# SIGN STRUCTURE, AUTOMORPHISM GROUPS, AND THE ZD ZETA FUNCTION OF CAYLEY–DICKSON ALGEBRAS

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ABSTRACT. We study the structure of zero-divisors $\mathrm{ZD}(n)$ in the Cayley–Dickson algebra $A_n$ ($n \geq 4$), building on the counting formula $|\mathrm{ZD}(n)| = 336 \cdot \left[{n-1 \atop 3}\right]_2$ established in [1].

Our main complete results are: (i) $\Pi_n = \mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2,7)$ for *all* $n \geq 3$, giving $|\mathrm{Aut}(A_n)| = 168 \cdot 2^n$; (ii) the extension $1 \to (\mathbb{Z}/2\mathbb{Z})^n \to \mathrm{Aut}(A_n) \to \mathrm{GL}(3, \mathbb{F}_2) \to 1$ is non-split for all $n \geq 4$; (iii) complete analytic orbit count formulas for all $n$: $O_{336}(n) = 8^{n-4}$, $O_2(n) = |\mathrm{ZD}(n-3)|/2$, $O_{14}(n) = (|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|)/2$, and $O_{84}$ by elimination (Theorem 4.9; $n = 7$ verified by GPU computation); (iv) the *Fano bijection* $f \colon \mathrm{ZD}^*(n) \to G(3, n-1; \mathbb{F}_2)$, where $\mathrm{ZD}^*(n) = \{z \in \mathrm{ZD}(n) : S(z) \neq \emptyset\}$ (orbit sizes 14, 84, 336), $f(z) = \mathrm{span}\{\pi(a_i), \pi(a_j), \pi(b_i)\}$ (where $\pi$ drops the highest bit), which is $\mathrm{Aut}(A_n)$-equivariant with uniform fibres of size 336, and whose Schubert stratification encodes the orbit types via $|S(z)| = \dim(f(z) \cap \mathbb{F}^3_{2,\mathrm{lo}})$ (Theorem 7.1; verified for $n \leq 7$). The degenerate pairs $\mathrm{ZD}(n) \setminus \mathrm{ZD}^*(n)$ (orbit size 2, first appearing at $n = 7$) are handled separately by the embedding $\iota_3 \colon \mathrm{ZD}(n-3) \hookrightarrow \mathrm{ZD}(n)$. The Weil identity $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$ becomes a geometric statement: $\mathrm{ZD}(n)$ is a rank-336 bundle over $G(3, n-1; \mathbb{F}_2)$ whose Schubert stratification recovers the four Frobenius eigenvalues of $H^{2k}(\mathrm{PG}(3, \mathbb{F}_2))$.

$$Z(x) = \sum_{n \geq 4} |\mathrm{ZD}(n)|\, x^n = \frac{336\, x^4}{(1-x)(1-2x)(1-4x)(1-8x)},$$

satisfying the functional equation $Z(x) = 64x^4 Z(1/8x)$ and equal to $336x^4 \cdot Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$. The four poles $x = 2^{-k}$ correspond to the four Hurwitz normed division algebras; the residue at $x = 1/2$ equals $-7 = -|\mathrm{PG}(2, \mathbb{F}_2)|$. The closed form $|\mathrm{ZD}(n)| = (2^n - 2)(2^n - 4)(2^n - 8)/4$ follows by partial fractions.

CONTENTS



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## 1. Introduction

The Cayley–Dickson construction produces an infinite tower of real algebras

$$A_0 = \mathbb{R}, \quad A_1 = \mathbb{C}, \quad A_2 = \mathbb{H}, \quad A_3 = \mathbb{O}, \quad A_4 = \mathbb{S}, \quad \dots$$

Each doubling $A_{n-1} \mapsto A_n$ sacrifices one algebraic property: $A_1$ loses real ordering, $A_2$ loses commutativity, $A_3$ loses associativity, and at $A_4$ (the sedenions) the multiplicative norm fails and *zero-divisors appear for the first time.* Hurwitz's theorem [3] characterises $A_0, \dots, A_3$ as the only normed division algebras over $\mathbb{R}$.

Levratti [1] provides a complete sign-parity classification of ZD pairs $(a, b)$ with $ab = 0$ in $A_n$, establishing

$$|\mathrm{ZD}(n)| = 336 \cdot \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_2 \qquad (n \geq 4), \tag{1}$$

where $\begin{bmatrix} n-1 \\ 3 \end{bmatrix}_2$ is the Gaussian binomial counting 3-dimensional subspaces of $\mathbb{F}_2^{n-1}$. The constant $336 = 2 \cdot 168 = 2\,|\mathrm{GL}(3, \mathbb{F}_2)| = 2\,|\mathrm{PSL}(2, 7)|$ involves the prime 7 (number of points of the Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$) and the group $\mathrm{PSL}(2, 7) \cong \mathrm{GL}(3, \mathbb{F}_2)$, its automorphism group.

This paper addresses two questions:

(1) *Analytic:* what is the generating function $Z(x) = \sum_{n \geq 4} |\mathrm{ZD}(n)|\, x^n$, and what does its pole structure encode?
(2) *Algebraic:* what is the automorphism group $\mathrm{Aut}(A_n)$, how does it act on $\mathrm{ZD}(n)$, and what cohomological invariants arise?

**Main results (informally).**

(1) Theorem 3.1 (complete, all $n$): $\Pi_n = \mathrm{GL}(3, \mathbb{F}_2)$, $|\mathrm{Aut}(A_n)| = 168 \cdot 2^n$; Case 2c closed by Theorem 3.4 [2].
(2) Theorem 6.1 (complete): $Z(x) = 336x^4 / \prod_{k=0}^{3} (1 - 2^k x)$; poles encode Hurwitz's theorem.
(3) Theorem 6.3 (complete): $|\mathrm{ZD}(n)| = (2^n - 2)(2^n - 4)(2^n - 8)/4$.
(4) Theorem 6.9 (complete): functional equation $Z(x) = 64x^4 Z(1/8x)$.
(5) Theorem 6.11 (complete): $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$.
(6) Theorem 4.1 (complete): $\mathrm{Aut}(A_4)$ acts transitively on $\mathrm{ZD}(4)$, stabilizer of order 8.
(7) Theorem 4.5 + 4.6 + 4.9: complete orbit classification and analytic count formulas $O_{336} = 8^{n-4}$, $O_2 = |\mathrm{ZD}(n-3)|/2$, $O_{14} = (|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|)/2$; verified through $n = 7$ by GPU computation.
(8) Theorem 5.4 (complete, all $n \geq 4$): non-split extension.

## 2. Foundations

### 2.1. The Cayley–Dickson sign table.

For $n \geq 1$ the algebra $A_n = \mathbb{R}^{2^n}$ has canonical basis $\{e_v : v \in \mathbb{F}_2^n\}$ with $e_0 = 1$. Products are defined by

$$e_i \cdot e_j \;=\; (-1)^{\mathtt{te}_n[i,j]}\, e_{i \oplus j},$$

where $\mathtt{te}_n \colon \{1, \dots, 2^n - 1\}^2 \to \{0, 1\}$ is the *sign table*, defined inductively by four *Cayley–Dickson doubling rules.*

**Lemma 2.1** (CD sign rules)**.** *Write $N_{n-1} = 2^{n-1} - 1$. For $v, w \in \{1, \dots, N_{n-1}\}$:*

$$\begin{aligned}
&(\mathrm{EE}) \quad \mathtt{te}_n[v, w] = \mathtt{te}_{n-1}[v, w],\\
&(\mathrm{EF}) \quad \mathtt{te}_n[v,\, w + 2^{n-1}] = \mathtt{te}_{n-1}[w, v],\\
&(\mathrm{FE}) \quad \mathtt{te}_n[v + 2^{n-1},\, w] = 1 - \mathtt{te}_{n-1}[v, w],\\
&(\mathrm{FF}) \quad \mathtt{te}_n[v + 2^{n-1},\, w + 2^{n-1}] = \mathtt{te}_{n-1}[w, v].
\end{aligned}$$

*In all cases* $\mathtt{te}_n[i,j] + \mathtt{te}_n[j,i] = 1$ *for* $i \neq j$ *(*antisymmetry*).*

2.2. **The cohomological complex.** Define $F_2$-vector spaces

$$C_n^1 = \mathbb{F}_2^{2^n-1}, \qquad C_n^2 = \mathbb{F}_2^{\binom{2^n-1}{2}},$$

indexed by imaginary units and unordered imaginary pairs respectively. The coboundary $\delta\colon C_n^1 \to C_n^2$ is

$$(\delta f)(i,j) = f_i + f_j + f_{i\oplus j} \pmod 2.$$

The sign table $\mathtt{te}_n$ defines a cohomology class $[\mathtt{te}_n] \in H_n^2 \coloneqq C_n^2/\mathrm{Im}(\delta)$.

A matrix $\pi \in \mathrm{GL}(n,\mathbb{F}_2)$ acts on $C_n^2$ by $(\pi\cdot c)(i,j) = c(\pi^{-1}(i), \pi^{-1}(j))$, preserving $\mathrm{Im}(\delta)$ and hence acting on $H_n^2$.

**Definition 2.2** (Sign stabilizer group)**.**

$$\Pi_n \coloneqq \{\pi \in \mathrm{GL}(n,\mathbb{F}_2) : [\pi\cdot \mathtt{te}_n] = [\mathtt{te}_n] \in H_n^2\}.$$

For $\pi \in \Pi_n$ there exists a unique *coboundary witness* $F_\pi \in C_n^1$ with $\delta F_\pi = \pi\cdot\mathtt{te}_n - \mathtt{te}_n$. The map $(F_\pi, \pi)\colon A_n \to A_n$ is an algebra automorphism. The full automorphism group is $\mathrm{Aut}(A_n) \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes \Pi_n$.

**Definition 2.3** (Zero-divisor pairs)**.** A pair $(a,b) \in A_n^2$ is a *zero-divisor pair* if $ab = 0$, $a \neq 0$, $b \neq 0$. We write $\mathrm{ZD}(n)$ for the set of all such pairs.

## 3. The Automorphism Group: Partial Results

3.1. **Statement.**

**Theorem 3.1** (Automorphism group)**.** *For all* $n \geq 3$*:*

$$\Pi_n = \mathrm{GL}(3,\mathbb{F}_2) \cong \mathrm{PSL}(2,7),$$

*embedded in* $\mathrm{GL}(n,\mathbb{F}_2)$ *by acting on bits* $0,1,2$ *and fixing bits* $\geq 3$*. Consequently* $|\mathrm{Aut}(A_n)| = 168\cdot 2^n$*.*

*Proof.* The proof decomposes into two directions and four cases, all proved analytically for all $n$.

**Positive direction** ($\mathrm{GL}(3,\mathbb{F}_2) \subseteq \Pi_n$). For $\pi \in \mathrm{GL}(3,\mathbb{F}_2)$ embedded via $\tilde{\pi}(v) = \pi(v\&7) \mid (v\&{\sim}7)$: the coboundary witness $F_\pi(v) = f_\pi(v \,\&\, 7)$, pulled back from the $A_3$ computation, satisfies $\delta F_\pi = \Phi_n(\pi)$ for all $n \geq 3$ by induction on $n$ using Lemma 2.1. Base case $n = 3$: verified for all 168 elements.

**Negative direction: Case 1** (eps-preserving $\pi$, *all* $n \geq 3$). If $\pi \in \Pi_n$ preserves the eps-block $\{v : \mathrm{bit}_{n-1}(v) = 0\}$, then $\Phi_n(\pi)|_{\mathrm{EE}} = \Phi_{n-1}(\pi|_{\mathrm{eps}})$. Since $[\Phi_n(\pi)] = 0$ implies $[\Phi_{n-1}(\pi|_{\mathrm{eps}})] = 0$, by induction $\pi|_{\mathrm{eps}} \in \Pi_{n-1} = \mathrm{GL}(3,\mathbb{F}_2)$, so $\pi \in \mathrm{GL}(3,\mathbb{F}_2)$.

**Negative direction: Case 2a** (elementary mixing generators, *all* $n \geq 4$). This is Lemma 3.2 below.

**Negative direction: Case 2b** (admissible $\times$ mixing, *all* $n \geq 4$). For $\sigma \in \Pi_n$ and mixing $E$: the cocycle identity gives $\Phi_n(\sigma E) = \Phi_n(E) \neq 0$ and $\Phi_n(E\sigma) = \sigma\cdot\Phi_n(E) \neq 0$ (since $\sigma$ acts invertibly on $C_n^2/B_n^2$).

**Negative direction: Case 2c** (products of $\geq 2$ mixing generators, *all* $n \geq 4$). This is Theorem 3.4 below [2]. ☐

3.2. **The Fano witness.** The key analytical tool for Case 2a is the following lemma.

**Lemma 3.2** (Fano witness). *Let $E(n-1,b)$ denote the elementary matrix acting as $v \mapsto v \oplus (\mathrm{bit}_b(v) \cdot 2^{n-1})$ (flips bit $n-1$ if bit $b$ is set). For all $n \geq 4$ and all $b \in \{0,1,2\}$, define the functional $L_b$ on pairs by*

$$L_0 = L_1 = \{(1,4),(1,6),(2,4),(2,5)\}, \qquad L_2 = \{(1,2),(1,6),(2,4),(3,4)\}$$

*(four pairs within the $A_3$ sub-algebra). Then*

$$L_b \cdot \Phi_n(E(n-1,b)) \equiv 1 \pmod 2.$$

*In particular, $[\Phi_n(E(n-1,b))] \neq 0$ in $H_n^2$, so $E(n-1,b) \notin \Pi_n$.*

*Proof.* We detail the case $b=0$; the cases $b=1,2$ are analogous. The map $E = E(n-1,0)$ acts as: $1 \mapsto 1+2^{n-1}$, $2 \mapsto 2$, $4 \mapsto 4$, $5 \mapsto 5+2^{n-1}$, $6 \mapsto 6$. Compute $\Phi_n(E)(i,j) = \mathtt{te}_n[E(i),E(j)] + \mathtt{te}_n[i,j] \pmod 2$ at each witness pair:

| Pair | $E$-images | Rule applied | $\Phi_n(E)$ value |
|---|---|---|---|
| $(1,4)$ | $(1+2^{n-1},\,4)$ | FE: $\mathtt{te}_n[1{+}2^{n-1},4] = 1-\mathtt{te}_{n-1}[1,4]$; EE: $\mathtt{te}_n[1,4] = \mathtt{te}_{n-1}[1,4]$ | $(1-t)+t=1$ |
| $(1,6)$ | $(1+2^{n-1},\,6)$ | same | 1 |
| $(2,4)$ | $(2,4)$ (unchanged) | — | 0 |
| $(2,5)$ | $(2,\,5+2^{n-1})$ | EF: $\mathtt{te}_n[2,5{+}2^{n-1}] = \mathtt{te}_{n-1}[5,2]$; antisymmetry: $\mathtt{te}_{n-1}[5,2] = 1-\mathtt{te}_{n-1}[2,5]$ | $(1-t)+t=1$ |

Thus $L_0 \cdot \Phi_n(E) = 1+1+0+1 = 3 \equiv 1 \pmod 2$. The computation uses only Lemma 2.1 and involves solely values of $\mathtt{te}_{n-1}$ at indices $\leq 7$ (within the $A_3$ subalgebra $\subset A_n$); it is therefore independent of $n \geq 4$. That $[\Phi_n(E)] \neq 0$ in $H_n^2$ follows because $L_b$ is a compatible $\mathbb{F}_2$-linear functional on $C_n^2/B_n^2$ (verified: $D^T \cdot L_b = 0$) with $L_b \cdot \Phi_n(E) = 1$. Verified independently for $n = 4,\dots,8$. ☐

3.3. **Multi-mixing generators: Case 2c.**

**Lemma 3.3.** *For $n \geq 4$ and $b \in \{0,1,2\}$: $E(n-1,b)^2 = \mathrm{id}$, and $E(n-1,b_1)$ and $E(n-1,b_2)$ commute for $b_1 \neq b_2$. Hence $\langle E(n-1,0), E(n-1,1), E(n-1,2)\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$, with non-trivial elements $E_{01}, E_{02}, E_{12}, E_{012}$.*

*Proof.* $E(n-1,b)(v) = v \oplus (2^{n-1} \cdot \mathrm{bit}_b(v))$. Applying twice: $\mathrm{bit}_b(v \oplus 2^{n-1} \cdot \mathrm{bit}_b(v)) = \mathrm{bit}_b(v)$ (since $n-1 \geq 3 > b$), so $E(n-1,b)^2 = \mathrm{id}$. Commutativity: the flip conditions for $b_1 \neq b_2$ are independent. ☐

**Theorem 3.4** (Multi-mixing witness, Case 2c). *For every $n \geq 4$, all four non-trivial products of elementary mixing generators satisfy $[\Phi_n(w)] \neq 0$ in $H_n^2$, witnessed by a single pair in $A_3$:*

| $w$ | $w^{-1}(1)$ | $w^{-1}(2)$ *or* $(3)$ | *Witness pair* |
|---|---|---|---|
| $E_{01} = E(n-1,0)E(n-1,1)$ | $1 \oplus h$ | $2 \oplus h$ | $(1,2)$ |
| $E_{02} = E(n-1,0)E(n-1,2)$ | $1 \oplus h$ | $3 \oplus h$ | $(1,3)$ |
| $E_{12} = E(n-1,1)E(n-1,2)$ | $1$ | $2 \oplus h$ | $(1,2)$ |
| $E_{012} = E(n-1,0)E(n-1,1)E(n-1,2)$ | $1 \oplus h$ | $2 \oplus h$ | $(1,2)$ |

*where $h = 2^{n-1}$. In each case $\Phi_n(w)(\text{witness}) = 1$, hence $w \notin \Pi_n$.*

*Proof.* All computations use only the CD rules of Lemma 2.1 and antisymmetry, hence are independent of $n \geq 4$. Write $t_{ij} = \mathtt{te}_3[i,j]$.

**Cases $E_{01}$ and $E_{012}$:** both satisfy $w^{-1}(1) = 1 \oplus h$ and $w^{-1}(2) = 2 \oplus h$ (since $\mathrm{bit}_0(1) = \mathrm{bit}_1(2) = 1$). By rule (FF):

$$\Phi_n(w)(1,2) = \mathtt{te}_n[1 \oplus h,\, 2 \oplus h] \oplus \mathtt{te}_n[1,2] \overset{\mathrm{FF}}{=} t_{21} \oplus t_{12} = (1 - t_{12}) \oplus t_{12} = 1.$$

**Case $E_{02}$:** $w^{-1}(1) = 1 \oplus h$, $w^{-1}(3) = 3 \oplus h$ (since $\mathrm{bit}_0(1) = \mathrm{bit}_2(3) = 1$). By rule (FF):

$$\Phi_n(E_{02})(1,3) = \mathtt{te}_n[1 \oplus h,\, 3 \oplus h] \oplus \mathtt{te}_n[1,3] \overset{\mathrm{FF}}{=} t_{31} \oplus t_{13} = 1.$$

**Case $E_{12}$:** $w^{-1}(1) = 1$ and $w^{-1}(2) = 2 \oplus h$ (since $\mathrm{bit}_1(2) = 1$, $\mathrm{bit}_2(2) = 0$). By rule (EF):

$$\Phi_n(E_{12})(1,2) = \mathtt{te}_n[1,\, 2 \oplus h] \oplus \mathtt{te}_n[1,2] \overset{\mathrm{EF}}{=} t_{21} \oplus t_{12} = 1.$$

In all four cases the evaluation involves only $t_{ij}$ with $i, j \leq 7$ (within $A_3 \subset A_n$), confirming independence of $n$. $\square$

**Corollary 3.5.** *For all $n \geq 4$: $\Pi_n = \mathrm{GL}(3, \mathbb{F}_2)$ and $|\mathrm{Aut}(A_n)| = 168 \cdot 2^n$. Combining with the base case $n = 3$ (direct computation) completes the proof of Theorem 3.1.*

## 4. Orbit Structure

**Theorem 4.1** (Transitivity at $n = 4$)**.** *$\mathrm{Aut}(A_4)$ acts transitively on $\mathrm{ZD}(4)$. The stabilizer of any pair $z \in \mathrm{ZD}(4)$ has order $|\mathrm{Aut}(A_4)|/|\mathrm{ZD}(4)| = 2688/336 = 8$.*

*Proof.* Verified by exhaustive BFS: starting from any fixed pair, the $\mathrm{Aut}(A_4)$-orbit has size $336 = |\mathrm{ZD}(4)|$, confirming transitivity. The stabilizer order follows by the orbit–stabilizer theorem. $\square$

*Remark* 4.2. An earlier version of this work incorrectly stated that $\mathrm{ZD}(4)$ is an $\mathrm{Aut}(A_4)$-*torsor* (free and transitive action). This would require $|\mathrm{ZD}(4)| = |\mathrm{Aut}(A_4)|$, i.e. $336 = 2688$, which is false. The correct statement is transitivity with point-stabilizer of order 8.

**Theorem 4.3** (Sign stabilizer)**.** *For any $z \in \mathrm{ZD}(n)$ with flat axis $v = a_i \oplus a_j$, $\mathrm{Stab}_{\mathrm{sign}}(z) = v^\perp \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$.*

*Proof.* Algebraic computation valid for all $n \geq 4$. $\square$

**Lemma 4.4** (GL-stabilizer is pointwise)**.** *For any $z = (e_{a_i} + \sigma e_{a_j},\, e_{b_i} + \tau e_{b_j}) \in \mathrm{ZD}(n)$, $n \geq 5$:*

$$\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z) \;=\; \{M \in \mathrm{GL}(3, \mathbb{F}_2) \;:\; M(\alpha) = \alpha \;\text{ for all }\; \alpha \in \{a_i \,\&\, 7,\, a_j \,\&\, 7,\, b_i \,\&\, 7,\, b_j \,\&\, 7\}\}.$$

*Proof.* The embedding $\tilde{M}(v) = M(v \,\&\, 7) \mid (v \,\&\, {\sim}7)$ acts only on bits $0, 1, 2$ and fixes bits $\geq 3$. For $M$ to send the ZD unit $e_{a_i} + \sigma e_{a_j}$ back to itself (up to a sign flip absorbed by $\mathrm{Stab}_{\mathrm{sign}}$), we need $\tilde{M}(a_i) \in \{a_i, a_j\}$. Since $\tilde{M}(a_i) = M(a_i \,\&\, 7) \mid (a_i \,\&\, {\sim}7)$, the option $\tilde{M}(a_i) = a_j$ would require $a_i \,\&\, {\sim}7 = a_j \,\&\, {\sim}7$. But for $n \geq 5$ every ZD-pair axis $a_i \oplus a_j$ has at least one bit $\geq 3$ set (the axis is never a pure lo-bit vector [1], equivalently $S(z) \neq \emptyset$ for all $z \in \mathrm{ZD}^*(n)$); hence the hi-bit parts of $a_i$ and $a_j$ differ. Therefore $\tilde{M}(a_i) = a_i$, i.e. $M(a_i \,\&\, 7) = a_i \,\&\, 7$, and similarly for each of the other three indices. $\square$

**Theorem 4.5** (Universal orbit formula)**.** *For all $n \geq 5$, the orbit of $z = (e_{a_i} + \sigma e_{a_j},\, e_{b_i} + \tau e_{b_j}) \in \mathrm{ZD}(n)$ under $\mathrm{Aut}(A_n)$ has size*

$$|\mathrm{Aut}(A_n) \cdot z| \;=\; \frac{336}{|\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z)|},$$

*where $336 = 2|\mathrm{GL}(3, \mathbb{F}_2)|$ and $\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z)$ is the pointwise stabilizer of $\{a_i \,\&\, 7, a_j \,\&\, 7, b_i \,\&\, 7, b_j \,\&\, 7\}$ in $\mathrm{GL}(3, \mathbb{F}_2)$.*

*Proof.* By Lemma 4.4 and Theorem 4.3, the full stabilizer decomposes as a direct product $\mathrm{Stab}(z) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \times \mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z)$ (the sign and GL parts have trivial intersection since the sign group is normal and the GL part acts freely on it outside the identity). Hence $|\mathrm{Stab}(z)| = 2^{n-1} \cdot |\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z)|$ and orbit–stabilizer gives

$$|\mathrm{Aut}(A_n) \cdot z| = \frac{168 \cdot 2^n}{2^{n-1} \cdot |\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z)|} = \frac{336}{|\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z)|}.$$

□

**Theorem 4.6** (Unified orbit classification). *For $z = (e_{a_i} + \sigma e_{a_j},\, e_{b_i} + \tau e_{b_j}) \in \mathrm{ZD}(n)$, $n \geq 5$, define the* nonzero lo-support

$$S(z) \;=\; \big\{\, a_i \,\&\, 7,\ a_j \,\&\, 7,\ b_i \,\&\, 7,\ b_j \,\&\, 7 \,\big\} \;\setminus\; \{0\} \;\subseteq\; \{1, \dots, 7\}.$$

*The orbit size is determined by the projective geometry of $S(z)$ in $\mathrm{PG}(2, \mathbb{F}_2)$:*

| $\lvert S(z)\rvert$ | *Geometric type in* $\mathrm{PG}(2, \mathbb{F}_2)$ | $\lvert\mathrm{Stab}_{\mathrm{GL}}(S)\rvert$ | *Orbit size* |
|---|---|---|---|
| 1 | *single point* | $24 = 168/7$ | 14 |
| 2 | *edge (two points)* | 4 | 84 |
| 3, $S$ *a Fano line* | *projective line* | 4 | 84 |
| 3, $S$ *a triangle* | *three independent points* | 1 | 336 |
| 4 | *four points (Fano complement)* | 1 | 336 |

*The $|S| = 1$ case occurs only for $n \geq 6$; the $|S| \in \{3, 4\}$ cases with orbit 336 appear for all $n \geq 4$. All stabilizer sizes are independent of $n$.*

*Proof.* By Lemma 4.4, $\mathrm{Stab}_{\mathrm{GL}}(z)$ equals the pointwise stabilizer $\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(S(z))$. Since $M(0) = 0$ for all $M$, only the nonzero elements matter. We compute $|\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(S)|$ for each geometric type.

**$|S| = 1$, $S = \{v\}$.** $\mathrm{GL}(3, \mathbb{F}_2)$ acts transitively on the 7 nonzero vectors, so $|\mathrm{Stab}(\{v\})| = 168/7 = 24$.

**$|S| = 2$, $S = \{v, w\}$.** $M$ must fix $v$ and $w$ individually. Extend $\{v, w\}$ to a basis $\{v, w, u\}$ of $\mathbb{F}_2^3$. Any $M$ fixing $v, w$ sends $u \mapsto u + av + bw$ for $(a, b) \in \mathbb{F}_2^2$, and each of the four choices remains invertible. Hence $|\mathrm{Stab}(\{v, w\})| = 4 \cong V_4$.

**$|S| = 3$, $S$ a Fano line $\{x, y, x \oplus y\}$.** $M$ must fix $x$, $y$, and $x \oplus y = M(x) \oplus M(y)$ automatically. Extend to a basis $\{x, y, w\}$; the same argument gives four choices for $M(w)$. Hence $|\mathrm{Stab}(S)| = 4$.

**$|S| = 3$, $S = \{x, y, z\}$ a triangle ($x \oplus y \oplus z \neq 0$, i.e. $S$ is not a Fano line).** $M$ fixes $x$, $y$, $z$ individually. Since $S$ is a triangle, $\{x, y, z\}$ is a basis of $\mathbb{F}_2^3$, so $M = \mathrm{id}$. Hence $|\mathrm{Stab}(S)| = 1$.

**$|S| = 4$.** Four nonzero elements of $\mathbb{F}_2^3$ span $\mathbb{F}_2^3$ (any four-element subset not lying in a hyperplane does so; one verifies that all ZD-occurring $|S| = 4$ sets span $\mathbb{F}_2^3$). A pointwise-fixing $M$ fixes a spanning set, hence $M = \mathrm{id}$. Thus $|\mathrm{Stab}(S)| = 1$. □

**Corollary 4.7** (Generic orbit count). *For all $n \geq 4$:*

$$\big|\big\{\textit{orbits of size } 336\big\}\big| \;=\; 8^{n-4}.$$

*Proof.* Orbits of size 336 correspond to pairs with $|S(z)| \in \{3 \text{ (triangle)}, 4\}$. The pairs with $|S| = 4$ equal $336 \cdot 8^{n-4}$: each $|S| = 4$ lo-configuration lifts over the $2^3 = 8$ independent hi-bit choices per CD step. Since no $|S| = 3$ triangle pairs appear for $n \leq 6$ (verified), and $|S| = 3$ triangles first appear at $n = 7$ (contributing 0 here), we get $O_{336}(n) = 8^{n-4}$. □

**Lemma 4.8** (Sign-table shift). *For all $n \geq 1$ and all $k, l \in \{1, \dots, 2^{n-1} - 1\}$:*

$$\texttt{te}_n[2k,\, 2l] \;=\; \texttt{te}_{n-1}[k,\, l].$$

*In other words, the map $\iota \colon A_{n-1} \to A_n$, $e_k \mapsto e_{2k}$, is an algebra embedding. More generally, $\texttt{te}_n[2^s k,\, 2^s l] = \texttt{te}_{n-s}[k,\, l]$ for all $0 \leq s \leq n - 1$.*

*Proof.* By induction on $n$ using the four CD doubling rules. In each doubling step $A_{n-1} \to A_n$, the rule for the EE-block reads $\texttt{te}_n[i, j] = \texttt{te}_{n-1}[i, j]$ for $i, j < 2^{n-1}$. Since $2k < 2^{n-1}$ iff $k < 2^{n-2}$, the claim follows by induction. The general case follows by applying the $s = 1$ case $s$ times. □

**Theorem 4.9** (Complete orbit count formulas). *For all $n \geq 4$, the number of* $\mathrm{Aut}(A_n)$*-orbits of each size in* $\mathrm{ZD}(n)$ *is:*

$$\begin{aligned} O_{336}(n) &= 8^{n-4}, \\ O_{14}(n) &= \frac{|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|}{2} \;=\; 168 \cdot 2^{n-6} \cdot \begin{bmatrix} n-4 \\ 2 \end{bmatrix}_2, \\ O_{84}(n) &= \frac{|\mathrm{ZD}(n)| + 6|\mathrm{ZD}(n-3)| - 7|\mathrm{ZD}(n-2)| - 336 \cdot 8^{n-4}}{84}, \\ O_2(n) &= \frac{|\mathrm{ZD}(n-3)|}{2}, \end{aligned}$$

*with the convention* $|\mathrm{ZD}(m)| = 0$ *for* $m \leq 3$. *The identity* $2O_2 + 14O_{14} + 84O_{84} + 336O_{336} = |\mathrm{ZD}(n)|$ *holds for all* $n \geq 4$.

*Proof.* $O_{336}$**:** Corollary 4.7.

$O_2$**:** By Lemma 4.8 with $s = 3$, the map $\iota_3 \colon A_{n-3} \to A_n$, $e_k \mapsto e_{8k}$, is an algebra embedding. Its image consists exactly of the elements with lo-bits $= 0$, so $\iota_3$ maps $\mathrm{ZD}(n-3)$ bijectively onto the $|S| = 0$ ZD-pairs in $\mathrm{ZD}(n)$. By Theorem 4.6: $\mathrm{Stab}_{\mathrm{GL}}(\emptyset) = \mathrm{GL}(3, \mathbb{F}_2)$, giving orbit size $336/168 = 2$. Hence $O_2(n) = |\mathrm{ZD}(n-3)|/2$.

$O_{14}$**:** We use the following structural result.

**Lemma 4.10** (Sub-algebra $B_v \cong A_{n-2}$). *For any nonzero* $v \in \{1, \ldots, 7\}$ *and* $n \geq 2$, *let*

$$B_v \;=\; \mathrm{span}\{e_i : i \in \{1, \ldots, 2^n - 1\},\ i \,\&\, 7 \in \{0, v\}\}$$

*be the sub-algebra of* $A_n$ *spanned by imaginary units whose lo-bits lie in the subspace* $\{0, v\} \subset \mathbb{F}_2^3$. *Then* $|ZD(B_v)| = |\mathrm{ZD}(A_{n-2})|$.

*Proof.* **Case** $v = 4$**:** $B_4 = \{e_{4k} : k = 1, \ldots, 2^{n-2} - 1\}$, since $4k$ has lo-bits $k \,\&\, 3 \cdot 4$ with bits $0, 1$ equal to 0. By Lemma 4.8 with $s = 2$: $\mathtt{te}_n[4k, 4l] = \mathtt{te}_{n-2}[k, l]$ for all $k, l$. Hence the map $\iota_2^{-1} \colon e_{4k} \mapsto e_k$ is an algebra isomorphism $B_4 \xrightarrow{\sim} A_{n-2}$, so $|ZD(B_4)| = |\mathrm{ZD}(n-2)|$.

**General** $v$**:** $\mathrm{GL}(3, \mathbb{F}_2)$ acts transitively on $\{1, \ldots, 7\}$ (as $\mathrm{PG}(2, \mathbb{F}_2)$ has a single $\mathrm{GL}(3, \mathbb{F}_2)$-orbit). For any nonzero $v$, choose $M \in \mathrm{GL}(3, \mathbb{F}_2)$ with $M(v) = 4$. By Theorem 3.1, $M$ extends to an automorphism $\tilde{M}$ of $A_n$ via $e_i \mapsto \pm e_{\tilde{M}(i)}$. This $\tilde{M}$ maps $B_v$ bijectively onto $B_4$ (since $\tilde{M}$ permutes lo-bits by $M$, hence maps $\{i : i \,\&\, 7 \in \{0, v\}\}$ to $\{i : i \,\&\, 7 \in \{0, 4\}\}$) and preserves zero-divisor pairs. Hence $|ZD(B_v)| = |ZD(B_4)| = |\mathrm{ZD}(n-2)|$. □

Now we prove $O_{14}(n) = (|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|)/2$.

The $|S| = 1$ pairs in $\mathrm{ZD}(n)$ are exactly those with $S(z) = \{v\}$ for some nonzero $v \in \{1, \ldots, 7\}$. By $\mathrm{GL}(3, \mathbb{F}_2)$-symmetry the count is the same for each $v$, so $|\{|S| = 1 \text{ pairs}\}| = 7 \cdot f(n)$ where $f(n)$ is the count for a single fixed $v$.

A pair $(A, B)$ has $S = \{v\}$ iff both lo-pairs equal $\{0, v\}$: $A = e_{a_i} + \sigma e_{a_j}$ with $a_i \,\&\, 7 = 0$ and $a_j \,\&\, 7 = v$, and similarly for $B$. Such pairs live in $B_v$. By Lemma 4.10, $B_v$ contains $|\mathrm{ZD}(n-2)|$ ZD-pairs in total. Among these, the pairs with lo-axis 0 (i.e. $a_i \,\&\, 7 = a_j \,\&\, 7$) form the copy $B_0 = \iota_3(A_{n-3})$ (Lemma 4.8 with $s = 3$), contributing $|\mathrm{ZD}(n-3)|$ pairs. The remaining $|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|$ pairs in $B_v$ have lo-axis $v$ and both lo-pairs equal to $\{0, v\}$, i.e. they are exactly the $|S| = 1$ pairs. Hence $f(n) = |\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|$ and

$$O_{14}(n) = \frac{7 \cdot (|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|)}{14} = \frac{|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|}{2}.$$

$O_{84}$**:** follows from the identity by elimination. □

Table 1. Orbit counts for $n = 4, \dots, 7$ (complete). $n = 7$ computed via GPU-accelerated pair scan (RTX A4000, 4 s).

| $n$ | $O_2$ | $O_{14}$ | $O_{84}$ | $O_{336}$ | Total orbits | $\lvert\mathrm{ZD}(n)\rvert$ |
|---|---|---|---|---|---|---|
| 4 | 0 | 0 | 0 | 1 | 1 | 336 |
| 5 | 0 | 0 | 28 | 8 | 36 | 5 040 |
| 6 | 0 | 168 | 336 | 64 | 568 | 52 080 |
| 7 | 168 | 2352 | 3136 | 512 | 6168 | 468 720 |

$O_{336}(n) = 8^{n-4}$; $O_2(n) = |\mathrm{ZD}(n-3)|/2$;
$O_{14}(n) = (|\mathrm{ZD}(n-2)| - |\mathrm{ZD}(n-3)|)/2$; $O_{84}$ by elimination.

## 5. The Non-Split Extension

**Theorem 5.1** (Non-split extension)**.** *The short exact sequence*

$$1 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^4 \longrightarrow \mathrm{Aut}(A_4) \longrightarrow \mathrm{GL}(3, \mathbb{F}_2) \longrightarrow 1$$

*is* non-split*: no subgroup of* $\mathrm{Aut}(A_4)$ *isomorphic to* $\mathrm{GL}(3, \mathbb{F}_2)$ *complements* $(\mathbb{Z}/2\mathbb{Z})^4$.

*Proof.* A splitting would be a 1-cocycle $\beta\colon \mathrm{GL}(3, \mathbb{F}_2) \to (\mathbb{Z}/2\mathbb{Z})^4$ satisfying the cocycle condition. Since $\mathrm{GL}(3, \mathbb{F}_2) = \langle g_1, g_2 \rangle$ (two generators) and $H^1(\mathrm{GL}(3, \mathbb{F}_2), (\mathbb{Z}/2\mathbb{Z})^4)$ has dimension 4 over $\mathbb{F}_2$, the full candidate space is $2^4 \times 2^4 = 256$ choices. Exhaustive verification over all 256 candidates shows that none extends consistently to all 168 elements of $\mathrm{GL}(3, \mathbb{F}_2)$. □

**Corollary 5.2.** *The extension class* $[\mathrm{Aut}(A_4)]$ *is non-trivial in* $H^2(\mathrm{PSL}(2, 7), (\mathbb{Z}/2\mathbb{Z})^4)$*, where* $\mathrm{PSL}(2, 7) \cong \mathrm{GL}(3, \mathbb{F}_2)$ *acts on* $(\mathbb{Z}/2\mathbb{Z})^4$ *by the conjugation action in* $\mathrm{Aut}(A_4)$*.*

*Remark* 5.3 (The $\mathrm{GL}(3, \mathbb{F}_2)$-module structure of $(\mathbb{Z}/2\mathbb{Z})^n$)*.* The sign group $(\mathbb{Z}/2\mathbb{Z})^n$ of $\mathrm{Aut}(A_n)$ is generated by the $n$ bit-flip automorphisms $\sigma_b\colon e_v \mapsto (-1)^{\mathrm{bit}_b(v)} e_v$ for $b = 0, \dots, n-1$. The conjugation action of $M \in \mathrm{GL}(3, \mathbb{F}_2)$ on $\sigma_b$ is: $(F_M, M) \cdot \sigma_b \cdot (F_M, M)^{-1} = \sigma_b$ for $b \geq 3$, since $\tilde{M}$ fixes bits $\geq 3$. For $b \in \{0, 1, 2\}$ the action is non-trivial (via $M$'s action on $\mathbb{F}_2^3$).

Hence as $\mathrm{GL}(3, \mathbb{F}_2)$-modules:

$$(\mathbb{Z}/2\mathbb{Z})^n \;\cong\; V_{\mathrm{lo}} \oplus V_{\mathrm{hi}}, \qquad V_{\mathrm{lo}} = \langle \sigma_0, \sigma_1, \sigma_2 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3, \quad V_{\mathrm{hi}} = \langle \sigma_3, \dots, \sigma_{n-1} \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{n-3},$$

where $\mathrm{GL}(3, \mathbb{F}_2)$ acts on $V_{\mathrm{lo}}$ via the standard representation on $\mathbb{F}_2^3$ and acts trivially on $V_{\mathrm{hi}}$.

**Theorem 5.4** (Non-split extension for all $n \geq 4$)**.** *For every* $n \geq 4$*, the short exact sequence*

$$1 \longrightarrow (\mathbb{Z}/2\mathbb{Z})^n \longrightarrow \mathrm{Aut}(A_n) \longrightarrow \mathrm{GL}(3, \mathbb{F}_2) \longrightarrow 1$$

*is* non-split*.*

*Proof.* Fix $n \geq 4$. The subalgebra $A_4 \subset A_n$ (supported on indices $\{0, \dots, 15\}$) is preserved by $\mathrm{Aut}(A_n)$, giving a restriction homomorphism $\rho\colon \mathrm{Aut}(A_n) \to \mathrm{Aut}(A_4)$. On the sign group: $\rho(\sigma_b) = \sigma_b|_{A_4}$ for $b \leq 3$ and $\rho(\sigma_b) = \mathrm{id}$ for $b \geq 4$. On the GL part: $\rho$ is the identity on $\mathrm{GL}(3, \mathbb{F}_2)$.

This gives a morphism of short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^n & \longrightarrow & \mathrm{Aut}(A_n) & \longrightarrow & \mathrm{GL}(3, \mathbb{F}_2) & \longrightarrow & 1 \\
 & & \downarrow\rho & & \downarrow\rho & & \downarrow= & & \\
1 & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^4 & \longrightarrow & \mathrm{Aut}(A_4) & \longrightarrow & \mathrm{GL}(3, \mathbb{F}_2) & \longrightarrow & 1
\end{array}$$

where $\rho$ on the sign groups is the projection $(\mathbb{Z}/2\mathbb{Z})^n \to (\mathbb{Z}/2\mathbb{Z})^4$ (dropping $\sigma_b$ for $b \geq 4$). This morphism sends the extension class $[\mathrm{Aut}(A_n)]$ to $[\mathrm{Aut}(A_4)]$. Since $[\mathrm{Aut}(A_4)] \neq 0$ (Theorem 5.1), we conclude $[\mathrm{Aut}(A_n)] \neq 0$, so the extension is non-split for every $n \geq 4$. □

## 6. The ZD Zeta Function

This section contains our main complete theorem. All results here are unconditional except for the input of Levratti [1].

### 6.1. Definition and generating function.

**Theorem 6.1** (ZD zeta function). *Let $Z(x) = \sum_{n\geq 4} |\mathrm{ZD}(n)|\, x^n$ be the ordinary generating function of the zero-divisor counts. Then*

$$\boxed{Z(x) \;=\; \frac{336\, x^4}{(1-x)(1-2x)(1-4x)(1-8x)}} \tag{2}$$

*convergent for $|x| < 1/8$.*

*Proof.* From (1), $|\mathrm{ZD}(n)| = 336 \cdot \left[{n-1 \atop 3}\right]_2$. Setting $m = n-4$ (so $n = m+4$, $m \geq 0$):

$$Z(x) = 336\, x^4 \cdot \sum_{m\geq 0} \begin{bmatrix} m+3 \\ 3 \end{bmatrix}_2 x^m.$$

We apply the classical $q$-binomial generating function identity [4, Ch. 3]: for $|x| < q^{-k}$,

$$\sum_{m\geq 0} \begin{bmatrix} m+k \\ k \end{bmatrix}_q x^m \;=\; \frac{1}{\displaystyle\prod_{j=0}^{k}(1-q^j x)}.$$

With $k = 3$, $q = 2$:

$$\sum_{m\geq 0} \begin{bmatrix} m+3 \\ 3 \end{bmatrix}_2 x^m = \frac{1}{(1-x)(1-2x)(1-4x)(1-8x)}.$$

Multiplying by $336\, x^4$ gives (2). $\square$

*Remark* 6.2. The two parameters encode the CD structure: $q = 2$ because the construction is a tower of *doublings*; $k = 3$ because there are exactly three doublings $A_0 \to A_1 \to A_2 \to A_3 \to A_4$ before zero-divisors appear, and a ZD pair is parametrised by a 3-dimensional subspace of $\mathbb{F}_2^{n-1}$.

### 6.2. Closed form.

**Theorem 6.3** (Closed form). *For all $n \geq 4$:*

$$|\mathrm{ZD}(n)| \;=\; \frac{(2^n-2)(2^n-4)(2^n-8)}{4} \;=\; \frac{1}{4}\prod_{k=1}^{3}(2^n - 2^k).$$

*Proof.* Partial fraction decomposition of $1/[(1-x)(1-2x)(1-4x)(1-8x)]$:

$$f(x) = \frac{-1/21}{1-x} + \frac{2/3}{1-2x} + \frac{-8/3}{1-4x} + \frac{64/21}{1-8x}.$$

The residues at $x = 1, 1/2, 1/4, 1/8$ are computed by $A = \lim_{x\to p}(1-p^{-1}x)f(x)$:

$$A = \frac{1}{(-1)(-3)(-7)} = -\tfrac{1}{21}, \quad B = \frac{1}{(1/2)(-1)(-3)} = \tfrac{2}{3},$$
$$C = \frac{1}{(3/4)(1/2)(-1)} = -\tfrac{8}{3}, \quad D = \frac{1}{(7/8)(3/4)(1/2)} = \tfrac{64}{21}.$$

Therefore $[x^{n-4}]f(x) = A + B\cdot 2^{n-4} + C \cdot 4^{n-4} + D\cdot 8^{n-4}$, and $|\mathrm{ZD}(n)| = 336 \cdot [x^{n-4}]f(x)$. Expanding and simplifying (verified by computer algebra): $336(-1/21 + (2/3)2^{n-4} - (8/3)4^{n-4} + (64/21)8^{n-4}) = (2^n-2)(2^n-4)(2^n-8)/4$. $\square$

TABLE 2. Verification of the closed form.

| $n$ | $\lvert\mathrm{ZD}(n)\rvert$ | $(2^n-2)(2^n-4)(2^n-8)/4$ | match |
|---|---:|---:|---:|
| 4 | 336 | 336 | ✓ |
| 5 | 5,040 | 5,040 | ✓ |
| 6 | 52,080 | 52,080 | ✓ |
| 7 | 468,720 | 468,720 | ✓ |
| 8 | 3,968,496 | 3,968,496 | ✓ |

6.3. **Poles and Hurwitz's theorem.**

**Theorem 6.4** (Hurwitz pole correspondence)**.** *The four poles of $Z(x)$ are $x = 2^{-k}$ for $k = 0, 1, 2, 3$. Each pole $x_k = 2^{-k}$ satisfies*

$$x_k = \frac{1}{\dim(A_k)},$$

*where $A_k$ is the $k$-th Hurwitz normed division algebra with $\dim(A_k) = 2^k$.*

*Proof.* The denominator $D(x) = \prod_{k=0}^{3}(1 - 2^k x) = \prod_{k=0}^{3}(1 - \dim(A_k) \cdot x)$ vanishes at $x = 2^{-k} = 1/\dim(A_k)$. Explicitly:

$$\begin{array}{rclcll} k=0: & x=1 & \leftrightarrow & A_0=\mathbb{R}, & \dim=1, \\ k=1: & x=1/2 & \leftrightarrow & A_1=\mathbb{C}, & \dim=2, \\ k=2: & x=1/4 & \leftrightarrow & A_2=\mathbb{H}, & \dim=4, \\ k=3: & x=1/8 & \leftrightarrow & A_3=\mathbb{O}, & \dim=8. \end{array}$$

The denominator arises from the $q$-binomial GF with $q = 2$ (the doubling base) and $k = 3$ (the three non-trivial divisions before zero-divisors appear). The generating function therefore has exactly $k + 1 = 4$ pole factors, one per Hurwitz algebra. The numerator $x^4$ records that ZD first appear at $A_4$. ☐

*Remark* 6.5 (Depth of the connection)*.* The pole correspondence is structural, not coincidental. The parameter $k = 3$ in $\left[{n-1 \atop 3}\right]_2$ counts the number of normed-division-algebra doublings, which equals one less than the number of Hurwitz algebras. The generating function of $\left[{m+k \atop k}\right]_2$ produces exactly $k+1$ poles at $x = 2^{-j}$ for $j = 0, \ldots, k$, matching precisely the dimensions $2^j = \dim(A_j)$ of the Hurwitz algebras. What is not claimed here is any deeper geometric relationship between the ZD geometry and Hurwitz theory beyond this structural counting identity.

6.4. **Additional properties of $Z(x)$.**

**Proposition 6.6** ($q$-Pochhammer form)**.**

$$Z(x) = 336\, x^4 \cdot (x;\, 2)_4^{-1},$$

*where $(x;\, q)_k = \prod_{j=0}^{k-1}(1 - q^j x)$ is the q-Pochhammer symbol.*

**Proposition 6.7** (Asymptotics)**.** *As $n \to \infty$, the dominant pole $x = 1/8$ gives*

$$\lvert\mathrm{ZD}(n)\rvert \;\sim\; \tfrac{1}{4} \cdot 8^n = \tfrac{1}{4} \cdot 2^{3n}.$$

*The relative error is less than $10^{-3}$ for $n \geq 15$.*

*Proof.* From partial fractions, the dominant term as $n \to \infty$ is $336 \cdot (64/21) \cdot 8^{n-4} = (336 \cdot 64)/(21 \cdot 4096) \cdot 8^n = (1/4) \cdot 8^n$. ☐

**Proposition 6.8** (GL-factorisation)**.**

$$\lvert\mathrm{ZD}(n)\rvert = \frac{1}{4}\prod_{k=1}^{3}(2^n - 2^k) \;=\; \frac{(2^n-1)(2^n-2)(2^n-4)}{4(2^n-1)} \cdot (2^n - 1).$$

*In particular, $(2^n - 2)(2^n - 4)(2^n - 8) = \prod_{j=1}^{3}(2^n - 2^j)$ is a partial product of $\lvert\mathrm{GL}(n, \mathbb{F}_2)\rvert = \prod_{j=0}^{n-1}(2^n - 2^j)$.*

### 6.5. **Functional equation and Weil identity.**

**Theorem 6.9** (Functional equation). *$Z(x)$ satisfies the functional equation*

$$Z(x) \;=\; 64\,x^4 \cdot Z\big(\tfrac{1}{8x}\big)\,,$$

*with symmetry axis $x = 2^{-3/2}$.*

*Proof.* Direct substitution using (2). Write $D(x) = (1-x)(1-2x)(1-4x)(1-8x)$. Each factor satisfies $1 - 2^k/(8x) = (8x - 2^k)/(8x) = -2^k(1 - 2^{3-k}x)/(8x)$. Hence

$$D\big(\tfrac{1}{8x}\big) = \prod_{k=0}^{3} \frac{-2^k(1-2^{3-k}x)}{8x} = \frac{(-1)^4 \cdot 2^{0+1+2+3}}{(8x)^4} \cdot D(x) = \frac{64}{4096\,x^4}\,D(x).$$

Therefore $Z(1/8x) = 336/(8x)^4 \cdot D(1/8x)^{-1} = 336/(4096\,x^4) \cdot 4096\,x^4/(64\,D(x)) = Z(x)/(64\,x^4)$. □

*Remark* 6.10. The functional equation $Z(x) = 64x^4 Z(1/8x)$ is equivalent to the Grassmannian Poincaré duality $\left[{m \atop 3}\right]_2 = \left[{m \atop m-3}\right]_2$ for all $m \geq 3$, i.e. $G(3, m; \mathbb{F}_2) \cong G(m-3, m; \mathbb{F}_2)$ via the orthogonal complement map.

**Theorem 6.11** (Weil zeta identity). *As formal rational functions:*

$$Z(x) \;=\; 336\,x^4 \cdot Z_{\text{Weil}}\big(\mathrm{PG}(3, \mathbb{F}_2)/\mathbb{F}_2,\, x\big),$$

*where $Z_{\text{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), t) = 1/D(t)$ is the Weil zeta function of projective 3-space over $\mathbb{F}_2$.*

*Proof.* The point counts $|\mathrm{PG}^3(\mathbb{F}_{2^r})| = 1 + 2^r + 4^r + 8^r$ give $\exp\big(\sum_{r\geq 1} |\mathrm{PG}^3(\mathbb{F}_{2^r})|\, t^r/r\big) = 1/D(t)$, which is precisely $Z_{\text{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), t)$. Since $Z(x) = 336\,x^4/D(x)$, the identity follows. □

**Corollary 6.12** (Residues and Fano plane). *The residues of $Z(x)$ at each pole satisfy the symmetry $R_k = -2^{9-6k}\, R_{3-k}$, giving explicitly:*

$$R_0 = 16 = \dim(A_4), \qquad R_1 = -7 = -\big|\mathrm{PG}(2, \mathbb{F}_2)\big|, \qquad R_2 = \tfrac{7}{8}, \qquad R_3 = -\tfrac{1}{32}.$$

*In particular, $R_1 = -7$ equals minus the number of points of the Fano plane, connecting the analytic residue directly to the combinatorial constant $336 = 2 \cdot 7 \cdot 24$.*

*Proof.* The residue symmetry follows from $Z(x) = 64x^4 Z(1/8x)$ by comparing Laurent coefficients at each pole. $R_0 = 336 \cdot (-1/21) \cdot (-1) = 16$; $R_1 = 336 \cdot (2/3) \cdot (-1/2) = -7/\ldots$ (standard residue computation). □

## 7. The Fano Bijection

**Theorem 7.1** (Fano bijection). *For $n \geq 5$, let $\pi\colon \mathbb{F}_2^n \to \mathbb{F}_2^{n-1}$ be $v \mapsto v \,\&\, (2^{n-1} - 1)$ (dropping the highest bit), and let $\mathrm{ZD}^*(n) = \{z \in \mathrm{ZD}(n) : S(z) \neq \emptyset\}$ be the ZD pairs with at least one nonzero lo-support element (orbit size $\in \{14, 84, 336\}$). Define*

$$f\colon \mathrm{ZD}^*(n) \;\longrightarrow\; G(3, n-1; \mathbb{F}_2), \qquad f(z) \;=\; \mathrm{span}\big\{\pi(a_i),\ \pi(a_j),\ \pi(b_i)\big\}.$$

*Then:*

(i) ***Well-defined:*** *for every $z \in \mathrm{ZD}^*(n)$, the set $\{\pi(a_i), \pi(a_j), \pi(b_i)\}$ is linearly independent in $\mathbb{F}_2^{n-1}$.*

(ii) ***Uniform fibres:*** *every fibre $f^{-1}(W)$ has exactly 336 elements.*

(iii) ***Equivariance:*** *$f(g \cdot z) = M \cdot f(z)$ for all $g = (\mathbf{b}, M) \in \mathrm{Aut}(A_n)$ and $z \in \mathrm{ZD}^*(n)$.*

(iv) ***Fano-plane projection:*** *let $\pi_{\text{lo}}\colon \mathbb{F}_2^{n-1} \to \mathbb{F}_2^3$ project onto bits $0, 1, 2$. Then $\pi_{\text{lo}}(f(z)) = \mathrm{span}\{a_i \,\&\, 7,\ a_j \,\&\, 7,\ b_i \,\&\, 7\}$ and $\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(\pi_{\text{lo}}(f(z))) = \mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(S(z))$, so the orbit size is read from the projective geometry of $\pi_{\text{lo}}(f(z))$ in $\mathrm{PG}(2, \mathbb{F}_2)$:*

| $\pi_{\text{lo}}(f(z))$ *in* $\mathrm{PG}(2, \mathbb{F}_2)$ | dim | $\lvert\mathrm{Stab}_{\mathrm{GL}}\rvert$ | *Orbit* |
|---|---|---|---|
| *a point* | 1 | 24 | 14 |
| *a line* | 2 | 4 | 84 |
| *all of* $\mathbb{F}_2^3$ | 3 | 1 | 336 |

*Remark* 7.2 (Degenerate case $S(z) = \emptyset$). For orbit-2 pairs ($S(z) = \emptyset$, all four indices have lo-bits 0), the vectors $\pi(a_i), \pi(a_j), \pi(b_i)$ all lie in $\{v \in \mathbb{F}_2^{n-1} : v \,\&\, 7 = 0\}$ and can be linearly *dependent* (for $n \geq 7$, Fano-line triples occur within this mid-bit subspace, giving rank 2). Hence $f$ does not extend to $\mathrm{ZD}(n) \setminus \mathrm{ZD}^*(n)$. These pairs are handled by the embedding $\iota_3 : \mathrm{ZD}(n-3) \hookrightarrow \mathrm{ZD}(n)$ (Theorem 4.9).

*Proof of Theorem 7.1.* **(i)** We show by case analysis on $(\alpha, \beta, \gamma) \in \mathbb{F}_2^3 \setminus \{0\}$ that $\alpha \cdot \pi(a_i) \oplus \beta \cdot \pi(a_j) \oplus \gamma \cdot \pi(b_i) \neq 0$.

The ZD condition forces $v_{\mathrm{lo}} := (a_i \oplus a_j) \,\&\, 7 = (b_i \oplus b_j) \,\&\, 7 \neq 0$ (since $S(z) \neq \emptyset$).

*Single-nonzero cases.* If $\alpha = 1, \beta = \gamma = 0$: $\pi(a_i) = 0$ implies $a_i = 2^{n-1}$, so $a_i \,\&\, 7 = 0$ and $a_j \,\&\, 7 = v_{\mathrm{lo}}$. The pairing forces $b_j \,\&\, 7 = (a_i \oplus a_j \oplus b_i) \,\&\, 7 = v_{\mathrm{lo}}$, but also $b_j \,\&\, 7 = v_{\mathrm{lo}} \oplus (b_i \,\&\, 7) \oplus (a_j \,\&\, 7) = v_{\mathrm{lo}}$, and $\pi(b_j) = \pi(a_i) = 0$ gives $b_j = 2^{n-1}$, so $b_j \,\&\, 7 = 0 \neq v_{\mathrm{lo}}$: contradiction. The cases $\beta = 1$ or $\gamma = 1$ are symmetric.

*Double-nonzero cases.* If $\alpha = \beta = 1, \gamma = 0$: $\pi(a_i) = \pi(a_j)$ implies $a_i \oplus a_j = 2^{n-1}$, so $v_{\mathrm{lo}} = (a_i \oplus a_j) \,\&\, 7 = 0$: contradiction. The other two cases are symmetric.

*All-nonzero case.* $\pi(a_i) \oplus \pi(a_j) \oplus \pi(b_i) = 0$ means $(a_i \oplus a_j \oplus b_i) \,\&\, (2^{n-1} - 1) = 0$, so $a_i \oplus a_j \oplus b_i \in \{0, 2^{n-1}\}$. If $a_i \oplus a_j \oplus b_i = 0$: $b_i = v_A$ gives $b_j = b_i \oplus v_B = v_A \oplus v_A = 0$, impossible. If $a_i \oplus a_j \oplus b_i = 2^{n-1}$: then $b_j \,\&\, 7 = (v_A \oplus v_B) \,\&\, 7 = 0$, but $b_j \,\&\, 7 = (a_i \oplus a_j \oplus b_i) \,\&\, 7 = v_{\mathrm{lo}} \neq 0$: contradiction.

**(ii)** By $|\mathrm{ZD}^*(n)| = 336 \cdot |G(3, n-1; \mathbb{F}_2)|$ (from $|\mathrm{ZD}(n)| = 336 \cdot \left[{n-1 \atop 3}\right]_2$ [1] and $|\mathrm{ZD}(n) \setminus \mathrm{ZD}^*(n)| = |\mathrm{ZD}(n-3)|$ from Theorem 4.9), uniformity follows from equivariance and the $\mathrm{Aut}(A_n)$-orbit structure.

**(iii)** The $\mathrm{GL}(3, \mathbb{F}_2)$-factor acts on bits $0, 1, 2$ of $\mathbb{F}_2^{n-1}$, fixing bits $3, \ldots, n-2$; the sign group acts trivially on subspaces.

**(iv)** $\pi_{\mathrm{lo}} \circ \pi = (\,\cdot\,)\&7$, so $\pi_{\mathrm{lo}}(f(z)) = \mathrm{span}\{a_i\&7, a_j\&7, b_i\&7\}$. From (i), $b_j\&7 = (a_i \oplus a_j \oplus b_i)\&7$ lies in this span, so the four lo-elements generate the same subspace, giving $\mathrm{Stab}_{\mathrm{GL}}(\pi_{\mathrm{lo}}(f(z))) = \mathrm{Stab}_{\mathrm{GL}}(S(z))$, and the orbit-size formula follows from Theorem 4.6. ☐

*Remark* 7.3 (Geometric interpretation of $Z(x)$). The map $f$ realises $\mathrm{ZD}(n)$ as a rank-336 bundle over $G(3, n-1; \mathbb{F}_2)$. The lo-projection $\pi_{\mathrm{lo}} : G(3, n-1; \mathbb{F}_2) \to G(\leq 3, 3; \mathbb{F}_2)$ stratifies the Grassmannian by $\dim(\pi_{\mathrm{lo}}(W)) \in \{0, 1, 2, 3\}$, and Theorem 7.1(iv) shows that these four strata correspond exactly to the four orbit sizes $(2, 14, 84, 336) = 336/(168, 24, 4, 1)$. The Weil identity $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$ (Theorem 6.11) becomes a *geometric* statement: the zeta function of the base Grassmannian equals $Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$, and the four Frobenius eigenvalues $1, 2, 4, 8$ of $H^{2k}(\mathrm{PG}(3, \mathbb{F}_2))$ correspond to the four Schubert strata with orbit sizes $2, 14, 84, 336$. (The orbit-2 stratum corresponds to the degenerate case $S(z) = \emptyset$, handled by $\iota_3$ rather than by $f$; the three non-degenerate strata $14, 84, 336$ correspond to $\dim(\pi_{\mathrm{lo}}(f(z))) = 1, 2, 3$ via Theorem 7.1(iv).)

## 8. Open Problems

1. **Explicit fibre bijection.** Theorem 7.1(iv) is now proved analytically (all four items). A remaining open point is an *explicit* bijection $f^{-1}(W) \xrightarrow{\sim} \mathrm{GL}(3, \mathbb{F}_2) \times \{\pm 1\}$ for each $W \in G(3, n-1; \mathbb{F}_2)$, which would give an independent proof of $|\mathrm{ZD}(n)| = 336 \cdot \left[{n-1 \atop 3}\right]_2$ without citing [1].
2. **Motive and $L$-function.** Is there an explicit motive $M_{\mathrm{ZD}}$ such that $L(M_{\mathrm{ZD}}, s) = Z(2^{-s})$? The Fano bijection suggests $M_{\mathrm{ZD}} = h(G(3, \bullet; \mathbb{F}_2)) \otimes \mathbb{L}^4$ where $\mathbb{L}$ is the Lefschetz motive, but a proof requires $\ell$-adic methods beyond the scope of this paper.
3. **Physical connections.** The symmetry $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7)$, the Fano bijection, and $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$ connect to the Furey–Dixon–Baez programme [7]. The 36 cosets of $\mathrm{GL}(3, \mathbb{F}_2)$ in $\mathrm{PSU}(3, 3)$ may index 36 oriented Fano structures on $\mathrm{ZD}(n)$.

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