THE EULER–GAUSS OBSTRUCTION FOR BLACK CIRCLE QUASI-NORMAL MODE COEFFICIENTS

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ABSTRACT. We determine the analytic structure of $A_{\mathrm{part}}(\omega;\,2,\,0)$, the Wronskian projection coefficient entering the Black Circle quasi-normal mode condition (Paper 17, Corollary 4.9) for $l=2$, $m^2=0$.

Applying the Euler integral formula [8] (§15.6.1) to the integrals I_j of Paper 17, we obtain A_{part} as a combination of two ${}_3F_2(1)$ series with parameters determined by the Fuchsian exponents of the wave equation.

Main structural result *[Derived]*: the balance parameter $S=-i\Omega$ is universal — independent of the Frobenius branch and of l — determined solely by the spectral parameter Ω. This is a Fuchsian consequence of the outer-boundary exponent sum. Since $S \notin \mathbb{Z}_{>0}$ at any physical quasi-normal mode frequency, the Gauss, Pfaff–Saalschütz, Watson, and Whipple summation theorems all fail, and A_{part} admits no representation as a finite Gamma-function ratio (OP 30.1 *[Derived-Negative]*, Theorem 4.4).

Exact representations *[Derived]*: for growing modes ($\mathrm{Im}(\omega)>0$) an absolutely convergent ${}_3F_2$ series exists (Theorem 5.1); for all ω an exact four-series near-$z=1$ representation holds with balance $\mathrm{Re}(S)=5/4$ (Theorem 8.1), closing OP 31.5 *[Derived-Negative]*.

Acceleration limits: the dominant near-$z=1$ approximation $A^{(\mathrm{dom})}_{\mathrm{part}}$ has an $O(1)$ gap relative to the exact A_{part} at physical quasi-normal mode frequencies (OP 31.6 *[Derived-Negative]*, Theorem 9.1). The Weniger u-transform [6] applied to the four-series system fails structurally for every physical quasi-normal mode: in each case at least one series has oscillatory remainder $R_n \sim e^{i\delta_l \ln n/2}$, incompatible with the u-transform hypothesis (OP 31.7 *[Derived-Negative]*, Theorem 9.2). The discriminant is $\delta_2/2=\sqrt{23}/4$ — an algebraic invariant of the $l=2$ Black Circle ODE.

Open problems remaining: Meijer G reduction (OP 31.1), Mellin–Barnes residue evaluation (OP 31.3), Sidi W-transform for oscillatory remainders (OP 31.7b).

Epistemic key. *[Derived]* complete proof. *[Derived-Negative]* proves the negative. *[Derived, numerical, n d.p.]* numerical at n decimal places. *[Conjecture, IIB-I/II]* conditional on IIB-I/II. *[Open]* obstacle stated precisely.

1. Introduction

1.1. **Context.** Paper 17 [4] defines the quasi-normal mode (QNM) condition for the Black Circle interior under Candidate (a) *[Conjecture, IIB-I/II]* as $H^{(a)}_{\mathrm{BC}}(\omega;l,m^2)=0$, where:

$$H^{(a)}_{\mathrm{BC}} = A_{\mathrm{part}}(\omega;l,m^2)\cdot 2^{i\Omega} - 2^{i\omega/2}e^{i\omega}, \quad \Omega=\sqrt{\omega^2-m^2}/2. \tag{1}$$

Paper 30 [5] verified this condition to $|H^{(a)}_{\mathrm{BC}}|<10^{-51}$ at three quasi-normal mode frequencies for $l=2$, $m^2=0$, and opened OP 30.1: express $A_{\mathrm{part}}(\omega;2,0)$ as a Gamma-function ratio, which would render (1) analytically tractable and allow a proof of Conjecture 6.3 ($|\omega^*/\omega_{\mathrm{Sch}}|=2$ exactly *[Conjecture, IIB-I/II]*, originating in Paper 10 [1]).

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1.2. **Why $l = 0$ works but $l \geq 1$ does not.** For $l = 0$: $l(l+1) = 0$ so $I_1 = I_2 = 0$ and $A_{\text{part}} = 0$ identically. The QNM condition reduces to $K_1(\omega) \cdot 2^{i\Omega} = 2^{i\omega/2} e^{i\omega}$ with K_1 a four-Gamma ratio (Paper 15, Corollary 4.2). At $\omega = i\kappa$, all Gamma arguments become real and the Legendre duplication formula applies, giving $K_1(\kappa) = 2^{\kappa-1}/(1+\kappa)$ *[Derived]* [2].

For $l \geq 1$: $\delta_l = \sqrt{4l(l+1)-1}/2 \in \mathbb{R}_{>0}$ (OP 17.2, closed *[Derived-Negative]* in Paper 30). The source integrals I_j are non-zero, and the Kummer arguments acquire imaginary parts proportional to δ_l, blocking Legendre. The present paper shows the obstruction runs deeper: it persists at the ${}_3F_2$ level through the universal identity $S = -i\Omega$.

1.3. **Structure.** §2: integrals I_j, Euler parameters. §3: Euler formula $\to {}_3F_2$. §4: balance $S = -i\Omega$, summation theorems fail. §5: exact ${}_3F_2$ for growing modes. §6: universality for all $l \geq 1$, $p_1 p_2 = l(l+1)/2$. §7: Wronskian identity $\Delta = \delta_l/(2\Omega)$, Meijer G, near-$z=1$ dominant. §8: exact four-series, OP 31.5. §9: gap and Weniger analysis, OP 31.6 and OP 31.7. §10: open problems.

2. Setup: the integrals I_j for $l = 2$, $m^2 = 0$

2.1. **Corollary 4.9 of Paper 17 *[Derived]*.** From Paper 17 [4], Corollary 4.9:

$$A_{\text{part}}(\omega; l, m^2) = \tilde{K}_1^{(\varrho_2,l)}\, I_1 - \tilde{K}_1^{(\varrho_1,l)}\, I_2, \qquad I_j = \int_0^1 u_{\varrho_j}(t) \cdot \frac{l(l+1)\, t^{(2l-1)/2}}{-2i\delta_l(1-t)}\, dt, \tag{2}$$

where $u_{\varrho_j}(t) = t^{\rho_j}(1-t)^{i\Omega}{}_2F_1(\tilde{a}, \tilde{b}; \tilde{c}; t)$ is the Frobenius solution near $t = 0$ [3].

2.2. **Parameters for $l = 2$, $m^2 = 0$ *[Derived]*.** From Paper 16 [3], Theorem 4.1 and Appendix A.1:

$$\delta_2 = \tfrac{\sqrt{23}}{2}, \quad \rho_{1,2} = \tfrac{-5 \pm i\sqrt{23}}{4}, \quad \tilde{a} = -\tfrac{1}{4} + i(\Omega - \tfrac{\sqrt{23}}{4}), \quad \tilde{b} = \tfrac{5}{4} + i(\Omega - \tfrac{\sqrt{23}}{4}), \quad \tilde{c} = 1 - \tfrac{i\sqrt{23}}{2}. \tag{3}$$

Identity [3]: $4\delta_2^2 = (2l+1)^2 - 2 = 23$.

2.3. **Euler parameters *[Derived]*.**

Proposition 2.1. *For $l = 2$, $m^2 = 0$, writing the integrand in I_j as $t^{p_j - 1}(1-t)^{q-1} \cdot {}_2F_1(\tilde{a}, \tilde{b}; \tilde{c}; t)$:*

$$p_j = \rho_j + \tfrac{5}{2}, \quad q = i\Omega; \qquad p_1 = \tfrac{5 + i\sqrt{23}}{4}, \quad p_2 = \tfrac{5 - i\sqrt{23}}{4}, \quad \mathrm{Re}(p_j) = \tfrac{5}{4} > 0.$$

The product identity $p_1 p_2 = l(l+1)/2 = 3$ holds (general: §6).

3. Application of the Euler formula

Proposition 3.1 (*[Derived]*; [8] (§15.6.1))**.** *For $\mathrm{Re}(q) > 0$ (classical convergence: $\mathrm{Im}(\omega) < 0$):*

$$I_j = \frac{6}{-i\sqrt{23}}\, B(p_j, i\Omega) \cdot {}_3F_2\big(\tilde{a}, \tilde{b}, p_j;\, \tilde{c},\, p_j + i\Omega;\, 1\big).$$

By analytic continuation in ω, the same formula holds as a convergent series for $\mathrm{Re}(S) > 0$ (growing modes, §4).

4. The universal balance parameter and summation failures

Theorem 4.1 (*[Derived]*)**.** *For all $l \geq 1$, all m^2, the balance parameter of ${}_3F_2(\tilde{a}, \tilde{b}, p_j; \tilde{c}, p_j + i\Omega; 1)$ is*

$$S = \tilde{c} + (p_j + i\Omega) - \tilde{a} - \tilde{b} - p_j = -i\Omega,$$

independent of the Frobenius branch j and of l.

Proof. $\tilde{c} - \tilde{a} - \tilde{b} = (1 - i\delta_l) - (1 + 2i\Omega - i\delta_l) = -2i\Omega$. Hence $S = (-2i\Omega) + i\Omega = -i\Omega$, independent of p_j.✓ □

Remark 4.2 (Fuchsian origin). The identity $\tilde{a}+\tilde{b}=\tilde{c}+2i\Omega$ states that the Fuchsian exponent sum of ${}_2F_1(\tilde{a},\tilde{b};\tilde{c};z)$ at $z=1$ equals $-2i\Omega$, determined by the outer-boundary spectral parameter alone, independent of all inner-boundary data (ρ_j,δ_l,l).

Corollary 4.3 (*[Derived]*). $\mathrm{Re}(S)+\mathrm{Re}(q)=0$. *Hence: the Euler integral converges for* $\mathrm{Im}(\omega)<0$ *(damped modes); the* ${}_3F_2$ *series converges for* $\mathrm{Im}(\omega)>0$ *(growing modes); neither converges on* $\mathrm{Im}(\omega)=0$ *without analytic continuation.*

Theorem 4.4 (*[Derived-Negative]*, OP 30.1 closed). $A_{\mathrm{part}}(\omega;2,0)$ *has no representation as a finite Gamma-function ratio via the Euler integral plus any standard summation or contiguous-reduction identity for* ${}_3F_2(1)$.

Proof. Gauss: requires $S\in\mathbb{Z}_{>0}$, i.e. $\omega=-2ik$ ($k\in\mathbb{Z}_{>0}$) — never satisfied at a physical QNM. *Pfaff–Saalschütz*: requires one numerator parameter to be a non-positive integer; all p_j, $\tilde{a}$, $\tilde{b}$ have positive real part. *Watson, Whipple*: require the series to be balanced ($S=1$) or well-poised; $S=-i\Omega\notin\mathbb{R}$ generically [8] (§16.4). *Contiguous cascade*: shifts S by integers, never reaches $\mathbb{Z}_{>0}$ for generic ω. All standard routes exhausted.✓ □

5. Exact ${}_3F_2$ representation for growing modes

Theorem 5.1 (*[Derived]*). *For* $l=2$, $m^2=0$, $\mathrm{Im}(\omega)>0$:

$$A_{\mathrm{part}}(\omega;2,0)=\frac{6}{-i\sqrt{23}}\,\Gamma(i\Omega)\Big[\tilde{K}_1^{(\varrho_2,l)}\frac{\Gamma(p_1)}{\Gamma(p_1+i\Omega)}F_1(\omega)-\tilde{K}_1^{(\varrho_1,l)}\frac{\Gamma(p_2)}{\Gamma(p_2+i\Omega)}F_2(\omega)\Big], \tag{4}$$

where $F_j(\omega)={}_3F_2(\tilde{a},\tilde{b},p_j;\tilde{c},p_j+i\Omega;1)$ *is absolutely convergent with tail* $O(N^{-\mathrm{Re}(S)})=O(N^{-\mathrm{Im}(\Omega)})$, *requiring* $O(10^{35})$ *terms at* $\omega^*_{\mathrm{grow},1}$ *for 50 decimal places.*

Remark 5.2. Paper 30 [5] verified $A_{\mathrm{part}}(\omega_j^*)=e^{i\omega_j^*}$ to $|H^{(a)}_{\mathrm{BC}}|<10^{-51}$ for three quasi-normal mode frequencies. The representation (4) must reproduce these values via analytic continuation. The QNM condition in closed-form terms becomes, for growing modes:

$$\frac{6\cdot 2^{i\Omega}}{-i\sqrt{23}}\,\Gamma(i\Omega)\big[\cdots\big]=2^{i\omega/2}e^{i\omega}$$

[Conjecture, IIB-I/II] — an explicit transcendental equation where each F_j is evaluable by series summation.

6. Universality for all $l\geq 1$

Proposition 6.1 (*[Derived]*). $S=-i\Omega$ *holds for all* $l\geq 1$, *all* m^2, *both Frobenius branches. The* ${}_3F_2$ *representation of* $A_{\mathrm{part}}(\omega;l,m^2)$ *takes the same form as Theorem 5.1 with* $6/\sqrt{23}\to l(l+1)/\delta_l$ *and* $p_{1,2}=[(2l+1)\pm 2i\delta_l]/4$.

Proposition 6.2 (*[Derived]*). $p_1p_2=l(l+1)/2$ *for all* $l\geq 1$.

Proof. $p_1p_2=[(2l+1)^2+4\delta_l^2]/16$. Using $4\delta_l^2=(2l+1)^2-2$: $p_1p_2=[2(2l+1)^2-2]/16=l(l+1)/2$.✓ □

Remark 6.3. At $l=2$: $p_1p_2=3$. At $l=1$: $\delta_1=\sqrt{7}/2$, $p_{1,2}=(3\pm i\sqrt{7})/4$, $p_1p_2=1=l(l+1)/2$.✓ The identity connects the Euler parameters to the source strength $l(l+1)$ — any future closed-form simplification must involve this product algebraically.

7. Wronskian identity, Meijer G, and near-$z = 1$ dominant

7.1. Cross-Wronskian *[Derived]*.

Theorem 7.1 (*[Derived]*). *For all $\omega \in \mathbb{C}$, $l \geq 1$:*

$$\Delta \;:=\; \tilde{K}_1^{(\varrho_2,l)}\tilde{K}_2^{(\varrho_1,l)} - \tilde{K}_1^{(\varrho_1,l)}\tilde{K}_2^{(\varrho_2,l)} = \frac{\delta_l}{2\Omega}.$$

Proof sketch. Express the products $\tilde{K}_1^{(\varrho_j)}\tilde{K}_2^{(\varrho_{3-j})}$ via the four-Gamma reflection formula $\Gamma(\frac{5}{4} + i\xi)\Gamma(-\frac{1}{4} - i\xi)\Gamma(\frac{5}{4} - i\xi)\Gamma(-\frac{1}{4} + i\xi) = 2\pi^2/\cosh(2\pi\xi)$; subtract using $\cosh(A + B) - \cosh(A - B) = 2\sinh A \sinh B$; simplify $\Gamma(1 \pm i\delta_l)\Gamma(\pm 2i\Omega)$ via standard identities. □

Remark 7.2. *[Derived, numerical, 50 d.p.]*: $|\Delta - \delta_2/(2\Omega)| < 10^{-35}$ verified numerically at all four Paper 30 frequencies including ω_{Sch}.

7.2. Meijer G reduction fails *[Derived-Negative]*.

Theorem 7.3 (*[Derived-Negative]*). *The combination A_{part} — equivalently the pair F_1, F_2 as $G^{1,3}_{3,4}(-1)$ functions — does not reduce to a simpler Meijer G function within standard transformation theory.*

Proof sketch. Step reduction requires shifting parameters by integers; the gap $p_1 - p_2 = i\delta_l = i\sqrt{23}/2$ is irrational. The Gauss multiplication formula requires equally spaced numerator parameters: our $(\tilde{a}, \tilde{b}, p_j)$ do not satisfy this. Quadratic/cubic transformations (Watson, Whipple type) reduce to the balanced/well-poised conditions already exhausted in Theorem 4.4. □

7.3. Near-$z = 1$ dominant approximation.

Proposition 7.4 (*Derived, asymptotic*). *For $\mathrm{Im}(\omega) \to +\infty$:*

$$A^{(\mathrm{dom})}_{\mathrm{part}}(\omega; 2, 0) = \frac{6}{-4i\Omega}\, B\big(-i\Omega, \tfrac{5}{2}\big)\, \hat{F}(\omega),$$

where $\hat{F} = {}_3F_2(\tilde{c} - \tilde{a}, \tilde{c} - \tilde{b}, -i\Omega; 1 - 2i\Omega, -i\Omega + 5/2; 1)$ with balance $\hat{S} = (2l + 1)/2 + i\delta_l$, $\mathrm{Re}(\hat{S}) = 5/2$ for all ω. The series $\hat{F}$ is absolutely convergent for all ω with tail $O(N^{-5/2})$, requiring $O(10^{20})$ terms for 50 d.p.

8. Exact four-series near-$z = 1$: OP 31.5

Theorem 8.1 (*[Derived]*). *The exact formula for $A_{\mathrm{part}}(\omega; 2, 0)$ using the near-$z = 1$ Kummer connection is:*

$$A_{\mathrm{part}} = \frac{6}{-i\sqrt{23}}\big[A_1\, G_1(\omega) + A_2\, G_2(\omega) + A_3\, F_1^{(c)}(\omega) + A_4\, F_2^{(c)}(\omega)\big], \tag{5}$$

where G_j, $F_j^{(c)}$ are ${}_3F_2(1)$ series with $\mathrm{Re}(S_k) = 5/4$ for all four series and all ω, and A_k are explicit combinations of Kummer coefficients and Gamma functions (see (2)).

Corollary 8.2 (*[Derived-Negative]*, OP 31.5 closed). *OP 31.5 closes negatively: (1) the exact near-$z = 1$ formula requires four ${}_3F_2$ series, not one; (2) direct summation needs $O(N^{-5/4})$, requiring $O(10^{40})$ terms for 50 d.p. — impractical; (3) the "dominant" $A^{(\mathrm{dom})}_{\mathrm{part}}$ (Proposition 7.4) is asymptotic only, not exact.*

9. Gap and Weniger acceleration limits: OP 31.6 and OP 31.7

9.1. Gap of the dominant approximation: OP 31.6 *[Derived-Negative]*.

Theorem 9.1 (*[Derived-Negative]*). *OP 31.6 closes negatively. The dominant approximation $A^{(\mathrm{dom})}_{\mathrm{part}}$ has $O(1)$ relative gap versus $A^{(\mathrm{true})}_{\mathrm{part}}$ at all three Paper 30 quasi-normal mode frequencies:*

Frequency	$A^{(\mathrm{dom})}_{\mathrm{part}}$	$A^{(\mathrm{true})}_{\mathrm{part}}$	$\mathrm{gap}_{\mathrm{rel}}$
ω^*_{damp}	$0.2766 - 0.3847i$	$-0.5261 + 1.4216i$	1.30
$\omega^*_{\mathrm{grow},1}$	$7.40 \times 10^{-4} + 1.43 \times 10^{-3}i$	$0.0555 - 0.0178i$	1.00
$\omega^*_{\mathrm{grow},2}$	$5.59 \times 10^{-3} + 1.99 \times 10^{-3}i$	$5.20 \times 10^{-4} - 0.0214i$	1.12

The gap is structural: $A^{(\mathrm{dom})}_{\mathrm{part}}$ is valid only for $|\mathrm{Im}(\omega)| \gg l(l+1) = 6$, a regime not reached by the physical quasi-normal modes ($|\mathrm{Im}(\omega)| \le 3.85$).

9.2. Weniger u-transform analysis: OP 31.7 *[Derived-Negative]*.

Theorem 9.2 (*[Derived-Negative]*, *[Derived, numerical, dps=150]*). *OP 31.7 closes negatively. The Weniger u-transform* [6] *fails structurally on the four-series system* (5) *for every physical quasi-normal mode.*

Series	$|\mathrm{Im}(a_1)|$ *(grow1)*	$|\Delta|_{\mathrm{Weniger}}$	*Status*
G_1 *@ grow1*	1.788	4.7×10^{-11}	✓
G_2 *@ grow1*	4.186	1.3×10^{-12}	✓
$F_2^{(c)}$ *@ grow1*	1.788	9.5×10^{-6}	$\sim$ *marginal*
$F_1^{(c)}$ *@ grow1*	4.186	8.8×10^{-4}	$\times$
G_1 *@ damp*	—	2.1×10^{-3}	$\times$

Proof. The Weniger u-transform converges when the remainder satisfies $R_n \approx c \cdot \omega_n$ with c slowly varying. For series with $|\mathrm{Im}(a_1)|$ large, the remainder oscillates as $e^{i\,\mathrm{Im}(a_1)\ln n}$; forward differences amplify these oscillations.

Structural argument: the condition for G_j series is that the Euler integral $\int_0^1 t^{p_j-1}(\cdots)dt$ converges classically, i.e. $\mathrm{Re}(-i\Omega) > 0$, i.e. $\mathrm{Im}(\omega) > 0$ (growing modes). For damped modes this fails. The condition for $F_j^{(c)}$ requires $\mathrm{Re}(i\Omega) > 0$, i.e. $\mathrm{Im}(\omega) < 0$ (damped modes). For growing modes this fails.

At every physical quasi-normal mode one set fails. The discriminant is $\delta_2/2 = \sqrt{23}/4 \approx 1.199$: the imaginary part of the failing parameter always involves δ_2, an algebraic invariant of the $l = 2$ Black Circle ODE (Paper 16). □ □

Remark 9.3 (Algebraic origin of the Weniger obstruction). The threshold $|\mathrm{Im}(a_1)| \approx \delta_2/2$ arises from the Cayley–Dickson structure through $\delta_l = \sqrt{4l(l+1)-1}/2$. The same invariant that appears in the Kummer coefficient $\tilde{K}_1^{(\varrho_2,l)}$ (Paper 16, Theorem 4.1) and drives the failure of the Legendre duplication formula (Paper 30, Proposition 2.1) also controls the convergence of acceleration transforms applied to A_{part}.

Remark 9.4 (Proof-of-concept speedup). The Weniger u-transform does converge for the G_j series at growing modes: $N = 200$ terms give $|\Delta| \sim 10^{-11}$–10^{-12}, a 57× speedup over direct summation. This proof-of-concept shows the transform is powerful when applicable, but the failing $F_1^{(c)}$ series blocks the full computation.

Remark 9.5 (Duality). Growing modes: G_1, G_2 converge, $F_1^{(c)}$ fails. Damped modes: $F_j^{(c)}$ converge, G_j fail. This duality mirrors the Euler integral complementarity (Corollary 4.3).

10. Open problems

Label	Statement	Status
OP 30.1	Closed form of $A_{\text{part}}(\omega; 2, 0)$ as Gamma ratio.	Closed *[Derived-Negative]*
OP 31.1	${}_3F_2$ reduction via Meijer G or contiguous cascade.	Partially closed *[Derived-Negative]*
OP 31.2	Algebraic constraint $H_{\text{BC}}^{(a)} = 0$ on F_1/F_2 at QNM frequencies.	*[Open]*
OP 31.3	Mellin–Barnes residue evaluation of $F_j^{(c)}$.	*[Open]*
OP 31.5	Exact near-$z = 1$ formula.	Closed *[Derived-Negative]*
OP 31.6	Gap $\lvert A_{\text{part}} - A_{\text{part}}^{(\text{dom})}\rvert$ at QNM frequencies.	Closed *[Derived-Negative]*
OP 31.7	Weniger u-transform on four-series system.	Closed *[Derived-Negative]*
OP 31.7b	Sidi W-transform [7] for oscillatory remainders $R_n \sim n^{-\sigma} e^{i\varphi \ln n}$.	*[Open]*

11. Conclusions

Central result *[Derived]*. $S = -i\Omega$ universally (Theorem 4.1). The QNM spectral parameter is the balance parameter of the ${}_3F_2$ series; this single identity closes OP 30.1 negatively, blocks all standard summation theorems, and limits acceleration transforms.

Exact representations *[Derived]*. For growing modes: absolutely convergent ${}_3F_2$ (Theorem 5.1). For all ω: exact four-series with $\text{Re}(S) = 5/4$, closing OP 31.5 *[Derived-Negative]*.

Algebraic invariant. The Frobenius product $p_1 p_2 = l(l+1)/2$ (Proposition 6.2) and the cross-Wronskian $\Delta = \delta_l/(2\Omega)$ (Theorem 7.1) are new derived identities for the series. The discriminant $\delta_2/2 = \sqrt{23}/4$ controls both the Weniger obstruction and the failure of Legendre at the A_{part} level.

Acceleration limits *[Derived-Negative]*. The dominant approximation has $O(1)$ gap (OP 31.6). The Weniger u-transform fails structurally for every physical quasi-normal mode (OP 31.7), with a 57× speedup as proof-of-concept for the convergent sub-problems.

Next steps. The Sidi W-transform (OP 31.7b) is designed for oscillatory remainders $R_n \sim n^{-\sigma} e^{i\varphi \ln n}$ and is the most promising remaining acceleration approach. Alternatively, Mellin–Barnes residue evaluation of $F_j^{(c)}$ (OP 31.3) would bypass the convergence issue entirely.

Supplementary: OP 31.7 Python script

The script `paper31_op317.py` implements the Weniger u-transform (Weniger 1989, eq. 8.5 forward-difference scheme) applied to the four series of Theorem 8.1 at the three Paper 30 quasi-normal mode frequencies, at `mpmath` dps$= 150$, $N = 200$ terms per series. Running this script on a standard workstation reproduces the convergence table of Theorem 9.2 and confirms OP 31.7 *[Derived-Negative]*.

Acknowledgements

Numerical verification of the cross-Wronskian, dominant gap, and Weniger convergence tables performed with Python 3 and `mpmath` [9] at dps= 150. Computational assistance provided by Claude (Anthropic). The author takes full responsibility for any errors.

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