HIGH-PRECISION QUASI-NORMAL MODE SPECTRUM OF THE BLACK CIRCLE INTERIOR

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Abstract. We resolve the outstanding problem of the Black Circle quasi-normal mode ratio $|\omega_{\rm BC}/\omega_{\rm Sch}| = 2.168 \pm 0.002$ reported in Paper 13, and establish a systematic high-precision QNM catalogue for $l = 2$, $m^2 = 0$ under Candidate (a) *[Conjecture, IIB-I/II]*. First, we show that the Kummer connection coefficient $\tilde{K}_1^{(\varrho_2,l)}$ (Paper 16, Theorem 4.1) is a four-Gamma ratio that admits no Legendre-duplication simplification for $l \geq 1$, closing OP 17.2 *[Derived-Negative]*. Second, using Paper 17's rootfinder at 50 decimal places with a corrected starting guess, we obtain the physical damped mode: $\omega^*_{\rm damp} = 1.92525087091330475008\ldots - 0.41596770795216527832\ldots i$, $|H^{(a)}_{\rm BC}(\omega^*)| = 6.68 \times 10^{-51}$ *[Derived, numerical, 50 d.p.]*, and identify a starting-guess error in Paper 17's reported value. Third, the ratio $|\omega^*_{\rm damp}|/|\omega_{\rm Sch}| = 1.9967 \pm 0.003$ is compatible with the Frobenius factor 2 predicted by Paper 10 to three significant figures (Conjecture 6.3 *[Conjecture, IIB-I/II]*). Fourth, a systematic scan of the upper half-plane (300 points, $\mathrm{Re} \in [0.5, 7.0]$, $\mathrm{Im} \in [0.1, 4.0]$, dps$=15$) followed by Newton rootfinding at 50 d.p. reveals two growing modes for $l = 2$, $m^2 = 0$:

$$\omega^*_{\rm grow,1} = 5.9730891600 + 2.8427523204\, i, \quad \tau_{\rm grow} = 0.3518\, r_s/c,$$
$$\omega^*_{\rm grow,2} = 4.7367434016 + 3.8456935744\, i, \quad \tau_{\rm grow} = 0.2600\, r_s/c,$$

closing OP 18.5 positively *[Derived, numerical, 50 d.p.]*. These timescales are incompatible with $\tau_{\rm grow}(l = 0, 1) \approx 1.36\, r_s/c$, establishing *[Derived-Negative]* that the quasi-universality of $\tau_{\rm grow}$ does not extend to $l = 2$; OP 18.6 (algebraic origin) remains *[Open]*.

Epistemic key. *[Derived]* — complete proof. *[Derived-Negative]* — proves the negative. *[Derived, numerical, n d.p.]* — numerical verification at n decimal places. *[Conjecture, IIB-I/II]* — conditional on Candidate (a) and IIB-I/II. *[Observation]* — numerically established, proof open. *[Open]* — obstacle stated precisely.

1. Introduction

1.1. **The 2.168 problem.** Paper 13 [2] computed the Black Circle quasi-normal mode (QNM) spectrum for $l = 2$, $m^2 = 0$ using the approximate condition $H^{\rm approx}_{\rm BC}(\omega) = 0$, obtaining Type II oscillatory modes with ratio

$$|\omega_{\rm BC}/\omega_{\rm Sch}(l = 2, n = 0)| = 2.168 \pm 0.002. \tag{1}$$

Paper 10 [1] had predicted a Frobenius factor of 2 from the algebraic structure of the $\mathcal{A}_3 \to \mathcal{A}_4$ transition. Papers 15–17 revealed that the computational framework of Paper 13 was incorrect for $l \geq 1$ due to a limit-circle singularity at $r = 0$ and the inhomogeneous nature of the physical wave equation. The quasi-normal mode problem for black holes has a long history [8, 9, 10, 12]; the Black Circle interior poses a novel variant due to its de Sitter geometry and algebraically constrained boundary conditions.

This paper closes the investigation with four results: (i) closure of OP 17.2, (ii) high-precision damped mode and errata for Paper 17, (iii) Conjecture 6.3 on the ratio, and (iv) systematic catalogue of growing modes for $l = 2$ closing OP 18.5.

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1.2. **Organisation.** Section 2: OP 17.2, Kummer coefficient. Section 3: damped mode at 50 d.p. Section 4: errata for Paper 17. Section 5: Conjecture 6.3. Section 6: upper-half-plane scan, growing modes, OP 18.5. Section 7: open problems.

2. Closed form of $\tilde{K}_1^{(\varrho_2,l)}$ and OP 17.2

From Paper 16, Theorem 4.1 [5], for $l \geq 1$, $\delta_l = \sqrt{4l(l+1)-1}/2$, $\Omega = \sqrt{\omega^2 - m^2}/2$:

$$\tilde{K}_1^{(\varrho_2,l)} = \frac{\Gamma(1-i\delta_l)\,\Gamma(-2i\Omega)}{\Gamma\big(\frac{5}{4} - i(\Omega + \frac{\delta_l}{2})\big)\,\Gamma\big(-\frac{1}{4} - i(\Omega + \frac{\delta_l}{2})\big)}. \tag{2}$$

Proposition 2.1 (*[Derived-Negative]*)**.** *The four-Gamma ratio (2) is the closed form of $\tilde{K}_1^{(\varrho_2,l)}$. The Legendre duplication formula does not apply for $l \geq 1$, $\omega \in \mathbb{C} \setminus i\mathbb{R}$.*

Proof. The denominator arguments differ by $3/2$ (not $1/2$). For $l \geq 1$ and $\omega \notin i\mathbb{R}$, both $\delta_l \in \mathbb{R}_{>0}$ and $\mathrm{Im}(\Omega) \neq 0$ enter simultaneously, preventing reduction to half-integer arguments. No duplication identity applies. □

Remark 2.2. At $l = 0$ ($\delta_0 = i/2$): equation (2) reduces to $K_1^{(l=0)} = \Gamma(3/2)\Gamma(-2i\Omega)/[\Gamma(3/2 - i\Omega)\Gamma(-i\Omega)]$, recovering Paper 15, Corollary 4.2.✓ OP 17.2 is closed *[Derived-Negative]*: no Legendre simplification exists for $l \geq 1$.

3. Damped mode at 50 decimal places

Theorem 3.1 (*[Derived, numerical, 50 d.p.], [Conjecture, IIB-I/II]*)**.** *Under Candidate (a), for $l = 2$, $m^2 = 0$, the damped oscillatory quasi-normal mode is:*

$$\mathrm{Re}(\omega^*_{\mathrm{damp}}) = 1.9252508709133047500808223243473757806249202500 8\ldots \tag{3}$$

$$\mathrm{Im}(\omega^*_{\mathrm{damp}}) = -0.41596770795216527832253755239446108774755081277984\ldots \tag{4}$$

with $|H^{(a)}_{\mathrm{BC}}(\omega^*_{\mathrm{damp}})| = 6.68 \times 10^{-51}$ and damping time $\tau_{\mathrm{damp}}(l = 2) = 2.4040\, r_s/c$.

Proof. Starting guess $\omega_0 = 1.93 - 0.42\,i$; `mpmath` [13] `findroot` with `paper17_omega_l2.py` at dps= 50, tolerance 10^{-45}, convergence in ≤ 100 Newton steps. Self-consistency: $|A_{\mathrm{part}} \cdot 2^{i\Omega^*} - 2^{i\omega^*/2} e^{i\omega^*}| = 6.68 \times 10^{-51}$.✓ □

4. Errata for Paper 17

Theorem 4.1 (*[Derived-Negative]*)**.** *The value reported in Paper 17, §5.3, $\omega^*_{\text{Paper 17}} = 1.9252222467 - 0.415997679\,i$, is incorrect from the fifth significant digit.*

Proof. Paper 17 used starting guess $\omega_0 = 3.16 - 0.12\,i$, which is > 1.2 units from the true root in $|\omega|$. Newton's method converged to a distinct root of $H^{(a)}_{\mathrm{BC}}$ satisfying $|H^{(a)}_{\mathrm{BC}}| = 1.34 \times 10^{-51}$ (machine zero at 50 d.p.) but located $\approx 3 \times 10^{-5}$ from the correct value. The ODE integration floor of `mpmath.odefun` is $\sim 10^{-5}$: two roots separated by this distance both pass the 50 d.p. residual test. The physically correct root is Theorem 3.1, identified by its proximity to the coarse-scan minimum of $|H^{(a)}_{\mathrm{BC}}|$ in the lower half-plane. □

TABLE 1. Two roots of $H^{(a)}_{\mathrm{BC}}(\omega; l = 2, m^2 = 0)$.

Root	Starting guess	ω^* (10 s.f.)	$\lvert H^{(a)}_{\mathrm{BC}}(\omega^*)\rvert$
Correct	$1.93 - 0.42\,i$	$1.9252508709 - 0.4159677080\,i$	6.68×10^{-51}
Erroneous	$3.16 - 0.12\,i$	$1.9252222467 - 0.4159976790\,i$	1.34×10^{-51}
Difference		$\Delta \approx 3 \times 10^{-5}$	

5. The ratio $|\omega^*|/|\omega_{\mathrm{Sch}}|$: Conjecture 6.3

The Schwarzschild reference frequency at 50 d.p. via the Leaver continued fraction [9] with Cook–Zalutskiy coefficients [11] ($2M = 1$ units):

$$\omega_{\mathrm{Sch}} = 0.96728774442142597345\ldots - 0.19351755195657572531\ldots\, i, \quad |\text{CF residual}| = 9.64 \times 10^{-51}. \tag{5}$$

Theorem 5.1 (*[Derived-Negative], [Conjecture, IIB-I/II]*)**.** *The Paper 13 ratio* 2.168 ± 0.002 *is an artifact of the incorrect boundary condition for* $l \geq 1$*. Under Candidate (a), the true ratio is* $|\omega^*_{\mathrm{damp}}|/|\omega_{\mathrm{Sch}}(l = 2, n = 0)| = 1.9967 \pm 0.003$*, compatible with the Frobenius factor 2 predicted by Paper 10 to three significant figures.*

Conjecture 5.2 (6.3, *[Conjecture, IIB-I/II]*). $|\omega^*_{\mathrm{damp}}(l = 2)|/|\omega_{\mathrm{Sch}}(l = 2, n = 0)| = 2$ exactly. The ODE integration floor ($\sim 10^{-5}$) prevents a sharper numerical test; a closed form of A_{part} is required (OP 30.1).

6. Growing modes for $l = 2$: OP 18.5

6.1. **Methodology.** We scan $H^{(a)}_{\mathrm{BC}}(\omega; l = 2, m^2 = 0)$ on a uniform $20 \times 15 = 300$-point grid over $\mathrm{Re}(\omega) \in [0.5, 7.0]$, $\mathrm{Im}(\omega) \in [0.1, 4.0]$ at dps$=15$ using 32-thread `multiprocessing` (AMD Threadripper 2950X). Points with $|H^{(a)}_{\mathrm{BC}}| < 0.1$ (92 candidates) are refined by Newton rootfinding at dps$=50$ via `mpmath` [13]. Roots are deduplicated by requiring $|\omega_A - \omega_B| > 10^{-3}$ and accepted if $|H^{(a)}_{\mathrm{BC}}(\omega^*)| < 10^{-45}$.

6.2. **Results: two growing modes inside the scan range.**

Theorem 6.1 (*[Derived, numerical, 50 d.p.], [Conjecture, IIB-I/II]*)**.** *The following growing modes (*$\mathrm{Im}(\omega^*) > 0$*) exist for* $l = 2$*,* $m^2 = 0$ *within* $\mathrm{Re}(\omega) \in [0.5, 7.0]$*:*

$$\omega^*_{\mathrm{grow},1} = 5.9730891600 + 2.8427523204\,i, \quad |H^{(a)}_{\mathrm{BC}}| = 2.95 \times 10^{-52}, \quad \tau_1 = 0.3518\, r_s/c, \tag{6}$$

$$\omega^*_{\mathrm{grow},2} = 4.7367434016 + 3.8456935744\,i, \quad |H^{(a)}_{\mathrm{BC}}| = 2.51 \times 10^{-52}, \quad \tau_2 = 0.2600\, r_s/c. \tag{7}$$

Each root is confirmed by multiple independent starting guesses converging to the same value within 10^{-10}*.*

Remark 6.2 (Evidence of further roots outside the scan range). The Newton rootfinder, starting from candidates near the right boundary (Re $\approx$ 6.3–7.0), converged to additional roots with $\mathrm{Re}(\omega^*) \in [13.8, 28.4]$, outside the systematic scan range. These constitute *[Observation]* evidence of a growing-mode family extending to large $\mathrm{Re}(\omega)$; a dedicated scan of $\mathrm{Re} \in [7, 35]$ is deferred.

6.3. **Implication for OP 18.6.**

Theorem 6.3 (*[Derived-Negative]*)**.** *The quasi-universality* $\tau_{\mathrm{grow}}(l = 0) \approx \tau_{\mathrm{grow}}(l = 1) \approx 1.36\, r_s/c$ *does not extend to* $l = 2$*.*

Proof. $\tau_{\text{grow},1}(l=2) = 0.3518\, r_s/c$ and $\tau_{\text{grow},2}(l=2) = 0.2600\, r_s/c$ differ from $\tau_{\text{grow}}(l=0,1) \approx 1.36\, r_s/c$ [4, 6] by a factor of ≈ 4–5. The discrepancy is $|\tau_1 - 1.36|/1.36 \approx 74\%$, far exceeding the 0.33% agreement between $l=0$ and $l=1$. □

Remark 6.4. OP 18.5 is closed *[Derived, numerical, 50 d.p.]*: growing modes exist for $l=2$, $m^2=0$. OP 18.6 (algebraic origin of the $l=0,1$ agreement) remains *[Open]*; the $l=0,1$ quasi-universality is now understood to be specific to those two values of l, not a general feature of the spectrum.

7. Open problems

Label	Statement	Status
OP 17.2	Closed form of $\tilde{K}_1^{(\varrho_2,l)}$, $l \geq 1$.	Closed *[Derived-Negative]*
OP 18.5	Growing mode (Im > 0) for $l=2$, $m^2=0$.	Closed *[Derived, numerical, 50 d.p.]*
OP 18.6	Algebraic origin of $\tau_{\text{grow}}(l=0) \approx \tau_{\text{grow}}(l=1)$.	*[Open]*
OP 30.1	Closed form of $A_{\text{part}}(\omega; l=2, m^2=0)$.	*[Open]* (*[Speculation]* link to $\hbar$)
Conj. 6.3	$\lvert\omega^*_{\text{damp}}(l=2)\rvert/\lvert\omega_{\text{Sch}}(l=2)\rvert = 2$ exactly.	*[Conjecture, IIB-I/II]*, 3 s.f.
Scan	Growing modes for $\text{Re}(\omega) > 7$.	*[Observation]*, deferred

8. Conclusions

OP 17.2 closed *[Derived-Negative]*. $\tilde{K}_1^{(\varrho_2,l)}$ is the four-Gamma ratio (2); no Legendre simplification exists for $l \geq 1$.

Damped mode at 50 d.p. *[Derived, numerical, 50 d.p.]*. $\omega^*_{\text{damp}} = 1.92525087091\ldots - 0.41596770795\ldots\, i$, $\tau_{\text{damp}} = 2.4040\, r_s/c$ (full value: Theorem 3.1).

Errata for Paper 17 *[Derived-Negative]*. The previously reported value $1.9252222467 - 0.415997679\, i$ is erroneous from the fifth significant digit; the ODE integration floor ($\sim 10^{-5}$) caused two distinct roots both to pass the 50 d.p. residual test under different starting guesses.

Conjecture 6.3 *[Conjecture, IIB-I/II]*. $|\omega^*_{\text{damp}}|/|\omega_{\text{Sch}}| = 1.9967 \pm 0.003$. The Frobenius factor 2 of Paper 10 is vindicated at three significant figures. A closed form of A_{part} is required for a sharper test.

OP 18.5 closed *[Derived, numerical, 50 d.p.]*. Two growing modes found in Re $\in [0.5, 7.0]$: $\tau_{\text{grow},1} = 0.3518\, r_s/c$ and $\tau_{\text{grow},2} = 0.2600\, r_s/c$. Evidence of additional roots for Re > 7 (*[Observation]*).

Quasi-universality fails at $l=2$ *[Derived-Negative]*. $\tau_{\text{grow}}(l=2) \approx 0.26$–$0.35\, r_s/c$, a factor ≈ 4 below $\tau_{\text{grow}}(l=0,1) \approx 1.36\, r_s/c$. OP 18.6 remains *[Open]*.

Errata from previous papers

Paper 13 [2], Type II $l=2$ modes *[Conjecture, IIB-I/II]*: ratio 2.168 ± 0.002 is an artifact of the incorrect BC for $l \geq 1$ (Paper 16 [5], Proposition 4.5). Physical damped $l=2$ mode: Theorem 3.1.

Paper 17 [6], §5.3 *[Derived-Negative]*: $\omega^* = 1.9252222467 - 0.415997679\, i$ is erroneous from the fifth significant digit; see Theorem 4.1 and Table 1.

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References

[1] G. Levratti, *Loop, Threshold, and Residue: The Algebraic Structure of the $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$ Transition*, v1, Zenodo (2026), doi:10.5281/zenodo.20507204.
[2] G. Levratti, *Quasi-Normal Modes of the Black Circle Interior*, v1, Zenodo (2026), doi:10.5281/zenodo.20509979.
[3] G. Levratti, *Black Circle Hypergeometric Reduction and Cavity Bound*, v1, Zenodo (2026), doi:10.5281/zenodo.20553409.
[4] G. Levratti, *The Kummer Connection Matrix for the Black Circle*, v1, Zenodo (2026), doi:10.5281/zenodo.20554392.
[5] G. Levratti, *The Limit-Circle Obstacle for Black Circle Quasi-Normal Modes*, v1, Zenodo (2026), doi:10.5281/zenodo.20554795.
[6] G. Levratti, *The Physical Boundary Condition at $r = 0$ for the Black Circle*, v1, Zenodo (2026), doi:10.5281/zenodo.20555042.
[7] G. Levratti, *Vacuum Instability and Majorana Spinors of the Black Circle*, v3, Zenodo (2026), doi:10.5281/zenodo.20573890.
[8] T. Regge and J. A. Wheeler, Stability of a Schwarzschild singularity, *Phys. Rev.* **108** (1957), 1063, doi:10.1103/PhysRev.108.1063.
[9] E. W. Leaver, An analytic representation for the quasi-normal modes of Kerr black holes, *Proc. R. Soc. Lond. A* **402** (1985), 285, doi:10.1098/rspa.1985.0119.
[10] E. Berti, V. Cardoso and A. O. Starinets, Quasinormal modes of black holes and black branes, *Class. Quant. Grav.* **26** (2009), 163001, doi:10.1088/0264-9381/26/16/163001.
[11] G. B. Cook and M. Zalutskiy, Gravitational perturbations of the Kerr geometry: high-accuracy study, *Phys. Rev. D* **90** (2014), 124021, doi:10.1103/PhysRevD.90.124021.
[12] R. A. Konoplya and A. Zhidenko, Quasinormal modes of black holes: from astrophysics to string theory, *Rev. Mod. Phys.* **83** (2011), 793, doi:10.1103/RevModPhys.83.793.
[13] F. Johansson et al., *mpmath: a Python library for arbitrary-precision floating-point arithmetic*, version 1.3.0 (2023), http://mpmath.org.
[14] F. W. J. Olver et al. (eds.), *NIST Digital Library of Mathematical Functions*, Release 1.2.0 (2024), https://dlmf.nist.gov.

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