(6) [24]. The series uses the zero-divisor *set* $\mathrm{ZD}^*(A_4)$ as a graph and the Clifford representation $\rho : \mathrm{ZD}^*(A_4) \to \mathrm{Cl}(0,7)$ [3]; the generation mechanism is different, and the series produces a *[Derived-Negative]* result for the analogous C_3 Fano-block mechanism (Paper 23 [11], Theorem 3.6).

Wilmot (arXiv:2505.11747, 2025). Derives a cardinality formula for the ZD set of A_n via quasi-octonion subalgebras, confirming 84 for $n = 4$ [25] by an independent method. Wilmot does not construct the adjacency spectrum or $\mathrm{PSL}(2,7)$ module structure.

Wilmot (arXiv:2512.07210, December 2025). Derives $\mathrm{Aut}(\mathbb{S}) \cong G_2 \times S_3$ [26]. This should be distinguished from $G_{84} = \mathrm{Aut}(\mathrm{ZD}^*(A_4))$ of order 2688: G_{84} is a *finite* group (automorphisms of the zero-divisor set as a graph), while $\mathrm{Aut}(\mathbb{S}) \cong G_2 \times S_3$ is an *infinite compact Lie group* (G_2 has Lie-algebra dimension 14; S_3 has order 6). No comparison of their "orders" in the finite-group sense is meaningful.

Aliferis–Leontaris–Vlachos (2017). This peer-reviewed paper studies $\mathrm{SL}(2,7)$ representations (the non-trivial double cover of $\mathrm{PSL}(2,7)$, of order 336) and their relevance to neutrino physics [27]. Its explicit character-table data provide the complex-representation structure of $\mathrm{PSL}(2,7)$ used in the series. In particular: over $\mathbb{C}$, the real 6-dimensional irrep decomposes as $\mathrm{irr}_6 = \mathrm{irr}_3 \oplus \overline{\mathrm{irr}_3}$ (two dual 3-dimensional complex irreps); and irr_8 is the unique irreducible 8-dimensional complex representation of $\mathrm{PSL}(2,7)$ — absolutely irreducible, not decomposable over $\mathbb{C}$. The splitting $\mathbf{8} = \mathbf{1}' \oplus \bar{\mathbf{1}}' \oplus \mathbf{3} \oplus \bar{\mathbf{3}}$ in [27] refers to a representation of $\mathrm{SL}(2,7)$, not of $\mathrm{PSL}(2,7)$; the two groups are distinct (orders 336 vs 168) and the splitting is not applicable to the series. The character-table entries for irr_6 and irr_8 of $\mathrm{PSL}(2,7)$ from [27] ground the series' identification $\mathrm{irr}_6 = \mathrm{Sym}^2(\mathrm{irr}_3)$ [8] and $\mathrm{irr}_8 = 4_s$ [12].

Okubo (1994). The construction of $\mathrm{Cl}(0,7) \cong M_8(\mathbb{R})$ from 8-component spinors [28] underpins the series' Clifford representation ρ [3] and the identification $\mathrm{irr}_8 = \mathrm{Cl}(0,7)$-spinor [12].

Tang (2026). An independent program derives a Yukawa-type correction to the Newtonian potential from the sedenion associator tensor $B_{\mu\nu}$, fitting galaxy rotation curves without dark matter [29]. This is a concurrent independent work. Tang's mechanism uses the non-associativity of multiplication; the series' IIB-II conjecture [2] uses the failure of the norm-composition identity at the $A_3 \to A_4$ threshold — fundamentally different identifications. Similarly, Dorofeev [30] identifies gravity with the non-associativity of the octonion algebra itself (associator → curvature).

Masi (2021). Uses $\mathrm{Aut}(\mathbb{A}_4) \supset G_2$ to propose an extension of the Standard Model [31]. The series works with $\mathrm{ZD}^*(A_4)$ (the 84-element vertex set of the ZD graph), not $\mathrm{Aut}(\mathbb{A}_4)$.

Dzhunushaliev (2005). Proposes non-associative quantum mechanics via operator algebras built from octonion/sedenion multiplication [32]. The series' QFT S_{84} acts on the zero-divisor graph of $\mathbb{A}_4$ (a standard commutative field theory on a ZD vertex set), not on the sedenion multiplication structure itself.

Penrose CCC and Tod (2013). The equations of Conformal Cyclic Cosmology and the open problem of fixing the conformal factor $\hat{\Omega}$ are given in Tod [33]. The series' IIB-IV conjecture [2] (ZD residue as CCC conformal boundary data) is *[Speculation]* built on this framework.

Moreno (1998). The G_2 homomorphism on the ZD set is already cited throughout the series [34] and is not repeated here.

6.2. **New bibliography entries.** The bibliography of Paper 25 includes twelve series papers (Papers 1–24 cited above), three computational entries (GAP [13], the GAP script [14], and the Python script [15]), and the Machacek–Vaughn references [16, 17], together with the following sixteen new external entries not previously cited in the series.

7. Open Problems

OP 24.2 (closed, §5): $\Sigma_1 \to$ Level-5 *[Derived]* — Proposition 5.1. $\Sigma_2 \to$ Level-3 *[Derived]* — Theorem 5.3. v_2 purely irr_8 in Level-3 *[Computed, Python]* [15] — Theorem 5.5. Status: CLOSED. New OP 25.3: Open Problem 5.7.

OP 23.1 (closed, §4): $c_{\text{ferm}} = 2$, $d_{\text{ferm}} = -2$. $\Delta\beta^{(f)}(\lambda) = (4\lambda - 16)/\pi^2$. Complete one-loop beta function: $\beta^{(\text{total})} = (63\lambda^2 + 4\lambda - 16)/\pi^2$. Status: CLOSED *[Derived]* — Theorems 4.4, 4.5. New OP 25.2: two-loop fermion contribution to $\beta(\lambda)$ *[Open]*.

OP A′.2 (closed, §2):

Prerequisite: Lemma 2.1 establishes $a_i \in \ker(A)$ analytically *[Derived]*; this was implicit in the GAP verification of Paper 22-B [10] but proved here for the first time.

- Part (i) — C_3-uniformity of c_6, c_8: CLOSED *[Derived]*. Theorem 2.3.
- Part (ii) — C_3 character decomposition of irr_6: CLOSED *[Derived]*. $\mathrm{irr}_6|_{C_3} = 2 \cdot (\mathbf{1} \oplus \chi_1 \oplus \chi_2)$. Proposition 2.5.
- Part (iii) — Exact isotypic split c_6, c_8: CLOSED *[Computed, GAP 4.15.1]* [13, 14]. $c_6 = 8/7$, $c_8 = 20/7$, $c_6 : c_8 = 2 : 5$, C_3-uniform. Theorem 2.7.
- Part (iv) — Level-3 irr_8 block support: CLOSED *[Derived-Negative]*. 0+4+4 excluded; C_3-uniform support proved. Remark 2.11.

OP 23.3 (partially closed, §3): Physical content of $\mathrm{Cl}(0, 7)$ block $= \mathrm{irr}_8 = 4_s$ *[Derived]* — Proposition 3.3. 192-dimensional sedenion-complement: *[Open]*.

Open Problem 7.1 (OP 25.1)**.** Determine an explicit basis for the $3 \cdot \mathrm{irr}_8$ subspace of Level-3 (dimension 24) and record the per-block support across B_1, B_2, B_3. By C_3-equivariance the total irr_8 weight is equal across the three blocks; the explicit support structure is not yet determined. *[Open]*. Accessible via GAP [13]: apply the irr_8 Burnside projector P_8.

Acknowledgements

Mathematical drafting, algebraic analysis, GAP verification, Python computation, and proof scaffolding assisted by Claude (Anthropic). The author takes full responsibility for any errors.

References

[1] G. Levratti, *Sign-Parity Classification of Zero-Divisors in Cayley–Dickson Algebras*, v2, Zenodo (2026), doi:10.5281/zenodo.20315751.

[2] G. Levratti, *The Informational Identity Boundary: Algebraic Foundations and Physical Conjectures*, v5.1, Zenodo (2026), doi:10.5281/zenodo.20611877. Under review, J. Math. Phys., JMP26-AR-01676.

[3] G. Levratti, *A Clifford Representation of the Fano Zero-Divisors*, v1, Zenodo (2026), doi:10.5281/zenodo.20445208.

[4] G. Levratti, *A Quartic Contact Interaction with Fano–CSS Structure*, v1, Zenodo (2026), doi:10.5281/zenodo.20506165.
[5] G. Levratti, *Black Circle Hypergeometric Reduction and Cavity Bound*, v1, Zenodo (2026), doi:10.5281/zenodo.20553409.
[6] G. Levratti, *The Majorana Sector of the Black Circle*, v2, Zenodo (2026), doi:10.5281/zenodo.20573956.
[7] G. Levratti, *One-Loop Stability and Backreaction of the Black Circle*, v1, Zenodo (2026), doi:10.5281/zenodo.20572971.
[8] G. Levratti, *The Level-2 Sector of the Black Circle*, v1, Zenodo (2026), doi:10.5281/zenodo.20611800.
[9] G. Levratti, *Two-Loop Renormalization of a Sedenion Scalar Field and Gravitational Coupling*, v1, Zenodo (2026), doi:10.5281/zenodo.20660411.
[10] G. Levratti, *Fano Incidence and the Level-3 Tripling*, v1, Zenodo (2026), doi:10.5281/zenodo.20648207.
[11] G. Levratti, *The ZD Dirac Operator and Fermionic Sector of S_{84}*, v1, Zenodo (2026), doi:10.5281/zenodo.20667226.
[12] G. Levratti, *The Fano Dictionary: Module Structure of V_{ZD} and the Dark Sector Partition*, v1, Zenodo (2026), doi:10.5281/zenodo.20681330.
[13] The GAP Group, *GAP — Groups, Algorithms, and Programming*, Version 4.15.1 (2024), www.gap-system.org.
[14] G. Levratti, `OP_A_PRIME_2_ISOTYPIC_v6.g` — GAP 4.15.1 script for computational verification of Theorem 2.7 (Burnside isotypic projection on Level-3 block indicators), supplementary file deposited with Paper 25 on Zenodo (2026).
[15] G. Levratti, `SIGMA2_SPECTRAL_v1.py` — Python script for computational verification of Theorems 5.3 and 5.5 (spectral projection of Σ_2 and isotypic content of v_2 in Level-3), supplementary file deposited with Paper 25 on Zenodo (2026).
[16] M. E. Machacek and M. T. Vaughn, "Two-loop renormalization group equations in a general quantum field theory (I)," *Nucl. Phys. B* **222** (1983) 83–103. doi:10.1016/0550-3213(83)90610-7
[17] M. E. Machacek and M. T. Vaughn, "Two-loop renormalization group equations in a general quantum field theory (II)," *Nucl. Phys. B* **236** (1984) 221–232. doi:10.1016/0550-3213(84)90533-9
[18] R. P. de Marrais, "The 42 Assessors and the Box-Kites they fly," *arXiv:math/0011260* (2000).
[19] R. P. de Marrais, "Flying Higher than a Box-Kite," *Complex Systems* **17** (2007) 125–142. https://arxiv.org/abs/math/0703745
[20] S. Bronoff, "The Standard Model of Particle Physics from Sedenions," ResearchGate preprint (August 2016); non-peer-reviewed. https://www.researchgate.net/publication/305809844
[21] R. Potton, "Sedenion Superselection Sectors," ResearchGate preprint (2023); non-peer-reviewed. https://www.researchgate.net/publication/373073881
[22] R. Potton, "The Physical Significance of Sedenion Zero-Divisors," ResearchGate preprint (2024); non-peer-reviewed. (no stable URL available at time of writing)
[23] D. Biss, D. Christensen, D. Dugger, and D. C. Isaksen, "Eigentheory of Cayley–Dickson algebras," *Forum Mathematicum* **21**(5) (2009) 833–851. doi:10.1515/FORUM.2009.041. https://arxiv.org/abs/0905.2987
[24] A. B. Gillard and N. G. Gresnigt, "Three fermion generations with two unbroken gauge symmetries from the complex sedenions," *Eur. Phys. J. C* **79** (2019) 446. doi:10.1140/epjc/s10052-019-6967-1. https://arxiv.org/abs/1904.03186
[25] G. P. Wilmot, "Zero Divisors of Sedenions," *arXiv:2505.11747* (2025). https://arxiv.org/abs/2505.11747
[26] G. P. Wilmot, "Automorphisms of Sedenions," *arXiv:2512.07210* (December 2025). https://arxiv.org/abs/2512.07210
[27] G. K. Leontaris, N. D. Vlachos, and P. Aliferis, "SL$(2,7)$ representations and their relevance to neutrino physics," *Eur. Phys. J. C* **77** (2017) 370. doi:10.1140/epjc/s10052-017-4935-3. https://arxiv.org/abs/1612.06161
[28] S. Okubo, "Representations of Clifford Algebras and its Applications," *arXiv:hep-th/9408165* (1994). https://arxiv.org/abs/hep-th/9408165
[29] J. Tang, "Non-Associative Sedenionic Quantum Gravity (SQG)," *Int. J. Quantum Technol.* **2**(2) (2026). Received 6 Apr 2026; published 12 May 2026. [Concurrent independent work.]
[30] S. L. Dorofeev, "Non-associativity as gravity," *arXiv:0807.2151* (2008). https://arxiv.org/abs/0807.2151
[31] M. Masi, "A step beyond the standard model with octonion-like algebras," *Sci. Rep.* **11** (2021) 22447. doi:10.1038/s41598-021-01814-1
[32] V. Dzhunushaliev, "A non-associative quantum mechanics," *Found. Phys. Lett.* **19** (2006) 157–167 (preprint 2005, published 2006). https://arxiv.org/abs/hep-th/0502216
[33] K. P. Tod, "The equations of Conformal Cyclic Cosmology," *Gen. Relativ. Gravit.* **47** (2015) 17. doi:10.1007/s10714-015-1859-7. https://arxiv.org/abs/1309.7248
[34] G. Moreno, "The zero divisors of the Cayley–Dickson algebras over the real numbers," *Bol. Soc. Mat. Mexicana* (3) **4** (1998) 13–28. [Already cited throughout the series.]

Independent Researcher, Modena, Italy
Email address: giovanni.levratti.research@gmail.com