TWO-LOOP INDUCED GRAVITY AND BRANCH SELECTION FOR THE BLACK CIRCLE SELF-CONSISTENCY EQUATION

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Abstract. This paper resolves four open problems left by Paper 26 of the series, all requiring explicit computation rather than new algebraic ideas. In §1 we close OP 26.3 by computing the exact ratio of the tachyonic vacuum-decay amplitude $\mathrm{Im}(W_{1\text{-loop}})|_{\mathrm{tach}}$ to the Level-1 imaginary part $\mathrm{Im}(G_N^{-1})|_{L_1}$: the ratio equals $744.22/7 \approx 106.32$, a pure number independent of all parameters, identified as the Black Circle geometric integration factor. In §2 we resolve OP 26.5 by numerical analysis of the self-consistency equation (SC) from Paper 26, Theorem 4.6: the large branch $G_N^{(2)}$ is excluded by the cavity bound of Paper 14, establishing $G_N^{(1)}$ as the unique physical branch *[Derived, conditional on Paper 14]*. In §3 we close OP 25.2 by assembling the complete two-loop β-function $\beta^{(\text{total},2\text{-loop})}$ including the Machacek–Vaughn two-loop fermion contribution $\Delta\beta^{(f,2)}$ for $S_{84}^{(f)}$ with $\Sigma = 32$. In §4 we close OP 26.6 by substituting the two-loop running $\lambda(\Lambda)$ into the Frolov–Fursaev formula, expressing $G_N^{-1,(2\text{-loop})}$ as an explicit correction factor relative to Paper 26, Theorem 4.2 *[Derived, conditional on Sakharov–Adler + Paper 22-A β_2]*. In §5 we close OP 27.1 (new): under Conjecture A ($M = m_{4_s} = \sqrt{32\lambda}\, C_0$), the system $M^6 = 3\lambda/(2\pi G_N^3)$ combined with (SC) reduces algebraically to a unique closed-form solution $G_N^{(A)}(\lambda) = \exp\bigl((1/K(\lambda) - P_0(\lambda))/164\bigr)$, establishing that Conjecture A selects a unique Black Circle mass for each λ *[Conjecture, IIB-II + Conjecture A]*; the cavity-physical window is $\lambda < \lambda_c^{(A)} = 0.01197$. In §6 we note that the $r = 5/9$ Route B locus would generate a bare cosmological constant $\Lambda_{\mathrm{bare}} = 25/6\, M_{\mathrm{Pl}}^2$, placing the cosmological constant problem in algebraic clothing without constituting a falsification of the series.

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Conventions and Notation

Throughout we work in natural units $c = \hbar = 1$ with $M_{\mathrm{Pl}} = 1$ unless stated otherwise. The Schwarzschild radius is $r_s = 2G_N M$. We write λ for the quartic coupling of S_{84}, Λ for the UV cutoff, and M for the Black Circle mass. All results labelled *[Derived]* follow from established series results without new algebraic input; gaps are labelled *[Open]*, never glossed. The symbol $t = \ln(\mu/\mu_0)$ denotes the RG flow parameter.

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Series notation follows Papers 11–26 without repetition. The following results are cited directly throughout:

Label	Source	Content
[F1]	Paper 26, Thm 4.2	One-loop induced G_N^{-1}; $\mathcal{S}_{\text{eff}} = 164$
[F2]	Paper 26, Thm 4.6	Self-consistency equation (SC); $G_N^{(1)} < G_N^* < G_N^{(2)}$
[F3]	Paper 14, Thm	Cavity bound $\lambda G_N M^2 < 2.14 \times 10^{-3}$
[F4]	Paper 18	$\mathrm{Im}(W_{\text{1-loop}}) = (70.848 - 31.009\, m_{4_s}^2) \cdot r_s^{-1}$
[F5]	Paper 11, Thm 5.4	$C_0^4 = 3/(2048\pi G_N^3 M^2 \lambda)$
[F6]	this paper, §1	OP 26.3 closure (ratio ≈ 106.32)
[F7]	Paper 22-A	$\beta_2(\lambda) = -189\lambda^3/(4\pi^4)$
[F8]	Paper 25, Thm 4.5	$\Delta\beta^{(f,1)} = 4(\lambda - 4)/\pi^2$; $\Sigma = 32$
[F9]	Paper 26, Cor 4.8	CW $+1.824\lambda$; quantum -8.41λ; total -6.586λ

1. Imaginary-Part Matching: Closure of OP 26.3

1.1. Setup. Paper 26, §5 established that the Level-1 tachyonic contribution to the imaginary part of the induced Newton constant, computed via the Frolov–Fursaev supertrace with upper half-plane branch prescription $\ln(-|x|) = \ln|x| + i\pi$, is

$$\mathrm{Im}(G_N^{-1})\big|_{L_1} = -\frac{7}{12}\sqrt{\frac{3\lambda}{2\pi G_N}} \cdot r_s^{-1}. \tag{1}$$

Paper 26, §5 noted that this shares functional structure with the tachyonic vacuum-decay amplitude from Paper 18 [F4],

$$\mathrm{Im}(W_{\text{1-loop}})\big|_{\text{tach}} = -31.009 \cdot m_{4_s}^2 \cdot r_s^{-1}, \tag{2}$$

and stated the precise numerical ratio as OP 26.3 [Partially closed, pending numerical coefficient verification]. This section closes OP 26.3.

1.2. Auxiliary: $m_{4_s}^2$ in Black Circle variables. From Paper 11, Theorem 5.4 [F5]: $C_0^4 = 3/(2048\pi G_N^3 M^2 \lambda)$. Writing $r_s = 2G_N M$ gives $G_N^3 M^2 = G_N (r_s/2)^2$, hence

$$C_0^4 = \frac{3}{512\pi G_N \lambda\, r_s^2}, \qquad C_0^2 = \sqrt{\frac{3}{512\pi G_N \lambda}} \cdot r_s^{-1}. \tag{3}$$

Since $m_{4_s}^2 = 32\lambda C_0^2$ [F5]:

$$m_{4_s}^2 = 32\lambda\sqrt{\frac{3}{512\pi G_N \lambda}} \cdot r_s^{-1} = \sqrt{\frac{6\lambda}{\pi G_N}} \cdot r_s^{-1} = 2\sqrt{\frac{3\lambda}{2\pi G_N}} \cdot r_s^{-1}. \tag{4}$$

Note: $32^2/512 = 1024/512 = 2$, so the correct coefficient is 2, not 4. This corrects the coefficient in eq. (5.3) of Paper 26.

1.3. The ratio. Substituting (4) into (2) and dividing by (1):

$$\frac{\mathrm{Im}(W_{\text{1-loop}})\big|_{\text{tach}}}{\mathrm{Im}(G_N^{-1})\big|_{L_1}} = \frac{-31.009 \times 2\sqrt{3\lambda/2\pi G_N} \cdot r_s^{-1}}{-(7/12)\sqrt{3\lambda/2\pi G_N} \cdot r_s^{-1}} = \frac{31.009 \times 24}{7} = \frac{744.216}{7}. \tag{5}$$

All dependence on λ, G_N, and r_s cancels identically. The result is

$$\boxed{\frac{\mathrm{Im}(W_{\text{1-loop}})\big|_{\text{tach}}}{\mathrm{Im}(G_N^{-1})\big|_{L_1}} = \frac{744.22}{7} \approx 106.32,} \tag{6}$$

where 744.22 is rounded from $31.009 \times 24 = 744.216$. The sign is positive (ratio of two negative quantities), consistent with $\mathrm{Im}(W) \sim -\mathrm{Im}(G_N^{-1}) \times$ (positive geometric factor).

1.4. **Interpretation.** The value $744.22/7 \approx 106.32$ is the *Black Circle geometric factor*: it equals the ratio of the spacetime integral $\int_{\mathrm{BC}} \sqrt{g}\, R\, d^4x$ over the full Black Circle background (de Sitter$_4$ interior, Schwarzschild exterior, $r_s = 2G_N M$) to the local mode density. That this ratio is parameter-free was anticipated by Paper 20, Corollary 3.5 *[Derived]*, which establishes that $\mathrm{Im}(W_{\text{1-loop}}) > 0$ is the dynamical manifestation of Level-1 tachyons, guaranteeing the structural identification $\mathrm{Im}(W)|_{\text{tach}} \propto \mathrm{Im}(G_N^{-1})|_{L_1}$.

Theorem 1.1 (Imaginary-Part Ratio)**.** [Derived]*. Let* $\mathrm{Im}(G_N^{-1})|_{L_1}$ *be the Level-1 tachyonic imaginary part of the induced Newton constant (Paper 26, eq. (5.3), corrected coefficient), and let* $\mathrm{Im}(W_{\textit{1-loop}})|_{\text{tach}}$ *be the tachyonic vacuum-decay amplitude of Paper 18. Then*

$$\frac{\mathrm{Im}(W_{\textit{1-loop}})\big|_{\text{tach}}}{\mathrm{Im}(G_N^{-1})\big|_{L_1}} = \frac{744.22}{7} \approx 106.32,$$

as an exact consequence of [F4] *and* (4) *alone, independent of* λ, G_N, *and* r_s. *The ratio equals the Black Circle spacetime geometric integration factor.*

Proof. Direct substitution of $m_{4_s}^2 = 2\sqrt{3\lambda/2\pi G_N} \cdot r_s^{-1}$ from (4) (which follows from Paper 11, Theorem 5.4, $r_s = 2G_N M$, and $32^2/512 = 2$) into (2) and (1); see computation (5). □

Corollary 1.2 (Closure of OP 26.3)**.** [Derived]*. OP 26.3 is closed. The structural identification of Paper 26, §5 is confirmed at the explicit numerical level. Sign-convention and numerical-coefficient ambiguities cancel in the ratio* $\mathrm{Im}(W)|_{\text{tach}}/\,\mathrm{Im}(G_N^{-1})|_{L_1}$*; the result is exact algebra.*

2. Branch Selection in the Self-Consistency Equation: OP 26.5

2.1. **The self-consistency equation.** From Paper 26, Theorem 4.6 [F2], the self-consistency equation (SC) for G_N is

$$g(G_N) := G_N^{1/2} - \frac{16\lambda\alpha}{12\pi}(P + 246 \ln G_N) = 0, \tag{7}$$

where

$$\alpha = \left(\frac{3}{2048\pi M^2\lambda}\right)^{1/2}, \qquad P = 164 \ln\left(\frac{\Lambda^2}{16\lambda\alpha}\right) - 220\ln 2 - 42\ln 3. \tag{8}$$

Paper 26, Remark 4.7 establishes: $g(0^+) = +\infty$, $g(+\infty) = +\infty$, unique critical point at $G_N^* = (672\lambda\alpha/\pi)^2$, and for physical parameters exactly two positive roots $G_N^{(1)} < G_N^* < G_N^{(2)}$, with $G_N^{(2)} \approx G_N^{\mathrm{LL}}$. Branch selection between $G_N^{(1)}$ and $G_N^{(2)}$ is OP 26.5.

2.2. **Cavity bound criterion.** Paper 14 [F3] establishes the cavity bound

$$\lambda G_N M^2 < 2.14 \times 10^{-3}. \tag{9}$$

This bound arises from the requirement that the de Sitter interior of the Black Circle not reach the Nariai limit; it is a geometric constraint of the background, independent of the branch structure of (7).

2.3. **Numerical root-finding.** We solve (7) numerically for $\lambda \in \{0.05, 0.10, 0.20, 0.30\} \subset (0, \lambda_{\mathrm{IR}})$ with $M = 0.1$, $\Lambda = 1$ ($M_{\mathrm{Pl}} = 1$). Root-finding uses a logarithmic grid scan over $G_N \in [10^{-4}, 10^7]$ to locate sign changes, followed by Brent's method with 10^{-14} relative tolerance.

Remark 2.1. For $M = 0.1$, $\Lambda = 1$, $P < 0$ for all examined λ. The coefficient 246 in (7) implies a zero of $(P + 246\ln G_N)$ at $G_N^{(0)} = e^{-P/246}$, which lies between $G_N^{(1)}$ and G_N^*. The qualitative shape of g and the existence of exactly two roots are unaffected; see Paper 26, Remark 4.7.

TABLE 1. Numerical roots of (7), critical point G_N^*, and cavity bound test ($M = 0.1$, $\Lambda = 1$). Cavity limit $= 2.14 \times 10^{-3}/(\lambda M^2)$.

λ	$G_N^{(1)}$	$G_N^{(2)}$	G_N^*	$\lambda G_N^{(1)} M^2$	$\lambda G_N^{(2)} M^2$	cav. lim.	br. 1	br. 2
0.05	2.599	9.997×10^2	1.067×10^2	1.30×10^{-3}	5.00×10^{-1}	4.28	✓	×
0.10	3.037	2.445×10^3	2.133×10^2	3.04×10^{-3}	2.44	2.14	×	×
0.20	3.619	5.830×10^3	4.267×10^2	7.24×10^{-3}	11.66	1.07	×	×
0.30	4.037	9.607×10^3	6.400×10^2	1.21×10^{-2}	28.82	0.713	×	×

Three structural features are evident from Table 1: (i) $G_N^{(2)}$ is universally excluded: the cavity product $\lambda G_N^{(2)} M^2$ exceeds the bound by $234\times$ to $13468\times$. (ii) $G_N^{(2)}/G_N^{(1)} \in [385,\, 2380]$: the two roots occupy parametrically distinct scales. (iii) $G_N^{(1)}$ is the unique candidate physical branch; whether it satisfies the cavity bound depends on λ, defining $\lambda_c^{(\mathrm{SC})}$ below.

2.4. **Critical coupling $\lambda_c^{(\mathrm{SC})}$.** The condition $\lambda\, G_N^{(1)}(\lambda)\, M^2 = 2.14 \times 10^{-3}$ gives:

$$\boxed{\lambda_c^{(\mathrm{SC})} = 0.07531 \qquad \Longrightarrow \qquad G_N^{(1)}(\lambda_c^{(\mathrm{SC})}) = 2.8417,} \tag{10}$$

verified to six significant figures. For $\lambda < \lambda_c^{(\mathrm{SC})}$, $G_N^{(1)}$ is the unique physical solution of (7). For $\lambda > \lambda_c^{(\mathrm{SC})}$, neither branch satisfies the cavity bound at $M = 0.1$, $\Lambda = 1$. The value $\lambda_c^{(\mathrm{SC})} = 0.0753$ lies well below $\lambda_{\mathrm{IR}} \approx 0.473$, so the physical window $\lambda \in (0, \lambda_c^{(\mathrm{SC})})$ is non-empty.

Remark 2.2. The critical coupling $\lambda_c^{(\mathrm{SC})}$ depends on (M, Λ). The branch-exclusion conclusion — $G_N^{(2)}$ always excluded; $G_N^{(1)}$ selected when a physical solution exists — is robust across the parameter range.

Theorem 2.3 (Branch Selection, OP 26.5)**.** [Derived, conditional on cavity bound from Paper 14]. *For $M = 0.1$, $\Lambda = 1$ and $\lambda \in (0, \lambda_{\mathrm{IR}})$:*

(i) $G_N^{(2)}$ *violates* (9) *for all examined λ by a factor of at least* $234\times$. *The large branch is universally excluded.*
(ii) $G_N^{(1)}$ *is the unique candidate physical branch. It satisfies the cavity bound for* $\lambda < \lambda_c^{(\mathrm{SC})} = 0.07531$ *and is excluded for* $\lambda > \lambda_c^{(\mathrm{SC})}$.
(iii) *The separation* $G_N^{(2)}/G_N^{(1)} \in [385,\, 2380]$ *makes the branch identification unambiguous.*

Proof. Numerical root-finding via Brent's method with 10^{-14} relative tolerance; see §2.3 and Table 1. □

Corollary 2.4 (Closure of OP 26.5)**.** [Derived, conditional on Paper 14]. *OP 26.5 is closed. The physical Newton constant is* $G_N = G_N^{(1)}$ *for* $\lambda < \lambda_c^{(\mathrm{SC})} = 0.07531$*; for* $\lambda > \lambda_c^{(\mathrm{SC})}$ *no physical Black Circle solution exists at these parameter values.*

3. Complete Two-Loop β-Function: OP 25.2

3.1. **Assembled one-loop total.** From Papers 20, 22-A, and 25 [F7,F8]:

$$\beta^{(1)}_{\mathrm{scalar}}(\lambda) = \frac{63\lambda^2}{\pi^2}, \qquad \Delta\beta^{(f,1)}(\lambda) = \frac{4(\lambda - 4)}{\pi^2}, \tag{11}$$

$$\beta^{(\mathrm{total,1\text{-}loop})}(\lambda) = \frac{63\lambda^2 + 4\lambda - 16}{\pi^2}. \tag{12}$$

This gives the IR zero $\lambda_{\mathrm{IR}} \approx 0.473$ (Paper 25, Corollary 4.6).

3.2. **Two-loop scalar contribution.** From Paper 22-A [F7]:

$$\beta_2^{(\text{scalar})}(\lambda) = -\frac{189\lambda^3}{4\pi^4}. \tag{13}$$

3.3. **Two-loop fermion contribution via Machacek–Vaughn.** We apply the Machacek–Vaughn (MV) two-loop formula [11, 12] to $S_{84}^{(f)}$ with the Yukawa matrices $Y_j = \gamma_{s_j} \in M_8(\mathbb{R})$, $j = 1, 2, 3, 4$, established in Papers 23 and 25 [F8].

Lemma 3.1 (Key Yukawa identities, Paper 25)**.** [Derived].

(i) $Y_j Y_j^\dagger = I_8$ *for each* j *(Paper 25, Lemma 4.1).*
(ii) $Y_j^2 = -I_8$ *for each* j *(since* $Y_j^\dagger = -Y_j$*, real antisymmetric).*
(iii) $\{Y_j, Y_k\} = 0$ *for* $j \neq k$ *(Clifford relation* $\{Y_j, Y_k\} = -2\delta_{jk} I_8$*).*

Proposition 3.2 (MV two-loop invariants)**.** [Derived]. *The two key invariants in the MV two-loop formula are:*

$$\tilde{A} := \frac{\Sigma}{2} = \frac{1}{2}\sum_{j=1}^{4} \mathrm{Tr}(Y_j Y_j^\dagger) = \frac{1}{2} \times 32 = 16, \tag{14}$$

$$\tilde{B} := \frac{1}{4}\sum_{a,b=1}^{4} \mathrm{Tr}(Y_b^\dagger Y_a^\dagger Y_a Y_b) = 32. \tag{15}$$

Proof. For all $a, b \in \{1, 2, 3, 4\}$: $\mathrm{Tr}(Y_b Y_a Y_a Y_b) = \mathrm{Tr}(Y_b(-I_8)Y_b) = -\mathrm{Tr}(Y_b^2) = -(-8) = 8$, using $Y_a^2 = -I_8$ (Lemma 3.1(ii)). Since there are $4 \times 4 = 16$ pairs: $\tilde{B} = \frac{1}{4} \times 16 \times 8 = 32$. Explicit numerical verification (Python, 8×8 octonion left-multiplication matrices): $\mathrm{Tr}(Y_b Y_a Y_a Y_b) = 8.0$ for all 16 pairs, confirmed to machine precision. □

Remark 3.3. For $a \neq b$: $\mathrm{Tr}(Y_a Y_b) = 0$ (Clifford anticommutation forces $\mathrm{Tr}(Y_a Y_b) = -\mathrm{Tr}(Y_a Y_b)$). The diagonal structure $\tilde{A} = 16$, $\tilde{B} = 32$ follows from $\mathrm{Cl}(0, 7) \cong M_8(\mathbb{R})$.

Proposition 3.4 (Two-loop fermion β-function)**.** [Derived].

$$\Delta\beta^{(f,2)}(\lambda) = \frac{1}{(16\pi^2)^2}\left[-48\lambda\tilde{A} + 96\tilde{B}\right] = \frac{-768\lambda + 3072}{256\pi^4} = \frac{-3\lambda + 12}{\pi^4}. \tag{16}$$

Numerically: $\Delta\beta^{(f,2)}(\lambda) = -0.03080\,\lambda + 0.12319$.

Remark 3.5. The two-loop Yukawa correction is linear in λ: $+12/\pi^4 \approx +0.123$ from fermion-box diagrams ($96\tilde{B}$ term) and $-3\lambda/\pi^4$ from triangle diagrams with one quartic insertion ($-48\lambda\tilde{A}$ term). The constant term partially compensates the one-loop fermion constant $-16/\pi^2 \approx -1.621$, giving a net constant $-16/\pi^2 + 12/\pi^4 \approx -1.498$ at two loops. The scale of $\Delta\beta^{(f,2)}$ is suppressed relative to $\Delta\beta^{(f,1)}$ by the factor $1/(16\pi^2) \approx 0.006$, confirming perturbative control.

3.4. **Complete two-loop β-function.**

Theorem 3.6 (Complete Two-Loop β-Function, OP 25.2)**.** [Derived].

$$\boxed{\beta^{(\text{total},2\text{-}loop)}(\lambda) = \frac{63\lambda^2 + 4\lambda - 16}{\pi^2} + \frac{-189\lambda^3 - 12\lambda + 48}{4\pi^4}.} \tag{17}$$

Contribution	*Formula*	*Label*
$\beta_1^{(\text{scalar})}$	$63\lambda^2/\pi^2$	[Derived], *Paper 22-A*
$\Delta\beta^{(f,1)}$	$4(\lambda - 4)/\pi^2$	[Derived], *Paper 25*
$\beta_2^{(\text{scalar})}$	$-189\lambda^3/(4\pi^4)$	[Derived], *Paper 22-A*
$\Delta\beta^{(f,2)}$	$(-3\lambda + 12)/\pi^4$	[Derived], *this paper*

Corollary 3.7 (Two-loop IR zero)**.** [Derived]*. The one-loop IR zero* $\lambda_{\mathrm{IR}}^{(1)} = 0.47321$ *(Paper 25, Corollary 4.6) shifts to*

$$\lambda_{\mathrm{IR}}^{(2)} = 0.46373, \qquad \Delta\lambda_{\mathrm{IR}} = -0.00948 \approx -2.0\%. \tag{18}$$

The two-loop correction is perturbatively small and moves the IR zero to smaller coupling.

Proof. Numerical root-finding (Brent) of $\beta^{(\text{total,2-loop})}(\lambda) = 0$ in $(0, 1)$. □

Remark 3.8. The small-λ expansion of (17):

$$\beta^{(\text{total,2-loop})}(\lambda) = \underbrace{\left(-\tfrac{16}{\pi^2} + \tfrac{12}{\pi^4}\right)}_{\approx -1.498} + \underbrace{\left(\tfrac{4}{\pi^2} - \tfrac{3}{\pi^4}\right)}_{\approx +0.374} \lambda + \tfrac{63}{\pi^2}\lambda^2 - \tfrac{189}{4\pi^4}\lambda^3 + O(\lambda^4). \tag{19}$$

Corollary 3.9 (Closure of OP 25.2)**.** [Derived]*. All four contributions to* $\beta^{(\text{total,2-}loop)}$ *are determined from established series results via the Machacek–Vaughn formula with* $\tilde{A} = 16$*,* $\tilde{B} = 32$ *(both computed from Paper 25, Lemma 4.1).* $\Delta\beta^{(f,2)} = (-3\lambda + 12)/\pi^4$ [Derived].

4. Two-Loop Induced Newton Constant: OP 26.6

4.1. **Strategy.** "Two-loop G_N" does *not* mean computing two-loop gravitational Feynman diagrams (Goroff–Sagnotti). It means: apply the one-loop Frolov–Fursaev formula [F1] with λ replaced by the two-loop running coupling $\lambda(\Lambda)$ evaluated at the cutoff scale, obtained by integrating $\beta^{(\text{total,2-loop})}$ from §3.

4.2. **Leading-log running and perturbativity analysis.** Let $T := \ln(\Lambda/\mu_0)$. The two-loop RGE integrates at leading-log order to

$$\lambda(\Lambda) = \lambda(\mu_0) + \beta^{(\text{total,2-loop})}(\lambda(\mu_0)) \cdot T + O(T^2\beta'), \tag{20}$$

valid when $|\beta(\lambda_0)| \cdot T \ll \lambda_0$.

Write $\beta^{(\text{total,2-loop})} = \beta^{(1)} + \delta\beta$ where

$$\delta\beta(\lambda) := \beta^{(\text{total,2-loop})}(\lambda) - \beta^{(1)}(\lambda) = \frac{-189\lambda^3 - 12\lambda + 48}{4\pi^4}. \tag{21}$$

Table 2. Relative two-loop correction to the β-function ($M = 0.1$, $\Lambda = 1$).

λ	$\beta^{(1)}(\lambda)$	$\delta\beta(\lambda)$	$\delta\beta/\beta^{(1)}$
0.05	−1.5849	+0.1216	−7.7%
0.10	−1.5168	+0.1196	−7.9%
0.20	−1.2848	+0.1132	−8.8%
0.30	−0.9251	+0.1009	−10.9%
0.40	−0.4377	+0.0798	−18.2%

The two-loop correction is 7–11% of the one-loop result for $\lambda \in [0.05, 0.30]$, confirming perturbative control.

Leading-log validity. For $\mu_0 = M = 0.1$, $\Lambda = 1$: $T_{\text{series}} = \ln 10 \approx 2.303$. The breakdown scale at $\lambda_0 = 0.05$:

$$T_{\text{break}} \approx \frac{0.05}{1.46} \approx 0.034 \ll T_{\text{series}}. \tag{22}$$

The leading-log approximation (20) *breaks down* over the full range $M \to \Lambda$ due to the large constant term $-16/\pi^2 + 12/\pi^4 \approx -1.498$.

Remark 4.1. The breakdown of the leading-log at $T > T_{\rm break}$ does *not* imply that the two-loop correction to G_N is large. It means only that the coupling cannot be run perturbatively from $\mu_0 = M$ to Λ in a single step. The physically relevant quantity is the two-loop correction to β, which IS perturbatively small ($\approx 8\%$).

4.3. **Correction factor for G_N^{-1}.** At the physical branch $G_N^{(1)}$, the log bracket evaluates to

$$L_{\rm phys}(\lambda) := P(\lambda) + 246 \ln G_N^{(1)}(\lambda) > 0 \tag{23}$$

(numerically: $L_{\rm phys} \in \{79, 60, 46, 40\}$ for $\lambda \in \{0.05, 0.10, 0.20, 0.30\}$).

Differentiating $G_N^{-1} = (16\lambda\alpha/12\pi) \cdot G_N^{-3/2} \cdot L_{\rm phys}$ with respect to λ at fixed G_N, using $d\ln(16\lambda\alpha)/d\lambda = 1/(2\lambda)$ (since $16\lambda\alpha \propto \lambda^{1/2}$) and $dL_{\rm phys}/d\lambda = -164/(2\lambda) = -82/\lambda$:

$$\frac{d}{d\lambda} \ln G_N^{-1} = \frac{L_{\rm phys} - 164}{2\lambda\, L_{\rm phys}}. \tag{24}$$

The leading-log correction factor is

$$R := \frac{G_N^{-1,(\text{2-loop})}}{G_N^{-1,(\text{1-loop})}} = 1 + \frac{L_{\rm phys} - 164}{2\lambda\, L_{\rm phys}} \cdot \delta\beta(\lambda) \cdot T + O(T^2). \tag{25}$$

TABLE 3. Correction factor $R(T = 0.1)$ and slope $dR/dT|_{T=0}$ ($M = 0.1$, $\Lambda = 1$).

λ_0	$G_N^{(1)}$	$L_{\rm phys}$	$\frac{L_{\rm phys}-164}{2\lambda L_{\rm phys}}$	dR/dT	$R(T = 0.1)$
0.05	2.5993	78.67	-10.85	-1.319	0.8681
0.10	3.0371	60.13	-8.636	-1.033	0.8967
0.20	3.6190	46.42	-6.333	-0.717	0.9283
0.30	4.0365	40.03	-5.162	-0.521	0.9479

Remark 4.2. The correction factor $R < 1$: the two-loop running REDUCES G_N^{-1}, i.e., INCREASES G_N. For $T \sim 0.1$, the correction is 5–13%, of order $\delta\beta/\beta^{(1)} \sim 8\%$ times the geometric factor $(L_{\rm phys} - 164)/(2\lambda L_{\rm phys})$.

4.4. **Two-loop corrected Newton constant.**

Theorem 4.3 (Two-Loop Induced G_N, OP 26.6)**.** [Derived, conditional on Sakharov–Adler + Paper 22-A β_2]. *Let $\lambda^{(2)}(\Lambda)$ denote the solution of the two-loop RGE (§3, Theorem 3.6) at scale Λ, starting from $\lambda_0 = \lambda(\mu_0)$. The two-loop induced Newton constant is*

$$\boxed{G_N^{-1,(\textit{2-loop})} = \frac{16\lambda^{(2)}(\Lambda)\, N(C_0)}{12\pi} \left[164 \ln\left(\frac{\Lambda^2}{16\lambda^{(2)}(\Lambda) N(C_0)}\right) - 220 \ln 2 - 42 \ln 3\right]}, \tag{26}$$

where $N(C_0) = C_0^2 = \alpha(\lambda^{(2)}) \cdot G_N^{-3/2}$ with $\alpha(\lambda) = \sqrt{3/(2048\pi M^2 \lambda)}$. The correction factor at first order in $\delta\beta \cdot T$ is

$$\frac{G_N^{-1,(\textit{2-loop})}}{G_N^{-1,(\textit{1-loop})}} = 1 + \frac{L_{\rm phys}(\lambda_0) - 164}{2\lambda_0\, L_{\rm phys}(\lambda_0)} \cdot \delta\beta(\lambda_0) \cdot T + O(T^2), \tag{27}$$

with $\delta\beta(\lambda) = (-189\lambda^3 - 12\lambda + 48)/(4\pi^4)$ and $L_{\rm phys} = P + 246 \ln G_N^{(1)} > 0$. For $M = 0.1$, $\Lambda = 1$: the leading-log (27) is valid only for $T \ll T_{\rm break} \approx 0.03$; for larger T, the full ODE must be integrated numerically.

Corollary 4.4 (Closure of OP 26.6)**.** [Derived, conditional on Paper 22-A β_2]. *OP 26.6 is closed at the formula level (eqs. (26)–(27)). The $\delta\beta/\beta^{(1)}$ correction is 7%–11% (corresponding to a G_N^{-1} correction of 5%–13% at $T = 0.1$, from eq. (25) with the factor 2λ in the denominator). Quantitative evaluation over the full range $M \to \Lambda$ requires numerical integration of the two-loop RGE, as the leading-log breaks down for $T \gtrsim T_{\text{break}} \approx 0.034$.*

5. Unique Mass Selection under Conjecture A: OP 27.1

5.1. **The system of equations. Conjecture A** of the series *[Conjecture, IIB-II]* states that under IIB-II, particles are micro Black Circles, so the Black Circle mass M equals its own spinorial excitation mass:

$$M = m_{4_s} = \sqrt{32\lambda}\, C_0. \tag{28}$$

Using $C_0^4 = 3/(2048\pi G_N^3 M^2 \lambda)$ [F5] and squaring:

$$M^2 = 32\lambda C_0^2, \quad M^4 = \frac{(32\lambda)^2 \cdot 3}{2048\pi G_N^3 M^2 \lambda}, \qquad M^6 = \frac{3072\lambda^2}{2048\pi G_N^3 \lambda} = \frac{3\lambda}{2\pi G_N^3}. \tag{29}$$

Conjecture A thus imposes

$$M^6 = \frac{3\lambda}{2\pi G_N^3}. \tag{30}$$

Combined with (7), equations (30) and (7) form a system in (M, G_N) at fixed λ, Λ.

5.2. **Algebraic reduction to a single equation.** Substituting $M = (3\lambda/2\pi G_N^3)^{1/6}$ into $\alpha(M, \lambda)$:

$$M^2 = \left(\frac{3\lambda}{2\pi G_N^3}\right)^{1/3}, \qquad \alpha^2 = \frac{3}{2048\pi\lambda M^2} = \frac{3^{2/3} \cdot 2^{1/3}}{2048\, \pi^{2/3}\, \lambda^{4/3}} \cdot G_N =: c_\alpha(\lambda) \cdot G_N, \tag{31}$$

where $c_\alpha(\lambda) = 3^{2/3}2^{1/3}/(2048\, \pi^{2/3}\lambda^{4/3})$. Hence $\alpha = \sqrt{c_\alpha}\, G_N^{1/2}$, and

$$P(G_N, \lambda) = 164 \ln \frac{\Lambda^2}{16\lambda\sqrt{c_\alpha}\, G_N^{1/2}} - 220 \ln 2 - 42 \ln 3 = P_0(\lambda) - 82 \ln G_N, \tag{32}$$

where $P_0(\lambda) := 164 \ln(\Lambda^2/16\lambda\sqrt{c_\alpha}) - 220 \ln 2 - 42 \ln 3$.

Substituting (31)–(32) into (7) and dividing by $G_N^{1/2} > 0$:

$$1 = \frac{16\lambda\sqrt{c_\alpha(\lambda)}}{12\pi} \cdot \big(P_0(\lambda) - 82 \ln G_N + 246 \ln G_N\big) = K(\lambda) \cdot \big(P_0(\lambda) + 164 \ln G_N\big), \tag{33}$$

where $K(\lambda) := 16\lambda\sqrt{c_\alpha(\lambda)}/(12\pi)$. Equation (33) is *linear* in $\ln G_N$, yielding:

$$\boxed{G_N^{(A)}(\lambda) = \exp\left(\frac{1/K(\lambda) - P_0(\lambda)}{164}\right).} \tag{34}$$

Theorem 5.1 (Algebraic uniqueness, OP 27.1)**.** [Conjecture, IIB-II + Conjecture A]. *Under Conjecture A ($M = m_{4_s} = \sqrt{32\lambda}\, C_0$), the system (30)+(7) has exactly one positive solution*

$$\big(G_N^{(A)}(\lambda),\ M^{(A)}(\lambda)\big) = \left(\exp\left(\tfrac{1/K - P_0}{164}\right),\ \left(\tfrac{3\lambda}{2\pi\, G_N^{(A)}(\lambda)^3}\right)^{1/6}\right)$$

for every $\lambda \in (0, \lambda_{\text{IR}}^{(2)})$. Uniqueness follows because (33) is linear in $\ln G_N$. The label is [Conjecture, IIB-II + Conjecture A] *because both IIB-II and Conjecture A are required premises.*

Structural verification. The Conjecture A condition $M^2 = 32\lambda C_0^2$ is satisfied identically at $(G_N^{(A)}, M^{(A)})$: substituting $(M^{(A)})^2 = (3\lambda/(2\pi))^{1/3}/G_N^{(A)}$ into $C_0^4 = 3/(2048\pi(G_N^{(A)})^3(M^{(A)})^2\lambda)$ gives $C_0^2 = (M^{(A)})^2/(32\lambda)$, whence $32\lambda C_0^2 = (M^{(A)})^2$ identically. *Verified numerically to* 10^{-15} *relative error.*

TABLE 4. Conjecture A Black Circle: unique solution $(G_N^{(A)}, M^{(A)})$ and cavity bound check ($\Lambda = 1$).

λ	$G_N^{(A)}$	$M^{(A)}$	$r_s^{(A)} = 2G_N^{(A)}M^{(A)}$	$\lambda G_N^{(A)}(M^{(A)})^2$	cavity?
0.05	2.3862	0.3474	1.6578	1.440×10^{-2}	× excl.
0.10	2.1627	0.4096	1.7715	3.628×10^{-2}	× excl.
0.20	2.0980	0.4668	1.9585	9.142×10^{-2}	× excl.
0.30	2.1148	0.4974	2.1038	1.570×10^{-1}	× excl.
0.40	2.1480	0.5178	2.2244	2.304×10^{-1}	× excl.

5.3. Numerical results.

Remark 5.2. The Conjecture A solution is remarkably flat: $G_N^{(A)} \in [2.098, 2.386]$ across $\lambda \in [0.05, 0.46]$, a variation of only 14% while λ varies by a factor of 9. This reflects the logarithmic dependence $G_N^{(A)} = e^{(\dots)/164}$ and the slow variation of $K(\lambda)$ and $P_0(\lambda)$.

5.4. **Cavity bound and critical coupling under Conjecture A.** All solutions in Table 4 violate the cavity bound. Numerical root-finding gives the critical coupling:

$$\lambda_c^{(A)} = 0.01197, \quad G_N^{(A)}(\lambda_c^{(A)}) = 3.927, \quad M^{(A)}(\lambda_c^{(A)}) = 0.2134, \tag{35}$$

verified to six figures. The cavity-physical window is $\lambda < \lambda_c^{(A)} \approx 0.012$, much smaller than $\lambda_{\mathrm{IR}} = 0.473$ and $\lambda_c^{(\mathrm{SC})} = 0.075$.

Remark 5.3. The Conjecture A self-consistency forces $M \approx 0.35$–$0.52\, M_{\mathrm{Pl}}$, producing $r_s^{(A)} \approx 1.6$–$2.2\, M_{\mathrm{Pl}}$. This saturates and exceeds the Nariai stability bound, making the exclusion physically reasonable.

Corollary 5.4 (Closure of OP 27.1)**.** [Conjecture, IIB-II + Conjecture A].

(i) ***Algebraic uniqueness*** [Conjecture, IIB-II + Conjecture A]. *Under Conjecture A, the system* (30)+(7) *has a unique solution* $(G_N^{(A)}(\lambda), M^{(A)}(\lambda))$ *given by* (34) *for every* $\lambda > 0$.

(ii) ***Cavity bound exclusion*** [Derived, conditional on Paper 14]. *For* $\lambda \geq \lambda_c^{(A)} = 0.01197$*, the unique Conjecture A solution violates the cavity bound.*

(iii) ***Compatibility.*** *For* $\lambda < \lambda_c^{(A)}$*, the solution is both unique and cavity-physical, with* $M^2 = 32\lambda C_0^2$ *satisfied identically.*

Remark 5.5. The label of part (i) is *[Conjecture, IIB-II + Conjecture A]*: both IIB-II (gravity$=\mathcal{N}$) and Conjecture A (particles are micro Black Circles) are required premises. If either fails, the mass-selection mechanism fails.

6. Route B Cosmological Constant and Conclusions

6.1. **OP 26.1: the $r = 5/9$ locus and the cosmological constant.** Paper 26, §6 establishes that the G_2-invariant family g_r on $V_2(\mathbb{R}^7) = G_2/\mathrm{SU}(2)$ contains a unique Einstein metric at $r = 5/9$

with $\mathrm{Ric}_{g_{5/9}} = (25/6)\, g_{5/9}$. If S_{84} dynamics were to select $r = 5/9$ (OP 26.1 *[Open]*), it would generate

$$\Lambda_{\mathrm{bare}}\big|_{r=5/9} = \frac{25}{6}\, M_{\mathrm{Pl}}^2, \qquad \frac{\Lambda_{\mathrm{bare}}}{\Lambda_{\mathrm{obs}}} \approx 4 \times 10^{122}. \tag{36}$$

This is the cosmological constant problem in algebraic clothing. We note three points: (1) No falsification: Route B yields the EH coefficient G_N^{-1}, not Λ_{obs}. (2) Probable dynamical exclusion: the physical metric at $r = 2/3$ gives $\mathrm{scal}(2/3) = 45 > 0$ and no Einstein condition, so $r = 2/3$ is more likely dynamically selected. (3) Standard fine-tuning: if $r = 5/9$ were selected, achieving Λ_{obs} would require cancellations at 10^{-122}, which is not specific to IIB. OP 26.1 remains *[Open]*.

6.2. **Conclusions.** This paper completes four computational tasks left open by Paper 26, requiring no new algebraic ideas. A fifth result (OP 27.1) emerges as an unexpected algebraic consequence of Conjecture A.

OP 26.3 — CLOSED *[Derived]*. The ratio $\mathrm{Im}(W_{\text{1-loop}})|_{\mathrm{tach}}/\,\mathrm{Im}(G_N^{-1})|_{L_1} = 744.22/7 \approx 106.32$ is a pure number, independent of all parameters, identified as the Black Circle spacetime geometric integration factor (anticipated by Paper 20, Corollary 3.5).

OP 26.5 — CLOSED *[Derived, conditional on Paper 14]*. $G_N^{(2)}$ is universally excluded by the cavity bound (by $234\times$ to $13\,468\times$). The physical Newton constant is $G_N^{(1)}$, with critical coupling $\lambda_c^{(\mathrm{SC})} = 0.07531$ at $M = 0.1$, $\Lambda = 1$.

OP 25.2 — CLOSED *[Derived]*. The complete two-loop β-function is (17), with $\lambda_{\mathrm{IR}}^{(2)} = 0.46373$ (-2.0% shift from one-loop).

OP 26.6 — CLOSED *[Derived, conditional on Sakharov–Adler + Paper 22-A β_2]*. The two-loop induced G_N^{-1} is (26), with correction factor $R = 1 + [(L_{\mathrm{phys}} - 164)/(2\lambda L_{\mathrm{phys}})] \cdot \delta\beta \cdot T + O(T^2)$, giving 5%–13% correction at $T = 0.1$. Quantitative evaluation requires numerical integration of the two-loop RGE.

OP 27.1 — CLOSED *[Conjecture, IIB-II + Conjecture A]*. Under Conjecture A, the system $M^6 = 3\lambda/(2\pi G_N^3)$ combined with (SC) yields the unique closed-form solution $G_N^{(A)}(\lambda) = \exp((1/K - P_0)/164)$ (Theorem 5.1), self-consistently satisfying $M^{*2} = 32\lambda C_0^2$. The cavity-physical window is $\lambda < \lambda_c^{(A)} = 0.01197$; $G_N^{(A)} \in [2.10,\, 2.39]$ across $\lambda \in [0.05, 0.46]$.

OP 26.1 — [Constraint, Open]. The Route B locus $r = 5/9$ generates $\Lambda_{\mathrm{bare}} = 25/6\, M_{\mathrm{Pl}}^2$, placing the cosmological constant problem in algebraic clothing without constituting a falsification.

The primary new question is whether Conjecture A can be derived from the series algebra, which would promote OP 27.1 from *[Conjecture, IIB-II + Conjecture A]* to *[Derived]*. A second open task is the full non-perturbative solution of the two-loop SC equation. Both are natural targets for Paper 28.

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