BLACK CIRCLE THERMODYNAMICS, FERMION BACKREACTION, AND THE NORM-GAP IDENTITY

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Abstract. Paper 26 [12] derived Newton's constant G_N algebraically from the sedenion scalar field theory S_{84} via the Sakharov–Adler mechanism; Paper 27 [13] resolved the branch structure of the self-consistency equation (SC) and closed four associated open problems. Paper 28 consolidates the downstream physical consequences. In §2 we compute the Bekenstein–Hawking entropy $S_{\mathrm{BH}}(M,\lambda)$ and Hawking temperature $T_H(M,\lambda)$ as explicit functions of $G_N^{(1)}(M,\lambda)$ from Paper 27 [13], verify the first law $T_H\, dS_{\mathrm{BH}} = dM$, and tabulate the two-loop correction from Paper 27 [13], Theorem 4.3. In §3 we examine OP 20.2 and derive a structural negative result: the tree-level GHY term $S_{\mathrm{GHY}} = -M$ is G_N-independent for Schwarzschild geometry because $r_s = 2G_N M$ cancels the $(8\pi G_N)^{-1}$ prefactor exactly *[Derived-Negative]*; the genuine quantum correction requires computing the boundary heat kernel of S_{84} *[Open]*. In §4 we evaluate OP 21.4 under Candidate (D) at $\lambda = \lambda_{\mathrm{IR}}^{(2)} = 0.46373$: the cavity bound $\lambda G_N M^2 < 2.14 \times 10^{-3}$ is violated for $M \geq 0.05$; a cavity-physical window exists for $M < M_c = 0.01927\, M_{\mathrm{Pl}}$ *[Derived, conditional on Papers 14 [8], 27 [13]]*. In §5 we partially close OP 23.2 by assembling the complete fermionic correction to G_N^{-1} from Paper 26 [12]: the log-term contribution $-4\ln(\Lambda^2/16\lambda N(C_0))$ and the constant term $+4\ln 2$ together give $\delta G_N^{-1}|_{\mathrm{ferm}} = -(16\lambda N(C_0)/3\pi)\ln(\Lambda^2/32\lambda N(C_0))$ *[Derived, conditional on Papers 19 [10], 26 [12]]*; the scheme-independent combination remains *[Open]*. In §6 we investigate the three algebraic instances of the number 4 in the series: the norm gap $\Delta N = 4$ (Paper 6 [6]), the kernel dimension $\dim\ker(L_a) = 4$ at $n = 4$ (Paper 1 [4], Theorem 5.17), and the spectral reduction $\mathcal{S}_{\mathrm{eff}} = 168 - 4 = 164$ (Paper 26 [12]). We show $\Delta N \neq \dim\ker$ for general $n \geq 4$ *[Derived-Negative]*, and state the $n = 4$ coincidence as an *[Observation]*. In §7 we write the IIB-IV dark-matter structural formula and obtain the order-of-magnitude estimate $m_{\mathrm{dark}} \approx 0.168\, M_{\mathrm{Pl}}$ *[Speculation]*. Open problems are collected in §8.

Contents

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Conventions and Notation

Natural units $c = \hbar = 1$ with $M_{\rm Pl} = 1$ unless stated otherwise. The Schwarzschild radius is $r_s = 2G_N M$. We write λ for the quartic coupling of S_{84}, Λ for the UV cutoff, M for the Black Circle mass. All results labelled *[Derived]* follow from established series results without new algebraic input; gaps are labelled *[Open]*, never glossed. Negative results are labelled *[Derived-Negative]* and treated as equally valuable. **Reference parameters:** $M = 0.1$, $\Lambda = 1$ (Planck units) unless otherwise noted.

1. Introduction

Paper 26 [12] established the central result of the current phase:

$$G_N^{-1} = \frac{16\lambda N(C_0)}{12\pi}\left[164 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 220\ln 2 - 42\ln 3\right], \tag{1}$$

where $\mathcal{S}_{\rm eff} = 164 = |\mathrm{GL}(3,\mathbb{F}_2)| - 4$ is the effective spectral invariant after the Majorana fermion correction (Paper 26 [12], Theorem 4.2, *[Derived, conditional on Sakharov–Adler]*). Paper 27 [13] resolved the branch structure: $G_N^{(2)}$ is universally excluded by the cavity bound of Paper 14 [8] (Theorem 2.3); the complete two-loop β-function is assembled (Theorem 3.6); and the two-loop G_N correction factor is derived (Theorem 4.3). With G_N as algebraic output rather than input, downstream computation becomes possible for the first time in the series.

Three targets are addressed. *Thermodynamics.* $S_{\rm BH} = 4\pi G_N M^2$ and $T_H = 1/(8\pi G_N M)$ become explicit functions of (M, λ); the first law serves as an analytic check. *Norm-gap identity.* Three algebraic instances of the number 4 are investigated: $\Delta N = 4$ (Paper 6 [6]), $\dim\ker(L_a) = 4$ (Paper 1 [4]), and $168 - 164 = 4$ (Paper 26 [12]). *IIB-IV structure.* The known $G_N^{(1)}$ places a structural constraint on the dark-matter energy density under [5].

Three open problems are targeted: OP 20.2 (quantum GHY), OP 21.4 (minimum Black Circle, Candidate D), OP 23.2 (complete fermionic backreaction). The outcomes are two partial closures, one negative structural result, one genuine negative computation, and a new open problem (OP 28.1).

2. Black Circle Thermodynamics

2.1. Entropy and temperature.

2.1. Entropy and temperature. The Black Circle is a Schwarzschild black hole of mass M with de Sitter interior governed by the condensate C_0 of S_{84}. The exterior metric is standard Schwarzschild with $G_N = G_N^{(1)}(M, \lambda)$ determined by the SC equation of Paper 26 [12], Theorem 4.6, uniquely selected by Paper 27 [13], Theorem 2.3 for $\lambda < \lambda_c^{(\rm SC)} = 0.07531$ at $M = 0.1$, $\Lambda = 1$.

The Bekenstein–Hawking entropy [1] and Hawking temperature [2] are:

$$S_{\rm BH}(M, \lambda) = \frac{A}{4G_N} = 4\pi G_N M^2, \qquad T_H(M, \lambda) = \frac{1}{8\pi G_N M}. \tag{2}$$

Both quantities depend on λ only through $G_N^{(1)}(M, \lambda)$.

2.2. First-law verification. *[Derived].* The first law requires $T_H\, dS_{\rm BH} = dM$:

$$T_H \cdot \frac{dS_{\rm BH}}{dM} = \frac{1}{8\pi G_N M} \cdot 8\pi G_N M = 1 = \frac{dM}{dM} \quad \checkmark.$$

Explicitly, $T_H \cdot S_{\rm BH} = M/2$, verified numerically in Table 1.

Proposition 2.1. [Derived]. *The product $T_H \cdot r_s = 1/(4\pi)$ is a universal constant independent of G_N, M, and λ:*

$$T_H \cdot r_s = \frac{1}{8\pi G_N M} \cdot 2G_N M = \frac{1}{4\pi} \approx 0.07958.$$

Proof. Direct substitution; $r_s = 2G_N M$ cancels G_N and M. ☐

Remark 2.2. Proposition 2.1 is a consequence of Schwarzschild geometry alone. Its series-specific content is the algebraic origin of G_N from the SC equation.

2.3. **Numerical tables.** All values use $M = 0.1$, $\Lambda = 1$. Values of $G_N^{(1)}$ from Paper 27 [13], Table 1 (Brent root-finding, tolerance 10^{-14}). $S_{\mathrm{BH}}^{(\mathrm{Pl})} := 4\pi M^2 = 4\pi \times 0.01 = 0.12566$ denotes the Planck-normalised entropy.

TABLE 1. One-loop Black Circle thermodynamics ($M = 0.1$, $\Lambda = 1$).

λ	$G_N^{(1)}$	S_{BH}	T_H	$T_H \cdot S_{\mathrm{BH}}$	$M/2$	$S_{\mathrm{BH}}/S_{\mathrm{BH}}^{(\mathrm{Pl})}$
0.05	2.5993	0.3266	0.1531	0.050000	0.0500	2.5993
0.10	3.0371	0.3817	0.1310	0.050000	0.0500	3.0371
0.20	3.6190	0.4548	0.1099	0.050000	0.0500	3.6190
0.30	4.0365	0.5072	0.09852	0.050000	0.0500	4.0365

Corollary 2.3. [Derived]. $S_{\mathrm{BH}}/S_{\mathrm{BH}}^{(\mathrm{Pl})} = G_N^{(1)}(M, \lambda)$. *The algebraic enhancement of the Black Circle entropy over the Planck-normalised value equals precisely the dimensionless Newton constant.*

Proof. $S_{\mathrm{BH}} = 4\pi G_N M^2$, $S_{\mathrm{BH}}^{(\mathrm{Pl})} = 4\pi M^2$; ratio $= G_N^{(1)}$. ☐

Remark 2.4. $S_{\mathrm{BH}}/M^2 = 4\pi G_N$ varies with λ; there is no fixed entropy-per-mass-squared. No algebraic relation between S_{BH} and $336 = 2|\mathrm{PSL}(2,7)|$ is observed.

Corollary 2.5. [Derived]. *The Black Circle free energy* $F := M - T_H S_{\mathrm{BH}} = M/2$ *is* G_N*-independent.*

Proof. $T_H \cdot S_{\mathrm{BH}} = M/2$ (Table 1); $F = M - M/2 = M/2$. ☐

Remark 2.6. The G_N-independence of $F = M/2$ is structurally parallel to Corollary 3.2 ($\delta S_{\mathrm{GHY}} = 0$ identically): both thermodynamic boundary contributions collapse to $M/2$ regardless of the algebraic origin of G_N from the SC equation.

2.4. **Two-loop running correction.** From Paper 27 [13], Theorem 4.3, the two-loop correction factor is

$$R(\lambda, T) = 1 + \frac{L_{\mathrm{phys}}(\lambda) - 164}{2\lambda\, L_{\mathrm{phys}}(\lambda)} \cdot \delta\beta(\lambda) \cdot T + O(T^2),$$

where $L_{\mathrm{phys}} = P + 246 \ln G_N^{(1)} > 0$, $\delta\beta = (-189\lambda^3 - 12\lambda + 48)/(4\pi^4)$, and $T = \ln(\Lambda_{\mathrm{high}}/\Lambda_{\mathrm{low}})$. Since $R < 1$, the two-loop running increases G_N (reduces G_N^{-1}): $G_N^{(2\mathrm{L})} = G_N^{(1)}/R$.

TABLE 2. Two-loop corrected thermodynamics ($T = 0.1$, $M = 0.1$, $\Lambda = 1$).

λ	$R(T{=}0.1)$	$G_N^{(2\mathrm{L})}$	$S_{\mathrm{BH}}^{(2\mathrm{L})}$	$T_H^{(2\mathrm{L})}$	δS_{BH}
0.05	0.8681	2.9942	0.3763	0.1329	+15.2%
0.10	0.8967	3.3870	0.4256	0.1175	+11.5%
0.20	0.9283	3.8985	0.4899	0.1021	+7.7%
0.30	0.9479	4.2584	0.5351	0.09344	+5.5%

Remark 2.7. At $\lambda = 0.05$: $\lambda G_N^{(1)} M^2 = 1.30 \times 10^{-3}$ and $\lambda G_N^{(2\mathrm{L})} M^2 = 1.50 \times 10^{-3}$, both below 2.14×10^{-3}. The two-loop correction does not change the branch classification.

Remark 2.8. Paper 27 [13], Theorem 4.3 notes that the leading-log expansion is valid only for $T \ll T_{\text{break}} \approx 0.034$. For $T = 0.1$, Table 2 is indicative; the full ODE integration of the two-loop RGE is required for quantitative results (OP 27.2).

3. The G_N-Independence of the Tree-Level GHY Term (OP 20.2)

3.1. **Setup.** The standard result of Gibbons and Hawking [3] for the on-shell GHY boundary contribution to the Euclidean partition function of Schwarzschild is $S_{\text{GHY}}^{\text{tree}} = -M$ (the negative ADM mass), obtained by evaluating the boundary term at the asymptotic boundary and subtracting the flat-space reference. Written explicitly at fixed mass M with $r_s = 2G_N M$:

$$S_{\text{GHY}}^{\text{tree}} = -\frac{r_s}{2G_N} = -\frac{2G_N M}{2G_N} = -M. \tag{3}$$

Proposition 3.1. [Derived]. *The tree-level GHY contribution for a Schwarzschild black hole of fixed mass M is independent of G_N:*

$$S_{\text{GHY}}(G_N) = -M \quad \textit{for all } G_N > 0.$$

Proof. From (3): the G_N in $r_s = 2G_N M$ and the G_N in the $(8\pi G_N)^{-1}$ prefactor cancel exactly. □

Corollary 3.2. [Derived-Negative]. *The prescription $\delta S_{\text{GHY}} := S_{\text{GHY}}(G_N + \delta G_N) - S_{\text{GHY}}(G_N)$ gives $\delta S_{\text{GHY}} = 0$ identically at all orders in δG_N, whenever M is fixed and $r_s = 2(G_N + \delta G_N)M$.*

Proof. $S_{\text{GHY}}(G_N + \delta G_N) = -M$ by Proposition 3.1 with $G_N \to G_N + \delta G_N$; subtraction gives zero. □

Remark 3.3. The structural zero of Corollary 3.2 reflects that $S_{\text{GHY}} = -M$ is the ADM mass. In induced gravity, the Einstein–Hilbert action and its GHY companion are generated simultaneously at one loop; there is no classical G_N from which to perturb. The correct computation for OP 20.2 is the *boundary contribution to the one-loop heat kernel* of S_{84} in curved spacetime with boundary at $r = r_s$ — the coefficient c_{bdry} in the Seeley–DeWitt expansion. The bulk part generates G_N^{-1} (Paper 26 [12], Theorem 4.2); the boundary part would generate a genuine quantum GHY term at one loop, requiring separate heat-kernel analysis for S_{84} in the Black Circle background. This has not been performed.

OP 20.2 (sharpened wording): Compute the boundary Seeley–DeWitt coefficient c_{bdry} for S_{84} in the Black Circle background (de Sitter interior, Schwarzschild exterior, matching at $r = r_s$). The result gives the genuine one-loop quantum GHY term, distinct from and not obtainable by $G_N \to G_N + \delta G_N$ in the classical formula. *[Open]*.

4. Minimum Black Circle: Candidate (D) at $\lambda_{\text{IR}}^{(2)}$ (OP 21.4)

4.1. **Setup.** OP 21.4 asks which of the minimum Black Circle candidates (A)–(D) of Paper 17 [9] is consistent with IIB first principles. Candidate (D) requires the physical coupling in the IR-asymptotic regime $\beta(\lambda) < 0$, i.e. $\lambda < \lambda_{\text{IR}}$. With the two-loop IR zero from Paper 27 [13], Theorem 3.6: $\lambda_{\text{IR}}^{(2)} = 0.46373$, we evaluate the SC equation at exactly $\lambda = \lambda_{\text{IR}}^{(2)}$ for $M \in \{0.05, 0.10, 0.20, 0.50\}$, $\Lambda = 1$.

4.2. **Numerical root-finding.**

TABLE 3. SC equation at $\lambda = 0.46373$, $\Lambda = 1$. Cavity limit $= 2.14 \times 10^{-3}/(\lambda M^2)$.

M	$G_N^{(1)}$	$\lambda G_N^{(1)} M^2$	Factor	Status
0.05	6.857	7.949×10^{-3}	3.71×	**FAIL**
0.10	4.558	2.114×10^{-2}	9.88×	**FAIL**
0.20	3.148	5.840×10^{-2}	27.3×	**FAIL**
0.50	2.199	2.549×10^{-1}	119×	**FAIL**

TABLE 4. Scan for cavity-physical M at $\lambda = 0.46373$.

M	$G_N^{(1)}$	$\lambda G_N^{(1)} M^2$	Factor	Status
0.030	9.397	3.922×10^{-3}	1.83×	FAIL
0.020	12.136	2.251×10^{-3}	1.052×	FAIL
0.019	12.539	2.099×10^{-3}	0.981×	OK
0.01927 ($= M_c$)	12.426	2.140×10^{-3}	1.000×	= bound
0.018	12.978	1.950×10^{-3}	0.911×	OK
0.015	14.584	1.522×10^{-3}	0.711×	OK
0.010	18.939	8.783×10^{-4}	0.411×	OK

4.3. **Cavity-physical window.** By bisection on $M \in [0.019, 0.020]$ (sign change confirmed, Brent tolerance 10^{-9}):

$$M_c = 0.01927, \quad G_N^{(1)}(M_c, \lambda_{\mathrm{IR}}^{(2)}) = 12.426, \quad \lambda G_N^{(1)} M_c^2 = 2.140 \times 10^{-3}.$$

Theorem 4.1. [Derived, conditional on Papers 14 [8], 27 [13]]. *Under Candidate (D), with* $\lambda = \lambda_{\mathrm{IR}}^{(2)} = 0.46373$*:*

(i) *For* $M \in \{0.05, 0.10, 0.20, 0.50\}$*, the cavity bound* $\lambda G_N^{(1)} M^2 < 2.14 \times 10^{-3}$ *is violated by factors* 3.71×–119×*. These parameter values are excluded.*

(ii) *A cavity-physical window exists for* $M < M_c = 0.01927\, M_{\mathrm{Pl}}$*. For* $M = 0.01$*:* $\lambda G_N^{(1)} M^2 = 8.78 \times 10^{-4} < 2.14 \times 10^{-3}$*. Candidate (D) is physically admissible in this regime.*

(iii) *OP 21.4 is* ***partially closed****: Candidate (D) requires a near-sub-Planck Black Circle (*$M < M_c \approx 0.019\, M_{\mathrm{Pl}}$*). Whether this constitutes a physical selection mechanism under IIB first principles remains* [Open].

Proof. (i) Brent root-finding, tolerance 10^{-12}, Table 3. (ii) Table 4, M_c by bisection to five significant figures. (iii) Algebraic selection requires Paper 27 [13], OP 27.3. ☐

Remark 4.2. The near-sub-Planck nature of the cavity-physical window is consistent with Paper 27 [13], §5: the Conjecture A window $\lambda < \lambda_c^{(A)} = 0.01197$ and all near-Planck solutions ($M \sim M_{\mathrm{Pl}}$) violate the cavity bound. Candidate (D) imposes an additional constraint $M < M_c$ from the IR side.

5. Complete Fermionic Correction to G_N^{-1}: Partial Closure of OP 23.2

5.1. **The two pieces.** Paper 26 [12] provides two algebraic contributions to the fermionic correction.

Piece 1 (Paper 26 [12], Corollary 4.5, *[Derived, conditional on Paper 19* [10]*]*): the constant (UV-finite) part,

$$\delta G_N^{-1}\big|_{\mathrm{ferm,const}} = +\frac{16\lambda N(C_0)}{3\pi} \ln 2 > 0. \qquad (4)$$

Piece 2 (Paper 26 [12], Theorem 4.2 proof): the Majorana loop reduces the leading-log coefficient $168 \to 164$, contributing explicitly

$$\delta G_N^{-1}\big|_{\text{ferm,log}} = \frac{16\lambda N(C_0)}{12\pi}(164-168)\ln\frac{\Lambda^2}{16\lambda N(C_0)} = -\frac{16\lambda N(C_0)}{12\pi}\cdot 4\cdot \ln\frac{\Lambda^2}{16\lambda N(C_0)}. \tag{5}$$

5.2. Combined formula.

Theorem 5.1. [Derived, conditional on Papers 19 [10], 26 [12]]. *The total Majorana fermion correction to G_N^{-1} from $S_{84}^{(f)}$ is*

$$\delta G_N^{-1}\big|_{\text{ferm}} = \frac{16\lambda N(C_0)}{12\pi}\left[-4\ln\frac{\Lambda^2}{16\lambda N(C_0)}+4\ln 2\right] = -\frac{16\lambda N(C_0)}{3\pi}\ln\frac{\Lambda^2}{32\lambda N(C_0)}. \tag{6}$$

Proof. Sum Pieces 1 and 2; apply $\ln(\Lambda^2/16\lambda N(C_0)) - \ln 2 = \ln(\Lambda^2/32\lambda N(C_0))$. ∎

Consistency check. *[Derived].* Adding $\delta G_N^{-1}|_{\text{ferm}}$ to the bosonic supertrace contribution reproduces Paper 26 [12], Theorem 4.2:

$$\begin{array}{rl}
\text{Bosonic:} & \dfrac{16\lambda N(C_0)}{12\pi}\left[+168\ln(\Lambda^2/16\lambda N(C_0)) \;-224\ln 2 - 42\ln 3\right] \\
\text{Fermionic:} & \dfrac{16\lambda N(C_0)}{12\pi}\left[\quad -4\ln(\Lambda^2/16\lambda N(C_0)) + 4\ln 2 \qquad\qquad\qquad\right] \\
\hline
\text{Total:} & \dfrac{16\lambda N(C_0)}{12\pi}\left[+164\ln(\Lambda^2/16\lambda N(C_0)) \;-220\ln 2 - 42\ln 3\right] \checkmark
\end{array}$$

TABLE 5. Components of $\delta G_N^{-1}|_{\text{ferm}}$ at $M = 0.1$, $\Lambda = 1$.

λ	$N(C_0)$	$\delta G_N^{-1}\|_{\text{const}}$	$\delta G_N^{-1}\|_{\text{log}}$	$\delta G_N^{-1}\|_{\text{ferm}}$
0.05	0.2304	$+1.356\times10^{-2}$	-3.308×10^{-2}	-1.952×10^{-2}
0.10	0.1290	$+1.518\times10^{-2}$	-3.456×10^{-2}	-1.938×10^{-2}
0.20	0.07015	$+1.651\times10^{-2}$	-3.558×10^{-2}	-1.908×10^{-2}
0.30	0.04860	$+1.716\times10^{-2}$	-3.603×10^{-2}	-1.887×10^{-2}

5.3. Numerical evaluation.

The full fermion correction is negative ($\delta G_N^{-1}|_{\text{ferm}} < 0$): the Majorana sector slightly reduces G_N^{-1} (increases G_N) relative to a hypothetical purely bosonic S_{84}. The constant piece is positive (Corollary 4.5) but dominated by the log-term reduction.

5.4. Status of OP 23.2.

Corollary 5.2. [Derived, conditional on Papers 19 [10], 26 [12]]. *Paper 26 [12], Corollary 4.5 and Theorem 4.2 are mutually consistent and together constitute the complete fermionic correction. No missing term is identified.*

Partial closure. OP 23.2 is closed for the explicit formula (Theorem 5.1). The scheme-independent combination — whether a Λ-free ratio $\delta G_N^{-1}|_{\text{ferm}}/G_N^{-1}|_{\text{total}}$ exists — remains *[Open]*: the ratio

$$\frac{\delta G_N^{-1}|_{\text{ferm}}}{G_N^{-1}|_{\text{total}}} = \frac{-4\ln(\Lambda^2/16\lambda N(C_0))+4\ln 2}{164\ln(\Lambda^2/16\lambda N(C_0)) - 220\ln 2 - 42\ln 3}$$

is Λ-dependent and does not simplify to a scheme-independent expression at this order.
OP 23.2 updated status: explicit formula *[Derived, conditional on Papers 19 [10], 26 [12]]*; scheme-independent combination *[Open]*.

6. The Three "4"s and the $\mathcal{S}_{\rm eff}$ Identity

6.1. **The three instances.** Three algebraically distinct "4"s appear in the series at $n = 4$:

[Z1] Paper 6 [6], Theorem 3.8: $\Delta N(a, b) = N(a)N(b) - N(ab) = 4$ for every ZD pair. *Origin:* ZD elements have norm 2 ($N(a) = N(b) = 2$, $N(ab) = 0$), so $\Delta N = 2 \cdot 2 - 0 = 4$.

[Z1] Paper 1 [4], Theorem 5.17: $\dim\ker(L_a) = 2^{n-2}$ for every ZD unit $a \in \mathcal{A}_n$, $n \geq 4$. At $n = 4$: $\dim\ker(L_a) = 4$. *Origin:* number of orbit representatives s with $c_s = \pm 2$ (active splits); equals the number of ZD partners of a.

[Z1] Paper 26 [12], Theorem 4.2: $\mathcal{S}_{\rm eff} = 168-4 = 164$. The reduction $4 = (4\text{ Majorana})\times(2+\mu_3)\times\frac{1}{2}$. *Origin:* 4 Majorana modes from 4_s sector (Paper 19 [10]); factor $2+\mu_3 = 2$ at Level 3 (algebraic, Proposition 6.3); Majorana factor $\frac{1}{2}$.

6.2. **Z1 and Z2: algebraically independent for $n \geq 4$.**

Proposition 6.1. [Derived-Negative]. *The equality $\Delta N(a, b) = \dim\ker(L_a)$ does* not *hold for general $n \geq 4$. It is a numerical coincidence at $n = 4$ only.*

Proof. For integer ZD elements $a = e_p + e_q$ (any $n \geq 4$): $N(a) = 2$, $N(ab) = 0$, so $\Delta N = 4$ for all $n \geq 4$. Meanwhile $\dim\ker(L_a) = 2^{n-2}$ (Paper 1 [4], Theorem 5.17). Equality $4 = 2^{n-2}$ holds only at $n = 4$. □

n	ΔN	$\dim\ker(L_a)$	Equal?
4	4	4	✓
5	4	8	×
6	4	16	×
7	4	32	×

Remark 6.2. [Z1] and [Z2] arise from independent algebraic mechanisms. [Z1] counts the norm residue of non-composition; [Z2] counts ZD partners via orbit-splitting. At $n = 4$ these counts agree by coincidence; for all $n \geq 5$ they diverge.

6.3. **The factor 2 in Z3 is algebraic.**

Proposition 6.3. [Derived]. *The factor $2 = m_{4_s}^2/m_0^2$ appearing in the $\mathcal{S}_{\rm eff}$ reduction is algebraic, not numerical:*

$$\frac{m_{4_s}^2}{m_0^2} = \frac{16\lambda N(C_0)(2+\mu_3)}{16\lambda N(C_0)} = 2 + \mu_3 = 2,$$

where $\mu_3 = 0$ is the adjacency eigenvalue at Level 3 (Paper 11 [7], Theorem 4.5) and $m_0^2 := 16\lambda N(C_0)$ is the common factor in $m_i^2 = m_0^2(2+\mu_i)$.

Proof. Define $m_0^2 := 16\lambda N(C_0)$ as the common factor in the mass formula $m_i^2 = m_0^2(2 + \mu_i)$. Then $m_{4_s}^2 = m_0^2(2+\mu_3) = 2m_0^2$, and the ratio $m_{4_s}^2/m_0^2 = 2 + \mu_3 = 2$ is an algebraic identity. □

6.4. **The Majorana count vs. $\dim\ker(L_a)$: an open gap.** The $\mathcal{S}_{\rm eff}$ reduction can be written:

$$\Delta\mathcal{S}_{\rm ferm} = \underbrace{4}_{\text{Majorana}} \times \underbrace{2}_{(2+\mu_3)} \times \underbrace{\tfrac{1}{2}}_{\text{Majorana}} = 4 = \dim\ker(L_a)\big|_{n=4}.$$

The "4" of [Z3] equals $\dim\ker(L_a)$ at $n = 4$. However: (i) the 4 Majorana modes are derived conditionally (Paper 19 [10], *[Derived, conditional on IIB-I/II +* FS$(4_s) = +1$*]*); (ii) the identification $\mathrm{irr}_8 \leftrightarrow \ker(L_a)$ appears as a dictionary entry in Paper 24 [11], not as a proved theorem; (iii) $\dim\ker(L_a)$ counts ZD partners; 4 Majorana modes counts spinorial degrees of freedom. The connection cannot be promoted to *[Derived]*.

6.5. The $\mathcal{S}_{\text{eff}}$ identity at $n = 4$.

Observation 6.4. *[Observation]*, $n = 4$ only. At $n = 4$ all three instances coincide:

$$\mathcal{S}_{\text{eff}} = |\mathrm{GL}(3, \mathbb{F}_2)| - \Delta N = |\mathrm{GL}(3, \mathbb{F}_2)| - \dim\ker(L_a)\big|_{n=4} = 164.$$

The chain Paper 1 [4] Thm 5.17 → Paper 19 [10] → Paper 26 [12] Thm 4.2 produces this identity structurally at $n = 4$, with the factor 2 from Proposition 6.3 being algebraic.

Remark 6.5. A general theorem $\mathcal{S}_{\text{eff}} = |\mathrm{GL}(3, \mathbb{F}_2)| - \dim\ker(L_a)$ would require dim ker to control the log coefficient of G_N^{-1} at all n. For $n \geq 5$ the 4_s sector does not exist and $\dim\ker(L_a) = 2^{n-2} \neq 4$; the identity is specific to the $\mathcal{A}_4$ structure.

OP 28.1 (new): Prove or disprove that the number of physical Majorana modes in S_{84} equals $\dim\ker(L_a) = 4$ via the Fano non-incidence map $\mathrm{irr}_8 \leftrightarrow \ker(L_a)$ (Paper 24 [11], Table 3.1). If true, Observation 6.4 becomes *[Derived]*. GAP-accessible via $\mathrm{Stab}_{G_{84}}(a)$-module analysis.

7. IIB-IV Gravitational Structure

7.1. Setup.
Under IIB-IV *[Conjecture, P2]*, the $|\mathrm{ZD}(4)| = 336$ signed ZD elements of $\mathcal{A}_4$ constitute the dark-sector seed of the next aeon. The sign-parity constraint $\pi_A\pi_B\pi_C = -1$ (Paper 1 [4], Theorem 7.18, *[Derived]*) is inherited: dark-matter ZD pairs are sign-parity antisymmetric, a falsifiable structural signature. The structural formula for dark-matter energy density is *[Speculation]*:

$$\rho_{\text{dark}} = \varepsilon \times 336,$$

where ε is the unknown energy per ZD pair.

7.2. Structural constraint on ε.
$G_N^{(1)}$ from Paper 27 [13] does not directly constrain ε; the SC equation determines $G_N(M, \lambda)$ but ε is an additional IIB-IV parameter not currently derivable.

From cosmological observation $\Omega_{\text{DM}}/\Omega_{\text{tot}} \approx 0.27$. Under the ansatz $\rho_{\text{total}} \approx \rho_{\text{Pl}} = M_{\text{Pl}}^4$ at the Planck epoch:

$$\varepsilon = \frac{0.27\, M_{\text{Pl}}^4}{336} \approx 8.04 \times 10^{-4}\, M_{\text{Pl}}^4.$$

Under the dimensional ansatz $\varepsilon = m_{\text{dark}}^4$:

$$m_{\text{dark}} = \varepsilon^{1/4} \approx 0.168\, M_{\text{Pl}}.$$

7.3. Comparison with m_{4_s}.
At $\lambda = 0.05$, $G_N^{(1)} = 2.599$, $M = 0.1$: $C_0 \approx 0.4800\, M_{\text{Pl}}$ (from $C_0^4 = 3/(2048\pi G_N^3 M^2 \lambda)$), $m_{4_s} = \sqrt{32\lambda}\, C_0 \approx 0.6072\, M_{\text{Pl}}$, and $m_{\text{dark}}/m_{4_s} \approx 0.277$, not a simple algebraic number. No parametric window where $m_{\text{dark}} = m_{4_s}$ was found.

Remark 7.1. *[Speculation]*, IIB-IV + QFT over $\mathcal{A}_4$. The estimate $m_{\text{dark}} \approx 0.168\, M_{\text{Pl}}$ rests on three unproven identification steps: (i) CCC boundary maps $\mathcal{A}_4$ ZD elements to dark-sector degrees of freedom; (ii) each ZD pair carries energy $\varepsilon = m^4$; (iii) the Planck-epoch density applies. None is derivable from the current series.

Remark 7.2. *[Conjecture, IIB-II, P2]*. Under IIB-II, G_N is universal: dark matter at aeon $k = 1$ couples to gravity with the same $G_N = G_N^{(1)}(M, \lambda)$. This universality constitutes the strongest structural output of IIB-IV currently within reach of the series.

Remark 7.3 (Holographic Connection 2 quantified). *[Conjecture, IIB-II, P2]*. Paper 2 [5], §4.2 identifies a structural analogy between the Bekenstein–Hawking entropy $S_{\text{BH}} = A/4G_N$ (determined by the horizon area alone, independent of gauge quantum numbers) and the Frobenius identity $\|L_a\|_F^2 = 2^n N(a)$ (Paper 2 [5], Paper 1 [4]; at $n = 4$: $\|L_a\|_F^2 = 16N(a)$, determined by $N(a)$ alone, independent of the specific products ae_k). Paper 2 [5] states explicitly: “the analogy is structural, not quantitative.”

With Corollary 2.3 ($S_{\mathrm{BH}}/S_{\mathrm{BH}}^{(\mathrm{Pl})} = G_N$) and Paper 26 [12], Corollary 4.4 ($G_N \propto N(C_0)$ under IIB-II), the analogy becomes quantitative:

$$S_{\mathrm{BH}} = 4\pi G_N M^2 \propto N(C_0)\cdot M^2 = \frac{\|L_{C_0}\|_F^2}{16}\cdot M^2 \quad \textit{[Conjecture, IIB-II]}$$

where $a = C_0/|C_0|$ is the condensate direction. The Bekenstein–Hawking entropy of the Black Circle is proportional to the squared Frobenius norm of the condensate left-multiplication, converting the structural holographic parallel of [5] into an explicit formula under IIB-II.

8. Open Problems

OP 28.1 (new, from §6). Prove or disprove that the number of physical Majorana modes of S_{84} equals $\dim\ker(L_a) = 4$ at $n = 4$ via the Fano non-incidence map $\mathrm{irr}_8 \leftrightarrow \ker(L_a)$ (Paper 24 [11], Table 3.1). GAP-accessible via $\mathrm{Stab}_{G_{84}}(a)$-module analysis. Closing this gap promotes Observation 6.4 to *[Derived]*.

OP 20.2 (sharpened, from §3). Compute the boundary Seeley–DeWitt coefficient c_{bdry} for S_{84} in the Black Circle background (de Sitter interior, Schwarzschild exterior, matching at $r = r_s$). Not GAP-accessible; requires heat-kernel analysis with boundary.

OP 21.4 (partially closed, residual). Theorem 4.1 shows Candidate (D) is cavity-physical for $M < M_c \approx 0.019\, M_{\mathrm{Pl}}$. The remaining question: does IIB first principles select a unique candidate from (A)–(D)? Resolution requires Paper 27 [13], OP 27.3: algebraic derivability of Conjecture A.

OP 23.2 (partially closed, residual). Explicit formula $\delta G_N^{-1}|_{\mathrm{ferm}}$ is now *[Derived]* (Theorem 5.1). Remaining: does a Λ-independent combination of fermion correction and total G_N^{-1} exist?

OP 27.2 (carried from Paper 27 [13]). Substitute two-loop running $\lambda^{(2)}(\Lambda)$ (Paper 27 [13], Theorem 3.6) into the SC equation and solve numerically for $G_N^{(1,\,2\text{-loop})}$. Leading-log breaks down for $T \gg T_{\mathrm{break}} \approx 0.034$.

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