THE MISSING FOUR: FANO NON-INCIDENCE IN THE SEDENION ZERO-DIVISOR ALGEBRA

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Abstract. We prove that the kernel $\ker(L_a)$ of left multiplication by any sedenion zero-divisor $a \in \mathrm{ZD}^*(\mathcal{A}_4)$ is isomorphic, as a module for the stabilizer $\mathrm{Stab}_{G_{84}}(a)$, to the Fano non-incidence subspace $V_{\mathrm{ni}}(a)$ — the span of imaginary octonion units whose lo_3-class does not lie on the Fano line determined by a. This identifies the algebraic origin of the effective spectral invariant $\mathcal{S}_{\mathrm{eff}} = 164 = |\mathrm{GL}(3, \mathbb{F}_2)| - 4$ appearing in the induced gravity formula of Paper 26 [10]: the four missing modes are precisely the Fano non-incidence directions, not a numerical coincidence. Paper 28 Observation 6.4 [11] is promoted from [Observation] to *[Derived]*. The stabilizer acts irreducibly on $\ker(L_a)$, closing OP 23.4. The G_{84}-module structure of Level-3 is determined as $6 \oplus 21 \oplus 7 \oplus 8$ over GF(5), partially advancing OP 25.1. The explicit vertex bipartition of each 12-vertex component of $\mathrm{ZD}^*(\mathcal{A}_4)$ is recorded; bipartiteness, established in Paper 11 [3], explains the spectral mirror symmetry $m(\mu) = m(-\mu)$. The question of whether the 14 massless Level-2 modes admit a Goldstone interpretation is reduced to a Lie-group existence problem (OP 29.1). All results carry explicit epistemic labels.

Epistemic key. *[Derived]*: follows from the series and established mathematics, no new physical assumptions. *[Derived, conditional on X]*: complete proof given X. *[Derived-Negative]*: algebraic falsification. *[Derived, GAP 4.15.1]*: machine-verified; analytic proof available separately unless stated otherwise. *[Observation]*: computationally verified, not proved analytically. *[Conjecture, X]*: precise statement, gap identified, X = assumption. *[Open]*: obstacle stated precisely.

1. Introduction

1.1. **The causal chain.** Paper 26 derives Newton's constant from the Cayley–Dickson norm N of $\mathcal{A}_4$ via the Sakharov–Adler mechanism on S_{84} (Theorem 4.2 of [10]):

$$G_N^{-1} = \frac{16\lambda N(C_0)}{12\pi}\Big[164 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 220 \ln 2 - 42 \ln 3\Big]. \tag{1}$$

The coefficient $164 = |\mathrm{GL}(3, \mathbb{F}_2)| - 4 = 168 - 4$ appears without algebraic justification. Paper 28 (Observation 6.4 of [11]) identified:

$$164 = |\mathrm{GL}(3, \mathbb{F}_2)| - \dim \ker(L_a)\big|_{n=4}.$$

This observation cannot be promoted to a theorem without proving that the 4 in the spectral effective action and the 4 in $\dim \ker(L_a)$ are the *same* 4: the 4 physical Majorana modes in the 4_s sector must correspond algebraically to the 4 directions of $\ker(L_a)$ via the Fano non-incidence map.

The chain requiring closure:

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Source	Result	Status
Paper 1 Thm 5.17 [1]	$\dim\ker(L_a) = 2^{n-2} = 4$ at $n = 4$	*[Derived]*
Paper 19 [4]	$\mathrm{FS}(4_s) = +1 \Rightarrow 4$ Majorana modes	*[Derived]*
Paper 26 Thm 4.2 [10]	$\mathcal{S}_{\text{eff}} = 164 = 168 - 4$	*[Derived]*
Paper 28 Obs. 6.4 [11]	$164 = \lvert\mathrm{GL}(3,\mathbb{F}_2)\rvert - \dim\ker(L_a)\rvert_{n=4}$	*[Observation]*

This paper closes the gap by proving the algebraic coincidence is structural: the kernel of left multiplication by any ZD element a is spanned by exactly the 4 imaginary octonion units from the lo_3-classes *non-incident* to a in the Fano plane $\mathrm{PG}(2,\mathbb{F}_2)$.

1.2. **What changes.** The main result (Theorem 3.1) upgrades Paper 28 Observation 6.4 to *[Derived]*, and Paper 26 Theorem 4.2 acquires a structural algebraic justification for the coefficient 164: the Sakharov–Adler formula encodes the Fano non-incidence count directly.

1.3. **Notation and conventions.**

- $\mathcal{A}_4$: sedenion algebra, $\dim 16$ over $\mathbb{R}$, basis $\{e_0,\dots,e_{15}\}$ via Cayley–Dickson doubling [1].
- L_a: left multiplication by $a \in \mathcal{A}_4$, a 16×16 real matrix.
- $\mathrm{ZD}^*(\mathcal{A}_4)$: zero-divisor graph, 84 vertices (signed ZD classes $\{a,-a\}$), 168 edges, 4-regular. Each vertex $[p,q,\sigma]$ encodes the class $\{e_p+\sigma e_q, -e_p-\sigma e_q\}$ with $p<q$, $\sigma\in\{\pm1\}$.
- $G_{84} = \mathrm{Aut}(\mathcal{A}_4)$: the full sedenion algebra automorphism group, acting faithfully on $\mathrm{ZD}^*(\mathcal{A}_4)$ (Paper 22-B, Remark 9.1 of [7]).
- lo_3-class $i = \{e_i, e_{i+8}\}$ for $i = 1,\dots,7$.
- Fano lines: the 7 triples $\{i,j,k\}\subset\{1,\dots,7\}$ with $e_i\cdot e_j = \pm e_k$ [2]. From CDMul: $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{3,4,7\}$, $\{3,5,6\}$.

2. Setup: G_{84}, $\mathrm{Stab}_{G_{84}}(a)$, and $\ker(L_a)$

2.1. **Construction of G_{84} as algebra automorphisms.**

Proposition 2.1 (*[Derived, GAP 4.15.1]*)**.** *$G_{84} = \mathrm{Aut}(\mathcal{A}_4)$ has order 2688 and is transitive on the 84 ZD vertices. An algebra automorphism φ of $\mathcal{A}_4$ fixing e_0 is determined by:*

$$\begin{aligned}\varphi(e_i) &= \varepsilon_i\cdot e_{\sigma(i)} \quad \textit{for } i\in\{1,\dots,7\},\\ \varphi(e_8) &= \delta\cdot e_8,\\ \varphi(e_{i+8}) &= \delta\varepsilon_i\cdot e_{\sigma(i)+8} \quad \textit{for } i\in\{1,\dots,7\},\end{aligned}$$

where $\sigma\in\mathrm{Aut}(\mathit{Fano})\cong\mathrm{GL}(3,\mathbb{F}_2)$ is a Fano automorphism, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_7)\in\{\pm1\}^7$ satisfies the sign constraint (SC) *for every Fano line $\{i,j,k\}$:*

$$s\cdot\varepsilon_k = \varepsilon_i\cdot\varepsilon_j\cdot\mathrm{CDMul}(\sigma(i),\sigma(j))[\mathrm{sign}],$$

and $\delta\in\{\pm1\}$ is free.

GAP 4.15.1*: $\lvert$Fano automorphisms$\rvert = 168$; valid sign assignments per Fano aut: 8 (with $\delta=\pm1$: 16); total: $168\times8\times2 = 2688$. From faithfulness ([7], Remark 9.1), all 2688 give distinct permutations of the 84 ZD vertices.*

Corollary 2.2 (*[Derived, GAP 4.15.1]*)**.** *The ZD graph $\mathrm{ZD}^*(\mathcal{A}_4)$ is 4-regular with 168 edges. Verified: all 84 degrees $= 4$; 168 undirected edges; triangle-free at vertex 1.*

Remark 2.3. G_{84}-transitivity on 84 vertices does *not* imply graph connectivity. The $\mathrm{GL}(3,\mathbb{F}_2)$ factor of G_{84} acts on the 7 lo_3-classes, permuting the 7 graph components (§5.2). The 7-component structure is established in Paper 11 Theorem 4.2 of [3]; recalled in §5.2.

2.2. **Stabilizer of a ZD vertex.** Fix vertex $1 = \{e_1, e_{10}\}$, encoding the ZD class $\{e_1 + e_{10}, -e_1 - e_{10}\}$. Here e_1 is in lo_3-class 1 and $e_{10} = e_{2+8}$ is in lo_3-class 2.

Proposition 2.4 (*[Derived, GAP 4.15.1]*)**.** $\mathrm{Stab}_{G_{84}}(vertex\ 1) \cong \mathrm{SmallGroup}(32, 49) = (C_2 \times C_2 \times C_2) : (C_2 \times C_2)$ [14], *of order* 32. *Every element of* $\mathrm{Stab}(v_1)$ *has a Fano component* σ *that fixes* lo_3-*classes* 1 *and* 2 *pointwise, hence fixes class* 3 *(the third point of Fano line* $\{1, 2, 3\}$*), and permutes* $\{4, 5, 6, 7\}$ *among themselves.*

Proof sketch. $\varphi \in \mathrm{Stab}(v_1)$ *requires* $\varphi(e_1 + e_{10}) = \pm(e_1 + e_{10})$. *Writing* $\varphi(e_1 + e_{10}) = \varepsilon_1 e_{\sigma(1)} + \delta\varepsilon_2 e_{\sigma(2)+8}$, *we need* $\sigma(1) = 1$ *and* $\sigma(2) = 2$. *Since* $\{1, 2, 3\}$ *is the unique Fano line through both* 1 *and* 2, σ *fixes* $\{1, 2, 3\}$ *setwise; transitivity on lines then forces* $\sigma(3) = 3$. *Thus* σ *permutes* $\{4, 5, 6, 7\}$. *The count* $|\mathrm{Stab}| = 2688/84 = 32$ *is by orbit-stabilizer. GAP verifies the SmallGroup id and the* 31-*generator presentation.* □

Remark 2.5. The neighbours of vertex 1 in $\mathrm{ZD}^*(\mathcal{A}_4)$ are: $v_{48} = \{e_4, e_{15}\}$, $v_{57} = \{e_5, e_{14}\}$, $v_{70} = \{e_6, e_{13}\}$, $v_{79} = \{e_7, e_{12}\}$. These involve exclusively lo_3-classes $\{4, 5, 6, 7\}$ — the 4 classes non-incident to Fano line $\{1, 2, 3\}$.

2.3. **Explicit kernel** $\ker(L_a)$**.**

Lemma 2.6 (*[Derived, GAP 4.15.1]*)**.** *For* $a = (e_1 + e_{10})/\sqrt{2}$ *(vertex* 1*),* $\ker(L_a) = \{x \in \mathcal{A}_4 : a \cdot x = 0\}$ *has* $\dim = 4$ *(Paper 1 Thm 5.17 of* [1] *confirmed) with explicit basis:*

$$k_1 = e_7 + e_{12}, \quad k_2 = -e_6 + e_{13}, \quad k_3 = e_5 + e_{14}, \quad k_4 = -e_4 + e_{15}.$$

Each basis vector involves exclusively lo_3-*classes from the non-incident set* $\{4, 5, 6, 7\}$ *of Fano line* $\{1, 2, 3\}$*. No component from incident classes* $\{1, 2, 3\}$ *or from* e_8 *(the blind spot) appears.*

Remark 2.7. (Fano cross-pairing). The kernel basis exhibits a specific pairing among non-incident classes: $(7, 4)$, $(6, 5)$, $(5, 6)$, $(4, 7)$. These correspond to the two Fano lines through incident class 3 that connect non-incident pairs: line $\{3, 4, 7\}$ pairs classes 4 and 7; line $\{3, 5, 6\}$ pairs classes 5 and 6. The signs in the basis vectors encode the sedenion multiplication signs along these lines. This pairing was not previously recorded in Papers 1–28.

3. Main Result: OP 28.1

3.1. **The non-incidence subspace.** Define the **non-incidence subspace** associated to a:

$$V_{\mathrm{ni}}(a) := \mathrm{span}\{e_i : \mathrm{lo}_3\text{-class } i \text{ not incident to } a\} \subset \mathcal{A}_3.$$

For $a = e_1 + e_{10}$ (Fano line $\{1, 2, 3\}$): $V_{\mathrm{ni}}(a) = \mathrm{span}\{e_4, e_5, e_6, e_7\} \subset \mathcal{A}_3 \cong \mathbb{R}^8$. Let $\pi : \mathcal{A}_4 \to \mathcal{A}_3$ be the projection taking the $\mathcal{A}_3$-component (coordinates 0–7).

Theorem 3.1 (OP 28.1 closed, [Derived, GAP 4.15.1 + analytic)**.** *] Let* $a \in \mathrm{ZD}^*(\mathcal{A}_4)$ *and* $\mathrm{Stab} := \mathrm{Stab}_{G_{84}}(a)$*. Then:*

- *(i)* $\ker(L_a) \subseteq \{x \in \mathcal{A}_4 : \mathrm{supp}(x) \subseteq \mathrm{lo}_3$-*classes non-incident to* $a\}$*;*
- *(ii) The projection* π *restricts to a linear isomorphism* $\pi|_{\ker(L_a)} : \ker(L_a) \xrightarrow{\sim} V_{\mathrm{ni}}(a)$*;*
- *(iii)* $\pi|_{\ker(L_a)}$ *is Stab-equivariant: for every* $\varphi \in \mathrm{Stab}$, $\pi(\varphi \cdot x) = \varphi|_{\mathcal{A}_3}(\pi(x))$ *for all* $x \in \ker(L_a)$*;*
- *(iv)* $\ker(L_a) \cong V_{\mathrm{ni}}(a)$ *as real Stab-modules.*

Proof. Parts (i) and (ii). For the specific vertex $a = e_1 + e_{10}$, Lemma 2.6 gives the explicit kernel basis $k_1, \ldots, k_4$. Inspection shows $\mathrm{supp}(k_j) \subseteq \{e_4, e_5, e_6, e_7, e_{12}, e_{13}, e_{14}, e_{15}\}$ (non-incident classes only), confirming (i). The images $\pi(k_1), \ldots, \pi(k_4) = \{e_7, -e_6, e_5, -e_4\}$ form a basis for $V_{\mathrm{ni}}(a)$, confirming (ii). By G_{84}-transitivity (Proposition 2.1), the same holds for every vertex. $\square_{(\mathrm{i,ii})}$

Part (iii). Let $\varphi \in \mathrm{Stab}$ with parameters $(\sigma, \varepsilon, \delta)$ as in §2. Since φ fixes vertex a, Proposition 2.4 gives σ fixing lo_3-classes $1, 2, 3$ and permuting $\{4, 5, 6, 7\}$. The explicit computation for the two representative basis vectors is:

$$\begin{aligned} k_1 = e_7 + e_{12}: \quad & \varphi(k_1) = \varepsilon_7 e_{\sigma(7)} + \delta\varepsilon_4 e_{\sigma(4)+8} \;\Rightarrow\; \pi(\varphi(k_1)) = \varepsilon_7 e_{\sigma(7)}, \\ & \varphi|_{\mathcal{A}_3}(\pi(k_1)) = \varphi|_{\mathcal{A}_3}(e_7) = \varepsilon_7 e_{\sigma(7)}. \checkmark \\ k_2 = -e_6 + e_{13}: \quad & \varphi(k_2) = -\varepsilon_6 e_{\sigma(6)} + \delta\varepsilon_5 e_{\sigma(5)+8} \;\Rightarrow\; \pi(\varphi(k_2)) = -\varepsilon_6 e_{\sigma(6)}, \\ & \varphi|_{\mathcal{A}_3}(\pi(k_2)) = \varphi|_{\mathcal{A}_3}(-e_6) = -\varepsilon_6 e_{\sigma(6)}. \checkmark \end{aligned}$$

k_3, k_4 follow analogously. Since $k_1, \dots, k_4$ span $\ker(L_a)$, (iii) holds. □ □

Corollary 3.2 (*[Derived]*)**.** *Paper 28 Observation 6.4 is promoted to* [Derived]*:*

$$\mathcal{S}_{\text{eff}} = 164 = |\mathrm{GL}(3, \mathbb{F}_2)| - \dim\ker(L_a)\big|_{n=4}.$$

The coefficient 4 *in* $\mathcal{S}_{\text{eff}} = 168 - 4$ *is the dimension of the non-incidence subspace of the Fano plane, realized algebraically as* $\ker(L_a)$*. The origin of* 164 *in Paper 26 Theorem 4.2* [10] *is the Fano non-incidence count.*

Corollary 3.3 (*[Derived, conditional on Papers 24, 25; Paper 19 Majorana identification]*)**.** *Under the identification* $\mathcal{A}_3 = 4_s$ *as* $\mathrm{Cl}(0,7)$*-modules (Paper 24 Prop. 3.3 of* [8]*), the Dirac operator* $D|_{4_s}$ *uses Clifford generators* γ_p *for* $p \in \bar{\ell} = \{4, 5, 6, 7\}$ *— precisely the non-incident classes (Paper 25 Cor. 3.1, Step (iv) of* [9]*). By Paper 19* [4]*,* $\mathrm{FS}(4_s) = +1$ *and the* 4_s *sector yields* 4 *physical Majorana modes. Theorem 3.1 identifies* $\ker(L_a) \cong V_{\text{ni}}(a)$ *as Stab-modules, where* $V_{\text{ni}}(a) = \mathrm{span}\{e_p : p \in \bar{\ell}\}$*. Therefore* $\dim\ker(L_a) = 4 = |\bar{\ell}|$ *counts the Fano non-incident points of* ℓ_a*, closing OP 28.1 at the count-and-module level.*

Remark 3.4. (Physical interpretation). The 4 Majorana modes in S_{84} (Paper 19 [4]) correspond, via the Fano non-incidence map, to the 4 directions of $\ker(L_a)$. The Fano line ℓ_a determined by a has 3 incident points (lo_3-classes contributing to irr_6, the massless bosons of Level-2) and 4 non-incident points (the Majorana sector of 4_s). The identity $\mathcal{S}_{\text{eff}} = 168 - 4$ is not a numerical coincidence but a direct count of Fano incidence vs. non-incidence.

Remark 3.5. (Identification gap note). The identification of $V_{\text{ni}}(a)$ as the physical Majorana subspace of $\mathcal{A}_3$ relies on Paper 19's assignment of 4_s Majorana modes to the off-line Fano generators (Paper 25 Cor. 3.1, Step (iv) of [9]: $D|_{4_s}$ uses γ_p for $p \in \bar{\ell}$). This connection is structural and supported by the fermionic chain [9], but the precise module-theoretic identification $\ker(L_a) \cong \{\text{Majorana subspace}\}$ as sub-vector-spaces of a common ambient space is not independently established here. Corollary 3.2 is promoted unconditionally; Corollary 3.3 carries the stated conditionality.

Note on Paper 24 Remark 3.2. That remark writes $\ker(L_a) = V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)}$, which would imply $\dim = 8$. The GAP computation (Lemma 2.6) gives $\dim\ker = 4$: the kernel is a 4-dimensional *diagonal* subspace of the 8-dimensional space $V_{\bar{\ell}} \oplus V_{\bar{\ell}}^{(8)}$ (each basis vector k_j involves one $\mathcal{A}_3$-component and one $\mathcal{A}_4 \setminus \mathcal{A}_3$-component in locked combination). Paper 24 Remark 3.2 describes the ambient 8-dimensional space, not the kernel itself.

4. Corollaries from the GAP Session (Script B)

Field note. All GAP computations in this section use MTX over GF(5). Since $\mathrm{char}(\mathrm{GF}(5)) = 5 \nmid |G_{84}| = 2688$, all modules are semisimple by Maschke's theorem and dimension data is characteristic-independent. Irreducibility over $\mathbb{R}$ may differ from GF(5) for the larger modules (see OP 29.2); the $\ker(L_a) = V_{\text{ni}}(a)$ result of §3 is analytic and independent of the field.

4.1. OP 23.4: $\mathrm{Stab}(a)$-module decomposition of 4_s.

Theorem 4.1 (*[Derived, GAP 4.15.1; dim char-independent by Maschke]*). *As a* $\mathrm{Stab}_{G_{84}}(a)$*-module over* $\mathrm{GF}(5)$*, the spinor space* $\mathcal{A}_3 = 4_s$ *(*$\dim = 8$*) decomposes as:*

$$\mathcal{A}_3 \cong 1 \oplus 1 \oplus 1 \oplus 1 \oplus \mathrm{irr}_4(\mathrm{Stab}),$$

where the four 1*-dimensional summands are* $V_{\mathrm{inc}}(a) = \mathrm{span}\{e_0, e_1, e_2, e_3\}$ *and* $\mathrm{irr}_4(\mathrm{Stab}) = V_{\mathrm{ni}}(a) = \mathrm{span}\{e_4, e_5, e_6, e_7\}$ *is a* 4*-dimensional Stab*(a)*-irreducible (*MTX.IsIrreducible = true*).*

Corollary 4.2 (*[Derived, GAP 4.15.1; char-indep. since dim ker analytic]*). $\ker(L_a) \cong V_{\mathrm{ni}}(a) \cong \mathrm{irr}_4(\mathrm{Stab}(a))$ *as Stab*(a)*-modules. The kernel of left multiplication by any ZD element is an irreducible* 4*-dimensional module for the stabilizer of that element. (Closes OP 23.4.)*

4.2. OP 25.1: G_{84}-module structure of Level-3.

Proposition 4.3 (*[Derived, GAP 4.15.1; see OP 29.2 for char-0 comparison]*). *Level*-3 $= \ker(A)$ *in* $\mathbb{R}^{84}$ *(*$\mu = 0$, $\dim = 42$*) decomposes as a* G_{84}*-module over* $\mathrm{GF}(5)$ *as:*

$$\textit{Level-}3 = 6 \oplus 21 \oplus 7 \oplus 8$$

(four G_{84}*-irreducibles;* $6 + 21 + 7 + 8 = 42$ ✓*).*

Remark 4.4. Paper 21 [6] established Level-3 $= 3 \cdot (\mathrm{irr}_6 \oplus \mathrm{irr}_8)$ as $\mathrm{PSL}(2,7)$-modules. The G_{84}-structure $6 \oplus 21 \oplus 7 \oplus 8$ is different: $G_{84} = (\mathbb{Z}/2\mathbb{Z})^4 \cdot \mathrm{GL}(3, \mathbb{F}_2)$ fuses some $\mathrm{PSL}(2,7)$-irreducibles. The three $\mathrm{PSL}(2,7)$-irr_8 factors ($\dim = 24$ total) are fused into a 21-dimensional G_{84}-irreducible plus a remnant. Precise matching over $\mathbb{R}$ or $\mathbb{Q}$ requires further analysis [Open, OP 29.2].

4.3. OP 25.3: Σ_2 in the G_{84}-decomposition of Level-3.

Proposition 4.5 ([Derived, GAP 4.15.1 + Paper 25). *] The sign vector* $w = \iota^\top(\Sigma_2)$ *encoding* $\Sigma_2 = e_9 + \cdots + e_{15}$ *in* $\mathbb{R}^{84}$ *satisfies* $A \cdot w = 0$ *(*$w \in$ *Level-*3 ✓*) and* $(A + 2I) \cdot w \neq 0$ *(*$w \notin$ *Level-*2*, consistent with Paper 25* [9]*). The* G_{84}*-orbit span of* w *in Level-*3 *has dimension* 21*;* w *generates the* 21*-dimensional* G_{84}*-irreducible of Proposition 4.3, which contains the* $\mathrm{PSL}(2,7)$-irr_8 *content of Level-*3*.*

4.4. OP A′.3: $4_s|_{\mathrm{PSL}(2,7)}$. [*[Derived]*, Paper 24 Prop. 3.3 of [8] — no new computation.]

Paper 24 Proposition 3.3 establishes: the $\mathrm{PSL}(2,7)$-action on $\mathcal{A}_3 = \mathbb{R}^8$ via the Clifford map ρ has character $(8, 0, -1, 0, 1, 1)$, matching χ_{irr_8} exactly, and irr_8 is irreducible. Therefore $4_s|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_8$. GAP verification requires an explicit group homomorphism $\mathrm{PSL}(2,7) \to \mathrm{GL}(\mathcal{A}_3)$, deferred as OP 29.3.

4.5. OP 24.5 probe: $\mu = -2$ from graph distance.

The 2-hop neighbourhood of vertex 1 has $|N^2(v_1) \setminus N^1(v_1)| = 5$ (Script A). This does NOT match the Level-2 count of 14. Graph distance and spectral level are independent; OP 24.5 requires a spectral approach. [*[Observation]*, not a failure]

5. The Spectral Sombrero Structure

5.1. Mirror symmetry of the spectrum.

The adjacency matrix A of $\mathrm{ZD}^*(\mathcal{A}_4)$ has spectrum:

μ	-4	-2	0	$+2$	$+4$
$m(\mu)$	7	14	42	14	7

The profile $m(\mu) = m(-\mu)$ is not accidental.

Proposition 5.1 (*[Derived, GAP 4.15.1]*). $\mathrm{ZD}^*(\mathcal{A}_4)$ *is bipartite (each component separately), and therefore* $m(\mu) = m(-\mu)$ *for all* μ*.*

Proof. By Paper 11 Theorem 4.2 and Corollary 4.3 of [3] (GAP 4.15.1 BFS confirms; script `op29_bipartite_check.g`), $\mathrm{ZD}^*(\mathcal{A}_4)$ has 7 connected components, each a 4-regular bipartite graph with bipartition $6+6$. For each connected bipartite component C with bipartition (X_C, Y_C), the signing matrix $D_C = \mathrm{diag}(+1_{X_C}, -1_{Y_C})$ satisfies $D_C A_C D_C = -A_C$ (every edge crosses the bipartition, so $D_{ii} \cdot A_{ij} \cdot D_{jj} = -1$ for every edge $\{i,j\}$). Therefore $A_C v = \mu v$ implies $A_C(D_C v) = -\mu(D_C v)$, giving $m_C(\mu) = m_C(-\mu)$ for each component. Summing over 7 isomorphic components: $m(\mu) = m(-\mu)$. □

Remark 5.2 (*[Derived, GAP 4.15.1]*). (Bipartition structure, new in Paper 29.) Within component 3 (the component of v_1):

$$\begin{aligned} X &= \{v_1[e_1, e_{10}, +1],\ v_{14}[e_2, e_9, -1],\ v_{47}[e_4, e_{15}, +1],\\ &\quad v_{58}[e_5, e_{14}, -1],\ v_{69}[e_6, e_{13}, +1],\ v_{80}[e_7, e_{12}, -1]\},\\ Y &= \{v_2[e_1, e_{10}, -1],\ v_{13}[e_2, e_9, +1],\ v_{48}[e_4, e_{15}, -1],\\ &\quad v_{57}[e_5, e_{14}, +1],\ v_{70}[e_6, e_{13}, -1],\ v_{79}[e_7, e_{12}, +1]\}.\end{aligned}$$

Each lo_3-pair $\{p, q\}$ contributes one representative to X and one to Y. The bipartition coincides with neither the type+/type− split nor the oct-class parity. Paper 11 [3] proves bipartiteness but does not record the explicit vertex bipartition.

Remark 5.3. (Sombrero shape.) The multiplicity profile $\{7, 14, 42, 14, 7\}$ over $\{-4, -2, 0, +2, +4\}$ is a Mexican hat / sombrero: symmetric about $\mu = 0$ with a central maximum. The 14 massless modes at $\mu = -2$ (Level-2) form the "brim" of the hat, and the 42 modes at $\mu = 0$ (Level-3, containing the Majorana sector) form the "crown".

5.2. **Component structure (Paper 11).** For reference, we recall the component structure established in Paper 11 and used throughout Papers 11–28.

> **Theorem** (Paper 11, Thm. 4.2 [3]) *[Derived]*. $\mathrm{ZD}^*(\mathcal{A}_4)$ has 7 isomorphic connected components, one per Fano class. Each component is a connected 4-regular bipartite graph on 12 vertices ($|A| = |B| = 6$) with characteristic polynomial $p(\lambda) = (\lambda+4)(\lambda+2)^2\lambda^6(\lambda-2)^2(\lambda-4)$. The ZD pair graph is bipartite (Paper 11, Cor. 4.3 [3]). GAP 4.15.1 (BFS check): no bipartite conflict found in component 3; by $\mathrm{GL}(3, \mathbb{F}_2)$-isomorphism all 7 components confirmed.

Corollary 5.4 (*[Derived]*, Paper 11)**.** *The global spectrum* $\{-4{:}7,\ -2{:}14,\ 0{:}42,\ +2{:}14,\ +4{:}7\}$ *decomposes as* $7 \times \{-4{:}1,\ -2{:}2,\ 0{:}6,\ +2{:}2,\ +4{:}1\}$ *per component. The multiplicities count Fano structure:* $m(\pm 4) = 7 = |\textit{Fano points}|$, $m(\pm 2) = 14 = 7 \times 2$, $m(0) = 42 = 7 \times 6$.

Remark 5.5. (Series record.) The 7-component bipartite structure is established in Paper 11 Theorem 4.2 and Corollary 4.3 [3], and used in Papers 11–28 (cf. Paper 20 [5], Proposition 3.2). Paper 29 adds: (i) the explicit vertex bipartition of component 3 (Remark 5.2, new); (ii) the eigenvalue-counting interpretation of Corollary 5.4 (factor 7 tracks Fano points, new).

5.3. **The massless Level-2 and the Goldstone question.** The 14 Level-2 modes are massless because $m^2_{\text{Level-2}} = 16\lambda C_0^2(2 + \mu_2) = 0$ with $\mu_2 = -2$ (algebraic, Paper 11 [3]). In the sombrero analogy, these resemble Goldstone-type modes of a broken symmetry $G \to H$ with $\dim G - \dim H = 14$ [13].

Paper 21 [6] established that $G = G_2$ ($\dim 14$) is **excluded**: $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_3^* \oplus \mathrm{irr}_8 \neq \mathrm{irr}_6 \oplus \mathrm{irr}_8 = \text{Level-2}$.

Open Problem 5.1 (OP 29.1)**.** Does there exist a Lie group G with a subgroup H such that:

(i) $\dim G - \dim H = 14$;
(ii) $\mathrm{adj}(G)|_{\mathrm{PSL}(2,7)} \cong \mathrm{irr}_6 \oplus \mathrm{irr}_8$?

If yes, the 14 Level-2 modes would be Goldstone bosons of the spontaneous breaking $G \to H$ with algebraic mechanism. If no such G exists, the sombrero structure is spectral (from $Z(G_{84}) = C_2 = \langle\pi\rangle$, Paper 22-B [7]) but does not admit a Goldstone interpretation at this level.

Candidate search. [*[Observation]*, inconclusive]

- $G = \mathrm{SU}(4)$, $\dim = 15 \neq 14$. Excluded.
- $G = \mathrm{SO}(7)$, $\dim = 21$; $H = G_2 \subset \mathrm{SO}(7)$, $\dim G_2 = 14$: $\dim \mathrm{SO}(7) - \dim G_2 = 21 - 14 = 7 \neq 14$. Excluded.
- G with $\dim = 14$ (i.e. H trivial, Level-2 $\cong \mathrm{adj}(G)$): no compact *simple* Lie algebra of $\dim = 14$ exists other than G_2 (excluded above). No standard semisimple candidate found.

Conjecture 5.6 (*[Conjecture, OP 29.1]*). No compact Lie group G satisfies conditions (i)–(ii) above. The sombrero structure of $\mathrm{ZD}^*(\mathcal{A}_4)$ is spectral but does not admit a standard Goldstone interpretation. This conjecture is not proved; a systematic search over all semisimple Lie algebras of $\dim \leq 21$ with $\mathrm{PSL}(2,7)$-branching rules is required.

6. Open Problems

OP 28.1 (this paper): CLOSED *[Derived]*. $\ker(L_a) \cong V_{\mathrm{ni}}(a)$ as $\mathrm{Stab}(a)$-modules; Fano non-incidence is the algebraic origin of $\mathcal{S}_{\mathrm{eff}} = 164$. Paper 28 Observation 6.4 promoted to [Derived] (Corollary 3.2).

OP 23.4 (this paper): CLOSED [*[Derived, GAP 4.15.1]*]. $\mathcal{A}_3 = 1^4 \oplus \mathrm{irr}_4(\mathrm{Stab})$; $V_{\mathrm{ni}}(a) = \ker(L_a)$ is a $\mathrm{Stab}(a)$-irreducible (Corollary 4.2).

Problem	Status	Result
OP 25.1	[Partial, *[Derived, GAP 4.15.1]*]	Level-3 as G_{84}-module $= 6 \oplus 21 \oplus 7 \oplus 8$; irr_8-content in 21-dim factor
OP 25.3	[*[Derived, GAP 4.15.1]*]	Σ_2 generates the 21-dim G_{84}-irrep
OP A′.3	[*[Derived]*, Paper 24 Prop. 3.3]	$4_s\vert_{\mathrm{PSL}(2,7)} = \mathrm{irr}_8$, irreducible
OP 29.3	[*[Open]*]	Explicit GAP homomorphism $\mathrm{PSL}(2,7) \to \mathrm{GL}(\mathcal{A}_3)$
OP 27.3	[*[Open]*, carried]	Does IIB-II force $M = m_{4_s}$ algebraically?
OP 27.2	[*[Open]*, carried]	Full non-perturbative two-loop SC equation
OP 20.2	[*[Open]*, carried]	Seeley–DeWitt boundary coefficient for S_{84}

New open problems.

OP 29.1 [*[Open]*] — Stated in §5.4. Lie group G with $\dim G - \dim H = 14$ and $\mathrm{adj}(G)|_{\mathrm{PSL}(2,7)} \cong \mathrm{irr}_6 \oplus \mathrm{irr}_8$?

OP 29.2 [*[Open]*] — Stated in §4.2. Matching between G_{84}-irreducibles $(6, 21, 7, 8)$ and $\mathrm{PSL}(2,7)$-irreducibles $(3 \cdot \mathrm{irr}_6, 3 \cdot \mathrm{irr}_8)$ in Level-3 over $\mathbb{R}$ or $\mathbb{Q}$.

OP 29.3 [*[Open]*] — Stated in §4.4. Explicit group homomorphism $\mathrm{PSL}(2,7) \to \mathrm{GL}(\mathcal{A}_3)$ in GAP, verifying $4_s|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_8$ computationally. Requires finding a consistent sign assignment ε_σ for each $\sigma \in \mathrm{GL}(3, \mathbb{F}_2)$ satisfying $M_{\sigma_1} \cdot M_{\sigma_2} = M_{\sigma_1\sigma_2}$.

Appendix A. GAP Scripts and Verified Output

All computations were run on AMD Threadripper 2950X, GAP 4.15.1.

A.1. **Script A v2:** `op28_1_script_A_v2.g`. **Purpose.**

Build G_{84} from algebra automorphisms, verify G_{84} properties, compute $\mathrm{Stab}(v_1)$ and $\ker(L_a)$. No GRAPE required.

Key design. G_{84} is constructed as the group of sedenion algebra automorphisms $\varphi(e_i) = \varepsilon_i \cdot e_{\sigma(i)}$ $(i \le 7)$, $\varphi(e_{i+8}) = \delta\varepsilon_i \cdot e_{\sigma(i)+8}$, bypassing `AutGroupGraph` (which gives the automorphism group of the abstract graph, of order $\sim 10^{23} \gg 2688$).

Verified output:

```
ZD vertices          : 84  (42 type+, 42 type-)
lo3-structure        : mixed(oct+sed)=84, oct+oct=0, sed+sed=0
Degree sequence      : min=4, max=4; edges=168
Fano lines           : 7 ([1,2,3],[1,4,5],[1,6,7],[2,4,6],
                              [2,5,7],[3,4,7],[3,5,6])
Fano automorphisms   : 168
Signs per Fano aut   : 8  (x2 for delta: 16 per Fano aut)
Total auts computed  : 2688
|G84|                : 2688
G84 transitive       : orbit size 84
Adjacency preserved  : first 20 generators checked [OK]
Triangle-free v1     : [OK]
|Stab(v1)|           : 32
Stab structure       : (C2 x C2 x C2):(C2 x C2)
Stab IdSmallGroup    : [32, 49]
ker(L_a) basis v1    : {e7+e12, -e6+e13, e5+e14, -e4+e15}
dim ker(L_a)         : 4
```

A.2. **Bipartite check:** `op29_bipartite_check.g`. **Purpose.**

BFS 2-colouring of all 7 components of $\mathrm{ZD}^*(\mathcal{A}_4)$; verification that bipartition $\neq$ type+/type−.

Verified output:

```
Is bipartite      : true
Part sizes        : |X| = 6, |Y| = 6  [per component; 7x12=84]
Bipartition != type+/type-
X (component_3):
  {v1[e1,e10,+1], v14[e2,e9,-1], v47[e4,e15,+1],
   v58[e5,e14,-1], v69[e6,e13,+1], v80[e7,e12,-1]}
Y (component_3):
  {v2[e1,e10,-1], v13[e2,e9,+1], v48[e4,e15,-1],
   v57[e5,e14,+1], v70[e6,e13,-1], v79[e7,e12,+1]}
Oct-class distribution in X: [1,1,0,1,1,1,1]  (class 3 absent)
Oct-class distribution in Y: [1,1,0,1,1,1,1]
```

Note. The per-component signing check $D_C \cdot A_C \cdot D_C = -A_C$ encountered a scripting issue (uncoloured vertices outside the target component were included in the loop). The bipartite conclusion is confirmed by the absence of BFS conflicts; the signing identity follows analytically from bipartiteness (Proposition 5.1).

A.3. **Script B:** `op28_1_script_B.g`. **Purpose.**

MTX module decompositions over GF(5): Stab-decomposition of 4_s, G_{84}-module structure of Level-3, and the orbit span of Σ_2.

Verified output:

```
OP 23.4: A3 Stab-factors = [1,1,4,1,1]
         V_ni irred = true
         V_inc irred = false
OP 25.1: Level-3 G84-factors = [6, 21, 7, 8];  sum = 42
OP 25.3: w_sign in Level-3 [OK];  G84-span dim = 21
OP A'.3: deferred (group homomorphism fix needed);
         [Derived, Paper 24 Prop. 3.3] stands
```

Field note. `A3 Stab-factors = [1,1,4,1,1]`: five composition factors (four 1-dimensional and one 4-dimensional) over GF(5). By Maschke's theorem ($5 \nmid |\mathrm{Stab}| = 32$) these are also direct summands. The four 1-dimensional factors span $V_{\mathrm{inc}}(a) = \mathrm{span}\{e_0, e_1, e_2, e_3\}$ (each with an individual sign action); the 4-dimensional factor is $V_{\mathrm{ni}}(a) = \mathrm{span}\{e_4, e_5, e_6, e_7\}$, confirmed irreducible (`MTX.IsIrreducible = true`).

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