INDUCED GRAVITY FROM THE SEDENION SCALAR FIELD THEORY S_{84}: NEWTON'S CONSTANT AS THE CAYLEY–DICKSON NORM

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Abstract. We apply the one-loop Sakharov–Adler induced gravity mechanism to the sedenion scalar field theory S_{84}, a quartic quantum field theory defined on the 84 signed zero-divisors of the Cayley–Dickson algebra A_4. Integrating out the 84 real scalar fields and 4 Majorana fermions in a curved spacetime background, we derive Newton's gravitational constant G_N as an explicit function of the Cayley–Dickson quadratic norm N evaluated on the condensate C_0, partially closing OP 20.3 of the series. The main result (Theorem 4.2) is

$$G_N^{-1} = \frac{1}{12\pi} \cdot 16\lambda\, N(C_0) \cdot \Big[164 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 220 \ln 2 - 42 \ln 3\Big],$$

where $\mathcal{S}_{\rm eff} = 164 = |\mathrm{GL}(3, \mathbb{F}_2)| - 4$ is the effective spectral invariant of the ZD adjacency graph after the Majorana fermion correction. The bosonic spectral invariant $\mathcal{S} = 168 = |\mathrm{GL}(3, \mathbb{F}_2)| = 2|\mathrm{ZD}^*(A_4)|$ encodes the Fano-incidence combinatorics of A_4 (Proposition 3.3). A key structural result (Lemma 3.1) is that the 14 massless Level-2 modes are gravitationally decoupled: the factor $(2 + \mu_2) = 0$ that protects their masslessness under triple algebraic protection of $\mathrm{PSL}(2, 7)$ simultaneously makes them gravitationally neutral at one loop. A complementary geometric derivation (Route B) uses the Riemannian structure $\mathrm{ZD}(\mathbb{S}) \cong V_2(\mathbb{R}^7) = G_2/\mathrm{SU}(2)$ (Reggiani [19]): the heat-kernel coefficient gives $G_N^{-1}|_B = \sqrt{6\pi}/128$. A self-consistency equation (Theorem 4.6) expresses G_N implicitly as a function of the Black Circle mass M and the coupling λ, making the Planck scale an algebraic output. Relation to the IIB-II conjecture, asymptotic safety, and the broader programme of identifying all four fundamental coupling constants with the Cayley–Dickson norm is discussed.

1. Deeper Motivation: The Four Forces and the Cayley–Dickson Tower

[Speculation, IIB-II] — stated here as motivating context, not as a result of this paper.

1.1. **The correspondence in the literature.** The idea that the four fundamental forces correspond to the four levels of the Cayley–Dickson tower appears in several prior works. Atiyah [12] conjectured that "the 4 division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ translate into the 4 basic forces," placing gravity at $n = 3$ (octonions) without derivation. Masi [18] derives gauge groups from automorphism groups $\mathrm{Aut}(A_n)$: $\mathrm{Aut}(A_1) \sim \mathrm{U}(1)$, $\mathrm{Aut}(A_2) \sim \mathrm{SU}(2)$, $\mathrm{Aut}(A_3) \sim G_2 \supset \mathrm{SU}(3)$, but derives no coupling constants. Singh [20] derives $\alpha = 1/137$ from the exceptional Jordan algebra on octonions, providing the first rigorous computation of a fundamental coupling constant from division algebra structure, though via a different mechanism (Jordan characteristic equation, not induced gravity). The Furey–Dixon–Gresnigt programme (SM from left ideals of Clifford algebras) has not addressed gravity.

1.2. **The precise IIB-II statement.** The table "$g_i^2 \sim N|_{A_i}$" is imprecise: the norm $N = \sum a_i^2$ is the *same* quadratic form at every CD level. What changes at each doubling is the *composition property*:

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n	Algebra	$N(ab) = N(a)N(b)$	Property lost
1	$\mathbb{C}$	✓	commutativity
2	$\mathbb{H}$	✓	associativity
3	$\mathbb{O}$	✓ (Born Rule)	alternativity
4	$\mathbb{S}$	**fails on ZD pairs**	norm composition

Where N composes ($n = 1, 2, 3$): gauge invariance — the gauge transformation is an isometry of N; the coupling constants g_1^2, g_2^2, g_3^2 measure this composition strength. Where N fails to compose ($n = 4$, ZD pairs): the residue of N evaluated on the condensate C_0 becomes G_N via the induced gravity mechanism derived here.

1.3. **Distinction from Atiyah.** Atiyah placed gravity at $n = 3$ ($\mathbb{O}$). IIB-II places it at $n = 4$ ($\mathbb{S}$), motivated algebraically by IIB-I: the $A_3 \to A_4$ transition is the Born Rule boundary — the first level where $N(ab) = N(a)N(b)$ fails. Gravity appears where the Born Rule fails, not one step before.

1.4. **What this paper contributes.** Unlike Atiyah (correspondence without derivation) and Masi (gauge groups from Aut, no coupling constants), this paper *derives* G_N from S_{84} via induced gravity, computes the coefficient explicitly, and connects the Level-2 gravitational decoupling (Lemma 3.1) to the same algebraic protection that makes Level-2 massless — one mechanism, two consequences. The extension that g_1^2, g_2^2, g_3^2 are also determinable from N at CD levels $n = 1, 2, 3$ is *[Speculation, IIB-II]* and is stated here as a research direction.

2. Introduction

2.1. **The IIB-II conjecture.** The IIB-II conjecture [2] states that Newton's gravitational constant G_N is not a fundamental constant but emerges from the Cayley–Dickson norm N of A_4 evaluated on the condensate C_0 of S_{84}: $G_N = f(N(C_0), \lambda)$ for a function to be derived from first principles (OP 20.3). The structural motivation: N is the only invariant of the CD tower surviving every doubling step, just as G_N is the only coupling universal for all matter.

2.2. **Two routes.** We develop two parallel derivations.

Route A (Sakharov–Frolov–Fursaev). The one-loop matter determinant in curved spacetime, following the graph-theory induced gravity framework of Kan, Kobayashi, and Shiraishi [16, 17], adapted to the ZD adjacency graph spectrum. The master formula is the Frolov–Fursaev supertrace $G_N^{-1} = (1/2\pi)\mathrm{str}[k_1 m^2 \ln(\Lambda^2/m^2)]$ applied to the S_{84} mass spectrum [11, 23].

Route B (σ-model on $V_2(\mathbb{R}^7)$). The σ-model with target $\mathrm{ZD}(\mathbb{S}) \cong V_2(\mathbb{R}^7) = G_2/\mathrm{SU}(2)$ (Reggiani [19]) induces an Einstein–Hilbert term with coefficient fixed by the Riemannian curvature of the coset, providing a geometric derivation complementary to Route A. Both routes give $G_N^{-1} \propto N(C_0)$.

2.3. **Novelty relative to prior non-associative gravity programs.**

Program	Gravity carrier	Algebra
Dorofeev 2006–2013 [14, 15]	Associator of octonion fields	A_3
Tang 2026 [21]	Sedenion associator tensor	A_4
Dzhunushaliev	Non-assoc. gauge structure	A_3
Vacaru 2024 [22]	Star-product R-flux deformation	(non-CD)
This paper	**Quadratic norm N on condensate**	A_4

The norm-carrier mechanism is, on current evidence, original.

3. The Spectral Invariant $\mathcal{S} = 168$

3.1. **Mass spectrum and minimal coupling. Foundation: Paper 11, Theorem 4.5** [3] *[Derived, conditional on Python+GAP]*. The complete mass spectrum of S_{84} around C_0, with PSL(2,7)-module structure, is established and requires no re-derivation in this paper:

Level	μ_i	PSL(2,7) irreps	$2{+}\mu_i$	Mult	m^2 ($\xi{=}0$)
L_1	-4	$\mathrm{irr}_7\,(\times 1)$	-2	7	$-32\lambda C_0^2$
L_2	-2	$\mathrm{irr}_6 \oplus \mathrm{irr}_8$	$\mathbf{0}$	14	$\mathbf{0}$
L_3	0	$3{\cdot}\mathrm{irr}_6 \oplus 3{\cdot}\mathrm{irr}_8$	2	42	$+32\lambda C_0^2$
L_4	$+2$	$\mathrm{irr}_7\,(\times 2)$	4	14	$+64\lambda C_0^2$
L_5	$+4$	$\mathrm{irr}_1 {\oplus} \mathrm{irr}_3 {\oplus} \mathrm{irr}_3'$	6	7	$+96\lambda C_0^2$

Note: $L_4 = 2$ copies of irr_7 (from $\det(\lambda I - M_7) = (\lambda+4)(\lambda-2)^2$); L_5 is the condensate direction, NOT irr_7. The 28-dimensional space of PSL(2,7)-invariant symmetric bilinear forms on $\mathbb{R}^{84}$ does not fix ξ algebraically (Paper 11, §6, *[Derived-Negative, GAP]*); we adopt $\xi = 0$ (minimal coupling), giving $k_1 = 1/6$ uniformly. Since $N(C_0) = C_0^2$ (condensate along the real axis):

$$m_i^2 = 16\lambda\, N(C_0)\,(2+\mu_i).$$

3.2. **Gravitational decoupling of Level-2.**

Lemma 3.1 (Level-2 gravitationally decoupled)**.** *[Derived]. The contribution of the 14 massless Level-2 modes ($\mu_2 = -2$, $\mathrm{irr}_6 \oplus \mathrm{irr}_8$) to the one-loop induced G_N^{-1} via the Frolov–Fursaev formula is exactly zero.*

Proof. By [3, Theorem 4.5], the mass eigenvalue at Level-2 is $m_2^2 = 16\lambda\, N(C_0)\,(2{+}\mu_2) = 16\lambda\, N(C_0)\cdot 0 = 0$. The Frolov–Fursaev weight $m_i^2 \ln(\Lambda^2/m_i^2)$ vanishes identically for $m_i^2 = 0$. Hence all 14 Level-2 modes contribute zero to G_N^{-1}. □

Remark 3.2. The factor $(2+\mu_2) = 0$ that confers $m_2^2 = 0$ via triple algebraic protection under PSL(2,7) (Papers 21–22A, [8, 9]) simultaneously decouples Level-2 from gravity at one loop. Massless scalars cannot generate a Newton constant from a mass-dependent formula — consistent with the equivalence principle at the algebraic level.

3.3. **The spectral invariant $\mathcal{S} = 168$.**

Proposition 3.3 (Spectral sum)**.** *[Derived]. Define the gravitational spectral invariant*

$$\mathcal{S} := \sum_{i=1}^{5} d_i\,(2+\mu_i),$$

where d_i is the multiplicity and μ_i the adjacency eigenvalue at Level-i. Then

$$\mathcal{S} = 2|\mathrm{ZD}^*(A_4)| + \mathrm{Tr}(A) = 2\times 84 + 0 = 168 = |\mathrm{GL}(3,\mathbb{F}_2)| = |\mathrm{PSL}(2,7)|.$$

Proof. $\mathrm{Tr}(A) = 0$ for any simple graph (no self-loops). $|\mathrm{ZD}^*(A_4)| = 84$ by [1, Theorem 2.1]. □

Level-by-level check:

Level	μ_i	d_i	$2+\mu_i$	$d_i(2+\mu_i)$	Module
L_1	-4	7	-2	-14	irr_7 (tachyonic)
L_2	-2	14	**0**	**0**	decoupled
L_3	0	42	2	$+84$	$3\cdot\mathrm{irr}_6\oplus 3\cdot\mathrm{irr}_8$
L_4	$+2$	14	4	$+56$	$\mathrm{irr}_7\,(\times 2)$
L_5	$+4$	7	6	$+42$	$\mathrm{irr}_1\oplus\mathrm{irr}_3\oplus\mathrm{irr}_3'$
Total		**84**		**168**	$=\lvert\mathrm{GL}(3,\mathbb{F}_2)\rvert$

Remark 3.4 (Algebraic origin of $\mathcal{S}=168$)**.** The identity $\mathcal{S}=168=|\mathrm{GL}(3,\mathbb{F}_2)|/2\times 2$ follows from $|\mathrm{ZD}^*(A_4)|=84=|\mathrm{GL}(3,\mathbb{F}_2)|/2$ [1, Theorem 5.17] and $\mathrm{Tr}(A)=0$. It is a consequence of the CD zero-divisor geometry of A_4, not an independent coincidence.

Remark 3.5 ($\mathrm{PSL}(2,7)$-module decomposition $42+84+42=168$)**.** *[Derived].* The spectral sum decomposes along $\mathrm{PSL}(2,7)$-module types:

$$\mathcal{S}=\underbrace{(-14+56)}_{\mathrm{irr}_7\text{-sector:}+42}+\underbrace{(0+84)}_{\mathrm{irr}_6\oplus\mathrm{irr}_8\text{-sector:}+84}+\underbrace{42}_{\text{condensate sector:}+42}=168.$$

The tachyonic irr_7 at L_1 (weight -14) and stable $\mathrm{irr}_7^{\oplus 2}$ at L_4 (weight $+56$) together yield $+42$, absorbing the tachyonic deficit within the irr_7-sector. Each sub-sum equals $|ZD(A_4)|=42$ (unsigned pairs, [1]). The $(\mathrm{irr}_6\oplus\mathrm{irr}_8)$-sector contributes $84=|\mathrm{ZD}^*(A_4)|$, the full signed ZD count — the bosonic gauge sector of the spectrum.

4. Route A: Sakharov–Frolov–Fursaev Induced Gravity

4.1. **The Frolov–Fursaev formula in the cutoff regime.** The Sakharov–Adler induced Newton constant from massive matter fields in a curved background [11, 23]:

$$\frac{1}{G_N}=\frac{1}{2\pi}\,\mathrm{str}\left[k_1\,m^2\ln\frac{\Lambda^2}{m^2}\right]+\mathcal{O}(\Lambda^0),\tag{1}$$

where $\mathrm{str}=\mathrm{Tr}_{\mathrm{bos}}-\mathrm{Tr}_{\mathrm{ferm}}$ (supertrace), $k_1=1/6$ for minimally coupled scalars ($\xi=0$), and Λ is the UV cutoff.

Cutoff regime. For UV finiteness one needs $\mathrm{str}(1)=\mathrm{str}(m^2)=\mathrm{str}(m^4)=0$. With 84 bosons and 4 Majorana fermions (= 2 Dirac): following [23] where each on-shell Majorana mode contributes $-1/2$ to $\mathrm{str}(1)$, $\mathrm{str}(1)=84-4\times(1/2)=82\neq 0$. (In the alternative 4D Euclidean convention with 4 real components per Majorana: $\mathrm{str}(1)=76\neq 0$.) In both cases $\mathrm{str}(1)\neq 0$, placing us in the **cutoff regime**: $G_N\sim\Lambda^2/m^2\times(\text{finite})$. We identify $\Lambda=M_{\mathrm{Pl}}$.

4.2. **Spectral log-sum $\mathcal{L}$.**

Remark 4.1 (Explicit computation of $\mathcal{L}$)**.** *[Derived].* Define $\mathcal{L}:=\sum_{i\neq L_2}d_i(2+\mu_i)\ln|2+\mu_i|$. For L_1 the weight $(2+\mu_1)=-2$ is negative; adopting the upper half-plane convention $\ln(-|x|)=\ln|x|+i\pi$ (consistent with the vacuum-decay treatment in Paper 18 [5]), the real part retains $\ln|2+\mu_1|=\ln 2$ with the signed prefactor. Level-2 is absent by Lemma 3.1. Collecting:

$$\begin{aligned}\mathcal{L}&=\underbrace{7(-2)\ln 2}_{L_1:\,-14\ln 2}+\underbrace{42(2)\ln 2}_{L_3:\,+84\ln 2}+\underbrace{14(4)\ln 4}_{L_4:\,+112\ln 2}+\underbrace{7(6)\ln 6}_{L_5:\,+42(\ln 2+\ln 3)}\\&=(-14+84+112+42)\ln 2+42\ln 3=\boxed{224\ln 2+42\ln 3.}\end{aligned}$$

Numerically: $\mathcal{L}\approx 201.4$. The L_1 term contributes *negatively* $(-14\ln 2)$; the imaginary part from L_1 is treated separately in §5.

4.3. **Main result (Route A).**

Theorem 4.2 (Induced G_N from S_{84})**.** *[Derived, conditional on Sakharov–Adler mechanism]. With $\xi = 0$ ($k_1 = 1/6$) and the mass spectrum of* [3, Theorem 4.5]*, the one-loop Frolov–Fursaev formula applied to S_{84} gives*

$$\boxed{\frac{1}{G_N} = \frac{1}{12\pi} \cdot 16\lambda\, N(C_0) \cdot \left[164 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 220 \ln 2 - 42 \ln 3\right] + \mathcal{O}(\lambda^2).} \tag{2}$$

Here $\mathcal{S}_{\text{eff}} = 164 = \mathcal{S} - 4$ is the effective spectral invariant after the Majorana fermion correction, $\mathcal{L} - 4\ln 2 = 220 \ln 2 + 42 \ln 3$ is the effective constant, and $N(C_0) = C_0^2$ is the Cayley–Dickson norm of the condensate ([3, Theorem 5.4]*).*

Proof. Substitute $m_i^2 = 16\lambda N(C_0)(2 + \mu_i)$ into (1). Level-2 decouples by Lemma 3.1. Scalar contribution:

$$\sum_{\text{bos}} d_i m_i^2 \ln \frac{\Lambda^2}{m_i^2} = 16\lambda N(C_0)\left[\mathcal{S} \ln \frac{\Lambda^2}{16\lambda N(C_0)} - \mathcal{L}\right].$$

Majorana fermion contribution (4 Majorana = 2 Dirac, $m_{4_s}^2 = 32\lambda N(C_0) = 2 \cdot 16\lambda N(C_0)$, [6]):

$$\sum_{\text{ferm}} m^2 \ln \frac{\Lambda^2}{m^2} = 2 \cdot 32\lambda N(C_0) \cdot \ln \frac{\Lambda^2}{32\lambda N(C_0)} = 16\lambda N(C_0) \cdot 4\left[\ln \frac{\Lambda^2}{16\lambda N(C_0)} - \ln 2\right].$$

Subtracting (str = bos − ferm):

$$\text{str} = 16\lambda N(C_0)\left[(\mathcal{S} - 4) \ln \frac{\Lambda^2}{16\lambda N(C_0)} - \mathcal{L} + 4 \ln 2\right].$$

With $k_1 = 1/6$, $\mathcal{S} - 4 = 164$, $\mathcal{L} - 4\ln 2 = 220 \ln 2 + 42 \ln 3$. □

Remark 4.3 (Fermion loop breaks the $|\mathrm{GL}(3, \mathbb{F}_2)|$ degeneracy)**.** The bosonic invariant $\mathcal{S} = 168 = |\mathrm{GL}(3, \mathbb{F}_2)|$. The fermion loop subtracts 4 from the log coefficient (2 Dirac × factor 2 from $m_{4_s}^2/m_0^2 = 2$), giving $\mathcal{S}_{\text{eff}} = 164 = |\mathrm{GL}(3, \mathbb{F}_2)| - 4$. The number 164 has no obvious independent algebraic interpretation; the breaking is a physical effect of the Majorana sector. The constant $220 \ln 2 + 42 \ln 3 = \mathcal{L} - 4 \ln 2$; numerically ≈ 198.6, so the leading-log overestimates G_N^{-1} by factor $e^{198.6/164} \approx 3.37$.

Corollary 4.4. *[Derived]. $G_N^{-1} \propto N(C_0) \cdot \lambda \cdot$ [log-terms]. Newton's constant is determined by the Cayley–Dickson norm of the condensate. OP 20.3 is closed at one loop in the cutoff regime.*

Corollary 4.5 (Majorana fermion correction, OP 23.2)**.** *[Derived, conditional on Paper 19* [6]*]. The constant (finite) part of the fermion correction is*

$$\delta G_N^{-1}\big|_{\text{ferm,const}} = +\frac{16\lambda N(C_0)}{3\pi} \ln 2 \ (> 0).$$

The fermion loop also reduces the leading-log coefficient 168 → 164 (see proof of Theorem 4.2). The net effect depends on the ratio $\ln(\Lambda^2/m_0^2)$ vs. $\ln 2$; consistent with $\beta^{(\text{total})} < 0$ for $\lambda < \lambda_{\text{IR}} \approx 0.473$ [10].

4.4. **Self-consistency equation and Planck scale as output.**

Theorem 4.6 (Self-consistency equation)**.** *[Derived, conditional on Sakharov–Adler +* [3, Theorem 5.4]*]. Define*

$$\alpha := \left(\frac{3}{2048\pi M^2 \lambda}\right)^{1/2}, \qquad P := 164 \ln \frac{\Lambda^2}{16\lambda\alpha} - 220 \ln 2 - 42 \ln 3.$$

Substituting $N(C_0) = \alpha\, G_N^{-3/2}$ *into Theorem 4.2, the implicit equation for* $G_N(M,\lambda)$ *is*

$$\boxed{g(G_N) := G_N^{1/2} - \frac{16\lambda\alpha}{12\pi}\big(P + 246\ln G_N\big) = 0} \tag{SC}$$

(where $246 = 164 \times 3/2$ *is the coefficient from the log split* $\ln(\Lambda^2/16\lambda N) = P_0 + \frac{3}{2}\ln G_N$*). In the leading-log approximation:*

$$G_N^{\mathrm{LL}} = \Big(\frac{16\lambda\alpha\, P}{12\pi}\Big)^2.$$

The Planck scale $M_{\mathrm{Pl}}^2 = G_N^{-1}$ *is an* output *of the ZD algebraic structure and the Black Circle mass* M*, not an input. This is OP 20.3 closed at leading-log order.*

Remark 4.7 (Shape of (SC) and OP 26.5)**.** *[Derived].* The function $g(G_N)$ satisfies $g(0^+) = +\infty$, $g(+\infty) = +\infty$, with unique critical point $G_N^* = (672\lambda\alpha/\pi)^2$ where $g'(G_N^*) = 0$. For physical parameters $g(G_N^*) < 0$, giving exactly *two* positive solutions $G_N^{(1)} < G_N^* < G_N^{(2)}$ with $G_N^{(2)} \approx G_N^{\mathrm{LL}}$. The cavity bound $\lambda G_N M^2 < 2.14 \times 10^{-3}$ [4] does not select a unique branch when $\sqrt{\lambda}|P| < 5.03$. Full uniqueness analysis is **OP 26.5** *[Open].*

Corollary 4.8 (Consistency with Paper 22-A)**.** *[Conjecture, IIB-II]. Under IIB-II,* $G_N^{-1} \propto N(C_0)$ *implies* $\delta G_N^{-1}/G_N^{-1} = \delta N(C_0)/N(C_0)$*.*

Step 1 — CW condensate shift *[Derived]. By* [7, Theorem 3.6]*:* $C_0^{(1)} = C_0(1 + 0.912\lambda)$*, hence* $\delta N_{\mathrm{CW}}/N(C_0) = (1 + 0.912\lambda)^2 - 1 = +1.824\lambda + O(\lambda^2)$*.*

Step 2 — Quantum norm correction *[Conjecture, IIB-II]. Paper 22-A* [9] *derives* $\langle N\rangle_{1\text{-loop}}/C_0^2 = -8.41\lambda$ *(loop integral at fixed condensate position, separate from Step 1).*

Total under IIB-II: $\delta G_N^{-1}/G_N^{-1}|_{\mathrm{total}} = +1.824\lambda + (-8.41\lambda) + O(\lambda^2) = -6.586\lambda$.

[Derived-Negative]: the pre-audit conjecture that CW alone reproduces -8.41λ *is falsified (*$+1.824\lambda \neq -8.41\lambda$*). Whether Paper 22-A's* -8.41λ *already subsumes the CW shift depends on conventions in* [9] *and is [Open].*

5. Tachyonic Level and OP 26.3

The 7 Level-1 modes have $m_1^2 = -32\lambda N(C_0) < 0$. We adopt the upper half-plane branch cut $\ln(-|x|) = \ln|x| + i\pi$ (consistent with Paper 18's treatment of vacuum decay [5]):

$$m_1^2 \ln\frac{\Lambda^2}{m_1^2} = (-32\lambda N(C_0))\Big[\ln\frac{\Lambda^2}{32\lambda N(C_0)} + i\pi\Big].$$

The imaginary part from the Frolov–Fursaev supertrace ($k_1 = 1/6$, $N(C_0) = C_0^2$):

$$\mathrm{Im}(G_N^{-1})\big|_{L_1} = \frac{k_1}{2\pi}\cdot 7\cdot(-32\lambda C_0^2)\cdot(+\pi) = -\frac{28\lambda C_0^2}{3}. \tag{5.1}$$

The negative sign is consistent with $\mathrm{Im}(G_N^{-1}) < 0$ implying $\mathrm{Im}(W) > 0$ (vacuum decay), since $W \sim -S_{\mathrm{EH}}/(16\pi G_N)$ gives $\mathrm{Im}(W) = -\mathrm{Im}(G_N^{-1})\cdot$(positive factor).

Substituting $m_{4_s}^2 = 32\lambda C_0^2$ [6]:

$$\mathrm{Im}(G_N^{-1})\big|_{L_1} = -\frac{7m_{4_s}^2}{24}. \tag{5.2}$$

Using $C_0^4 = 3/(2048\pi G_N^3 M^2\lambda)$ [3, Theorem 5.4] and $r_s = 2G_N M$:

$$\boxed{\mathrm{Im}(G_N^{-1})\big|_{L_1} = -\frac{7}{6}\sqrt{\frac{3\lambda}{2\pi G_N}}\cdot r_s^{-1}.} \tag{5.3}$$

Structural match with Paper 18 [5]**.** Paper 18 gives $\mathrm{Im}(W_{1\text{-loop}}) = (70.848 - 31.009\cdot m_{4_s}^2)\cdot r_s^{-1}$, containing a background de Sitter QNM term ($70.848\cdot r_s^{-1}$) and a tachyonic L_1 term ($-31.009\,\cdot$

$m_{4_s}^2 \cdot r_s^{-1}$, negative = growing instability). Equations (5.3) and the Paper 18 tachyonic term share identical structure (both $\propto m_{4_s}^2 \cdot r_s^{-1}$, same sign), with ratio $744.22/7 \cdot r_s^{-1} \approx +106.32 \cdot r_s^{-1}$ (positive, consistent with $-\mathrm{Im}(G_N^{-1}) \propto +\mathrm{Im}(W)$), which is the geometric factor from the Black Circle spacetime integral.

OP 26.3 — PARTIALLY CLOSED *[Derived, conditional on sign convention and numerical coefficient verification].* Paper 20, Corollary 3.5 [7] establishes that $\mathrm{Im}(W_{\text{1-loop}}) > 0$ is the dynamical manifestation of Level-1 tachyons, guaranteeing the structural identification. Full numerical closure requires verifying the factor $106.32 \cdot r_s^{-1}$ from first principles.

6. Route B: σ-model on $V_2(\mathbb{R}^7) = G_2/\mathrm{SU}(2)$

6.1. **Geometric setup (Reggiani 2024).** Reggiani [19] establishes: $\mathrm{ZD}(\mathbb{S}) \cong V_2(\mathbb{R}^7) = G_2/\mathrm{SU}(2)$ as a Riemannian manifold of dimension 11 ($\neq 13$). One-parameter family of G_2-invariant metrics g_r on $V_2(\mathbb{R}^7)$:

- Physical metric (from sedenion norm): $r = 2/3$, $\mathrm{Ric}_{g_r} = (15r/2)Y^5 \otimes Y^5 + (-3r/2 + 5)\sum_{i\neq 5} Y^i \otimes Y^i$, $\mathrm{scal}(2/3) = 50 - 5 = \mathbf{45} > \mathbf{0}$.
- Einstein metric: $r = 5/9$, $\mathrm{Ric} = (25/6)g$, $\Lambda_{\text{cosm}} = 25/6$.
- Non-negative sectional curvature iff $0 < r \leq 4/9$.

The positive scalar curvature at $r = 2/3$ gives $G_N^{-1} > 0$ via the σ-model — consistent with physical gravity.

6.2. **Volume and Route B induced G_N.** The tangent space decomposes as $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ (1-dim Y^5 plus 10-dim remainder); the metric scales $g_r = r\cdot g_0|_{\mathfrak{m}_1} + g_0|_{\mathfrak{m}_2}$, giving $\mathrm{vol}(V_2, g_r) = r^{1/2}\,\mathrm{vol}(V_2, g_0)$.

The reference volume via the Stiefel fibration $S^5 \hookrightarrow V_2(\mathbb{R}^7) \to S^6$:

$$\mathrm{vol}(V_2, g_0) = \mathrm{vol}(S^6) \times \mathrm{vol}(S^5) = \frac{16\pi^3}{15} \times \pi^3 = \frac{16\pi^6}{15}.$$

At $r = 2/3$:

$$\mathrm{vol}(V_2, g_{2/3}) = \left(\tfrac{2}{3}\right)^{1/2} \times \frac{16\pi^6}{15} = \frac{16\pi^6\sqrt{6}}{45}. \tag{6.3}$$

The heat-kernel a_1 coefficient (constant scalar curvature):

$$\frac{1}{G_N^{(\sigma)}}\bigg|_{r=2/3} = \frac{\mathrm{scal} \times \mathrm{vol}(V_2, g_{2/3})}{(4\pi)^{11/2}} = \frac{45 \times \frac{16\pi^6\sqrt{6}}{45}}{2048\,\pi^{11/2}} = \boxed{\frac{\sqrt{6\pi}}{128}} \approx 0.0339. \tag{6.4}$$

(In units $M_{\text{Pl}} = 1$, conditional on the Reggiani normalization $r_{\text{ref}} = 1$; full closure of OP 26.4 awaits verification of this normalization against [19].)

At the Einstein locus $r = 5/9$: $G_N^{-1}|_{r=5/9} = 55\sqrt{5\pi}/6912 \approx 0.0315$, which is 7.6% smaller than the $r = 2/3$ value. Whether S_{84} dynamics selects $r = 5/9$ is **OP 26.1** *[Open].*

7. Asymptotic Safety: Tension and Resolution

The induced-gravity results of §§4–6 are valid at any fixed UV cutoff Λ and are *independent* of asymptotic safety (AS). This section records the AS tension and its natural resolution; the full RG analysis is deferred to OP 26.2.

7.1. **Tension with the DEP bound.** Donà, Eichhorn, and Percacci [13] derive necessary conditions for a UV fixed point $G_* > 0$:

$$N_s + 2N_D < 4N_V + 46 \quad \text{(weaker)}, \tag{3}$$

$$N_s + 2N_D < 4N_V + 31 \quad \text{(sharper)}. \tag{4}$$

For ungauged S_{84}: $N_s = 84$, $N_D = 2$, $N_V = 0$, giving $N_s + 2N_D = 88 > 46$. Both bounds are violated; the ungauged theory is excluded from the AS fixed-point regime.

7.2. **Resolution via gauging $G_2 \times \mathrm{U}(1)$.** The isometry group of $V_2(\mathbb{R}^7) = G_2/\mathrm{SU}(2)$ is $G_2 \times \mathrm{U}(1)$ [19]. Gauging adds $N_V = \dim(G_2 \times \mathrm{U}(1)) = 14 + 1 = 15$ vector bosons. With $N_V = 15$: $4N_V + 46 = 106 > 88$ ✓ and $4N_V + 31 = 91 > 88$ ✓. Both DEP bounds are satisfied. Gauging G_2 alone gives $N_V = 14$, satisfying the weaker bound ($88 < 102$, ✓) but violating the sharper ($88 > 87$, $\times$); the single U(1) from the S^1 isometry factor is the minimal addition.

[Conjecture, IIB-II] — the DEP bounds are necessary conditions, not sufficient; whether the fixed point exists is **OP 26.2** *[Open]*.

Remark 7.1 (Separation of concerns)**.** Even if OP 26.2 shows the absence of a UV fixed point, the Route A and Route B results for G_N^{-1} remain valid in the cutoff regime $\Lambda = M_{\mathrm{Pl}}$. AS compatibility would extend validity to arbitrarily high energies.

8. Open Problems

OP 26.1 (Einstein locus selection) *[Open]*. Does S_{84} dynamics prefer $r = 5/9$ over $r = 2/3$? If so, $\Lambda_{\mathrm{cosm}} = 25/6\, M_{\mathrm{Pl}}^2$ is an algebraic output with no free parameters.

OP 26.2 (Full AS analysis) *[Open]*. Does the $G_2 \times \mathrm{U}(1)$-gauged extension of S_{84} possess a UV fixed point?

OP 26.3 (Im-part matching) *[Partially closed, §5]*. Structural match with Paper 18 tachyonic term guaranteed by [7, Corollary 3.5]; full numerical coefficient verification pending.

OP 26.4 (Volume of $V_2(\mathbb{R}^7)$) *[Partially closed, §6, conditional on Reggiani normalization]*. $\mathrm{vol}(V_2, g_{2/3}) = 16\pi^6\sqrt{6}/45$ derived; cross-check against [19] pending.

OP 26.5 (Branch selection in (SC)) *[Open]*. The self-consistency equation has two positive solutions; uniqueness of the physical branch is not established.

OP 26.6 (Two-loop induced G_N) *[Open]*. With β_2 from [9] and fermion correction from [10], the two-loop G_N is computable in principle.

OP 26.7 (Coupling hierarchy) *[Speculation, IIB-II]*. Generalization of IIB-II to g_1^2, g_2^2, g_3^2 from N at CD levels $n = 1, 2, 3$ — future paper.

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