

# BEFORE THE ZERO-DIVISORS: RESIDUES, AUTOMORPHISMS, AND GAUGE COUPLING CONSTANTS IN CAYLEY–DICKSON ALGEBRAS

GIOVANNI LEVRATTI

ABSTRACT. Paper 26 [6] derived Newton’s constant  $G_N$  from the failure of the Cayley–Dickson norm  $N$  to compose at level  $n = 4$  (sedenions), where zero-divisors first appear. Open Problem 26.7 asks whether the gauge coupling constants  $g_1^2, g_2^2, g_3^2$  emerge, by the same Sakharov–Adler mechanism, from  $N$  at the three levels  $n = 1, 2, 3$  where  $N$  composes perfectly and no zero-divisors exist. This paper builds the algebraic case.

We compute all four residues of the ZD zeta function  $Z(x) = 336x^4/[(1-x)(1-2x)(1-4x)(1-8x)]$  at its poles  $x = 2^{-k}$ ,  $k = 0, 1, 2, 3$ , obtaining  $R_0 = 16$ ,  $R_1 = -7$ ,  $R_2 = 7/8$ ,  $R_3 = -1/32$ , and derive a closed-form formula. The functional equation  $Z(x) = 64x^4Z(1/8x)$  imposes the symmetry  $R_k = -2^{9-6k}R_{3-k}$ , pairing  $(\mathbb{R}, \mathbb{O})$  and  $(\mathbb{C}, \mathbb{H})$ . The alternating-sum identity  $1 - 7 + 14 - 8 = 0$  is a corollary of the residue theorem on  $\mathbb{P}^1$ .

A new algebraic observation:  $R_k \cdot \dim(\mathcal{A}_k)^4 = 16 \cdot (-1)^k \cdot s_k$  with  $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$ , which equal  $\dim(\mathbb{R})$ ,  $|\mathrm{PG}(2, \mathbb{F}_2)|$ ,  $\dim G_2 = \dim \mathrm{Aut}(\mathbb{O})$ ,  $\dim \mathbb{O}$ . The identity  $s_2 = 2s_1$  reflects the fact that  $\dim(G_2) = 2 \cdot |\mathrm{Fano}|$  because  $\dim(\mathrm{SO}(7)/G_2) = |\mathrm{Im}(\mathbb{O})| = 7$ . The functional equation pairs the electroweak levels  $(\mathbb{C}, \mathbb{H})$  with  $|R_1/R_2| = 8 = \dim(\mathbb{O})$ , and the (gravity-scale, strong) levels  $(\mathbb{R}, \mathbb{O})$  with  $|R_0/R_3| = 512 = 2^9$ .

We also establish: (i) the spectral transition  $\mathrm{Spec}(M_a)|_{n \leq 3} = \{1^{2^n}\} \rightarrow \{0^4, 1^8, 2^4\}$  at  $n = 4$ , the algebraic signature of the gauge-to-gravity transition; (ii) that  $\ker(L_a) = 0$  at  $n \leq 3$  forces all fermionic corrections to the spectral invariants to zero; (iii) that the AMT  $\mathbb{E}_v[N(av)] = N(a)$  holds point-by-point at  $n \leq 3$  but only on average at  $n \geq 4$ .

Under IIB-II-strong [*Speculation*, *IIB-II-strong*], we propose [*Conjecture*, *IIB-II-strong*] that the Sakharov–Adler loop at level  $k$  runs over a scalar field on  $\mathrm{Aut}(\mathcal{A}_k)$ , with spectral invariant  $\mathcal{S}_k = \dim \mathrm{Aut}(\mathcal{A}_k)$  in the absence of fermionic corrections, and identify the open problems required to complete the derivation of  $g_1^2, g_2^2, g_3^2$ .

**Epistemic key.** [*Derived*] complete proof. [*Observation*] numerically or algebraically established, no proof of deeper reason. [*Conjecture*, *IIB-II-strong*] mechanism proposed, gap identified. [*Speculation*, *IIB-II-strong*] structurally motivated. [*Open*] obstacle stated precisely.

## 1. INTRODUCTION: THE LEFT SIDE OF THE TOWER

**1.1. The Cayley–Dickson tower.** The Cayley–Dickson construction produces the tower  $\mathcal{A}_0 = \mathbb{R} \subset \mathcal{A}_1 = \mathbb{C} \subset \mathcal{A}_2 = \mathbb{H} \subset \mathcal{A}_3 = \mathbb{O} \subset \mathcal{A}_4 = \mathbb{S} \subset \dots$ . The first four algebras — the Hurwitz normed division algebras [11] — satisfy

$$N(ab) = N(a)N(b) \quad \text{for all } a, b \in \mathcal{A}_k, \quad k \leq 3.$$

At  $n = 4$  this law fails on the 336 zero-divisor pairs of  $\mathbb{S}$ .

**1.2. The IIB-II perspective on the tower.** [*Speculation*, *IIB-II-strong*]; synthesis of Papers 2, 26, 32 [2, 6, 8]:

---

*Date:* June 2026.

*2020 Mathematics Subject Classification.* 17A35, 11M41, 22E70, 81V22.

*Key words and phrases.* Cayley–Dickson algebras; ZD zeta function; gauge couplings; induced gravity; Sakharov–Adler mechanism; Fano plane; automorphism groups; Furey programme; IIB conjecture.

| Level   | Algebra      | $N(ab)=N(a)N(b)$ | Property lost $(\mathcal{A}_{k-1} \rightarrow \mathcal{A}_k)$ | Force   |
|---------|--------------|------------------|---------------------------------------------------------------|---------|
| $n = 1$ | $\mathbb{C}$ | ✓                | ordering                                                      | EM      |
| $n = 2$ | $\mathbb{H}$ | ✓                | commutativity                                                 | weak    |
| $n = 3$ | $\mathbb{O}$ | ✓                | associativity                                                 | strong  |
| $n = 4$ | $\mathbb{S}$ | fails on ZD      | alternativity/Born Rule                                       | gravity |

### 1.3. OP 26.7.

*Remark 1.1* (Relation to Furey’s programme). Furey’s programme [9, 10] derives the *gauge symmetry groups*  $U(1)_Y \times SU(2)_L \times SU(3)_C$  from the algebraic structure of  $\mathcal{A}_3 = \mathbb{O}$ . This paper addresses a distinct question: the *numerical values* of the coupling constants  $g_1^2, g_2^2, g_3^2$ , which Furey’s programme takes as experimental inputs. The two programmes are distinct and compatible [2]: gauge symmetries are level-specific multiplication structures; coupling constants may emerge from the same algebras via the Sakharov–Adler mechanism applied to  $\text{Aut}(\mathcal{A}_k)$ .

OP 26.7 [*Speculation, IIB-II-strong*]; Papers 2, 26 [2, 6]: do the gauge coupling constants  $g_1^2, g_2^2, g_3^2$  emerge from  $N$  at levels  $n = 1, 2, 3$  via the Sakharov–Adler mechanism, with the role of  $\text{ZD}^*(\mathcal{A}_4)$  replaced by  $\text{Aut}(\mathcal{A}_k)$ ?

## 2. THE ZD ZETA FUNCTION AS THE SIGNATURE OF $N$

**2.1. Definition, poles, and the Frobenius connection.** From Paper 3 [3]:

$$Z(x) = \sum_{n \geq 4} |\text{ZD}(n)| x^n = \frac{336x^4}{(1-x)(1-2x)(1-4x)(1-8x)}.$$

The four poles  $x_k = 2^{-k}$  satisfy  $x_k = 1/\dim(\mathcal{A}_k)$ . By the Weil factorisation  $Z(x) = 336x^4 \cdot Z_{\text{Weil}}(\text{PG}(3, \mathbb{F}_2), x)$  (Paper 4 [4]), the Frobenius eigenvalues of  $\text{PG}(3, \mathbb{F}_2)$  are  $\{1, 2, 4, 8\}$ , giving the four poles directly.

*Observation 2.1* ([*Observation*]). The Frobenius identity  $\|L_a\|_F^2 = 2^k N(a)$  ([1] Thm. 5.2; [2] Rem. 2.3) gives  $\|L_{e_j}\|_F^2 = 2^k$  for unit basis elements. Hence  $x_k = 2^{-k} = 1/\|L_{e_j}\|_F^2$ : the poles of  $Z(x)$  are the reciprocals of the Frobenius norms of unit basis elements at each Hurwitz level.

**2.2. Partial fraction decomposition** [*Derived*].

$$\frac{1}{(1-x)(1-2x)(1-4x)(1-8x)} = \frac{-1/21}{1-x} + \frac{2/3}{1-2x} + \frac{-8/3}{1-4x} + \frac{64/21}{1-8x}.$$

Verification: coefficients sum to  $1 = f(0)$ . ✓

**2.3. All four residues** [*Derived*].

**Proposition 2.2** ([*Derived*]). *The residues of  $Z(x)$  at its four poles are:*

$$R_0 = 16, \quad R_1 = -7, \quad R_2 = \frac{7}{8}, \quad R_3 = -\frac{1}{32}.$$

*Proof.*  $R_k = \text{Res}_{x=2^{-k}} Z(x) = 336 \cdot (2^{-k})^4 \cdot [\text{coefficient at } (1-2^k x)]$ . Substituting:  $R_0 = 336 \cdot (-1/21) = -16$  — wait; correcting sign:  $R_k = \lim_{x \rightarrow 2^{-k}} (x - 2^{-k}) Z(x)$ , and  $(1 - 2^k x) = -2^k (x - 2^{-k})$ , giving  $R_k = 336 \cdot (2^{-k})^4 / \prod_{j \neq k} (1 - 2^j x)$ .

$$k = 0 : 336 / [(-1)(-3)(-7)] = 336 / (-21) \dots$$

By the cover-up method from §2.2:  $R_0 = 336 \cdot (-1/21) = -16 \dots$  using  $(1 - x) = -(x - 1)$ :  $R_0 = 336 \cdot (A)$  where  $A = -1/21$  and the sign from  $(1 - x) = -(x - 1)$  gives residue at  $x = 1$  equal to  $336 \cdot 1 / ((-1)(-3)(-7))^{-1} \dots$

Let me use the explicit calculation:  $R_k = 336 \cdot 2^{-4k} / \prod_{j \neq k} (1 - 2^{j-k})$ .

$$k = 0 : \prod_{j=1,2,3} (1 - 2^j) = (-1)(-3)(-7) = -21, \quad R_0 = 336/(-21) \dots$$

Taking the sign convention  $\text{Res}_{x=a} f(x) = \lim_{x \rightarrow a} (x-a) f(x)$ : at  $x = 1$ ,  $(1-x) = -(x-1)$  introduces a sign:  $R_0 = 336/[(1-x)(-3)(-7)] = 336/(-21) \dots$

Actually, directly from the partial fractions:  $Z(x) = 336x^4 \cdot f(x)$  with coefficient  $A = -1/21$  at  $(1-x)^{-1}$ .  $\text{Res}_{x=1}[1/(1-x)] = -1$  (since  $1/(1-x) = -(1/(x-1))$ ). Hence  $R_0 = 336 \cdot 1^4 \cdot (-1) \cdot (-1/21) = 336/21 = 16$ . ✓

Similarly  $R_1 = 336 \cdot (1/2)^4 \cdot (-1/2) \cdot (2/3) = -7$ . ✓

$R_2 = 336 \cdot (1/4)^4 \cdot (-1/4) \cdot (-8/3) = 7/8$ . ✓

$R_3 = 336 \cdot (1/8)^4 \cdot (-1/8) \cdot (64/21) = -1/32$ . ✓ □

*Remark 2.3 ([Observation]).*  $R_0 = 16 = \dim(\mathbb{S}) = \dim(\mathcal{A}_4)$ . Under IIB-II-strong *[Speculation, IIB-II-strong]*, the pole associated with  $\mathbb{R}$  (the simplest algebra) encodes the dimension of the algebra where  $N$  first fails to compose. The two extremes of the Hurwitz tower are linked by  $Z(x)$ .

*Remark 2.4 ([Observation]).*  $R_1 = -7 = -|\text{PG}(2, \mathbb{F}_2)|$ . This follows from the Weil factorisation (Paper 4 [4]): the residue of  $Z_{\text{Weil}}(\text{PG}(3, \mathbb{F}_2), x)$  at the Frobenius eigenvalue  $x = 1/q = 1/2$  equals  $-|\text{PG}(2, \mathbb{F}_2)|$ , the number of points of the Fano plane.

#### 2.4. Closed-form formula and functional equation *[Derived]*.

**Proposition 2.5** (*[Derived]*). For  $k = 0, 1, 2, 3$ :

$$R_k = \frac{-336 \cdot 2^{-5k}}{\prod_{j=0, j \neq k}^3 (1 - 2^{j-k})}.$$

*Proof.* Standard partial-fraction residue at  $x = 2^{-k}$ ; verified for all four values in Proposition 2.2. ✓ □

**Proposition 2.6** (*[Derived]*).  $Z(x) = 64x^4 Z(1/8x)$ , with residue symmetry  $R_k = -2^{9-6k} R_{3-k}$  for  $k = 0, 1, 2, 3$ .

*Proof.* Direct substitution:  $64x^4 \cdot 336(1/8x)^4 / [(1 - 1/8x) \dots] = 336x^4 / [(8x - 1) \dots / (-8)^4]$ , which simplifies to  $Z(x)$ . ✓ The residue symmetry follows by comparing Laurent coefficients of both sides at  $x = 2^{-k}$ . ✓ □

*Observation 2.7 ([Observation]).* The functional equation pairs  $(\mathbb{C}, \mathbb{H})$  (tight:  $|R_1/R_2| = 8$ ) and  $(\mathbb{R}, \mathbb{O})$  (loose:  $|R_0/R_3| = 512 = 2^9$ ). Under IIB-II-strong *[Speculation, IIB-II-strong]*,  $(\mathbb{C}, \mathbb{H})$  is the electroweak pair (the two forces that unify), encoded as the tight pair of  $Z(x)$ .

**Corollary 2.8** (*[Derived]*).  $\sum_{k=0}^3 \sigma_k = 0$ , where  $\sigma_k := R_k \cdot \dim(\mathcal{A}_k)^4$ . Equivalently,  $1 - 7 + 14 - 8 = 0$ .

*Proof.* Define  $g(x) := Z(x)/x^4 = 336/[(1-x)(1-2x)(1-4x)(1-8x)]$ . Since  $g(x) \rightarrow 0$  as  $x \rightarrow \infty$ , the function has no pole at  $\infty$  in  $\mathbb{P}^1(\mathbb{C})$ . By the residue theorem on  $\mathbb{P}^1$ ,  $\sum_{k=0}^3 \sigma_k = \sum_k \text{Res}_{x=2^{-k}} g(x) = 0$ . ✓

The equivalence  $1 - 7 + 14 - 8 = 0$  follows from  $\sigma_k = 16(-1)^k s_k$  (Observation 2.10). ✓ □

*Remark 2.9 ([Derived]).* The identity  $1 - 7 + 14 - 8 = 0$  is not numerical coincidence: it is the analytic statement that  $g(x)$  is a proper rational function (degree 0 numerator, degree 4 denominator), equivalently that  $Z(x)$  grows no faster than  $x^4$  as  $x \rightarrow \infty$ .

## 2.5. A new algebraic observation.

*Observation 2.10 ([Observation]).* Define  $\sigma_k := R_k \cdot (2^k)^4$  and  $s_k := |\sigma_k|/16$ .

| $k$ | $R_k$ | $\dim(\mathcal{A}_k)^4$ | $\sigma_k$ | $s_k$                                |
|-----|-------|-------------------------|------------|--------------------------------------|
| 0   | 16    | 1                       | 16         | $1 = \dim \mathbb{R}$                |
| 1   | -7    | 16                      | -112       | $7 =  \mathrm{PG}(2, \mathbb{F}_2) $ |
| 2   | 7/8   | 256                     | 224        | $14 = \dim G_2$                      |
| 3   | -1/32 | 4096                    | -128       | $8 = \dim \mathbb{O}$                |

That is,  $\sigma_k = 16(-1)^k s_k$  with  $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$ , and  $s_k$  equals  $\dim(\mathcal{A}_0)$ ,  $|\mathrm{PG}(2, \mathbb{F}_2)|$ ,  $\dim \mathrm{Aut}(\mathbb{O})$ ,  $\dim(\mathcal{A}_3)$  respectively.

*Remark 2.11 ([Observation]).* The identity  $s_2 = 2s_1$ , i.e.  $\dim(G_2) = 2 \cdot |\mathrm{Fano}|$ , is not accidental:  $G_2 \subset \mathrm{SO}(7)$  [14, 15] with  $\dim(\mathrm{SO}(7)/G_2) = 21 - 14 = 7 = |\mathrm{Im}(\mathbb{O})| = |\mathrm{Fano}|$ . The Fano plane has exactly as many points as the codimension of  $G_2$  in  $\mathrm{SO}(7)$ .

*Remark 2.12 ([Speculation, IIB-II-strong]).* The appearance of  $s_2 = 14 = \dim G_2 = \dim \mathrm{Aut}(\mathbb{O})$  at the  $\mathbb{H}$ -pole ( $k = 2$ ) is a cross-level imprint: the quaternionic pole carries the dimension of the octonionic automorphism group. Under IIB-II-strong,  $G_2$  is the first “large” gauge group of the tower (containing  $\mathrm{SU}(3)_C$ ), and its dimension is encoded in  $Z(x)$  via the global residue structure. OP 33.3: algebraic proof of this cross-level appearance.

## 3. AUTOMORPHISM GROUPS AND THE SPECTRAL TRANSITION

### 3.1. Automorphism groups of the Hurwitz algebras.

| $n$ | Algebra      | $\mathrm{Aut}(\mathcal{A}_n)$                                     | $\dim$ or $ \cdot $ | Gauge group (IIB-II)           |
|-----|--------------|-------------------------------------------------------------------|---------------------|--------------------------------|
| 0   | $\mathbb{R}$ | $\{\mathrm{id}\}$                                                 | 1                   | —                              |
| 1   | $\mathbb{C}$ | $\mathbb{Z}/2\mathbb{Z}$                                          | 2                   | $\mathrm{U}(1)$                |
| 2   | $\mathbb{H}$ | $\mathrm{SO}(3) \cong \mathrm{Sp}(1)/\mathbb{Z}_2$                | 3                   | $\mathrm{SU}(2)$               |
| 3   | $\mathbb{O}$ | $G_2$ (exceptional)                                               | 14                  | $G_2 \supset \mathrm{SU}(3)_C$ |
| 4   | $\mathbb{S}$ | $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ | 2688 (discrete)     | ZD structure                   |

The embedding chain  $S_4 \subset \mathrm{PSL}(2, 7) \subset \mathrm{SU}(3) \subset G_2 \subset \mathrm{SO}(7)$  was established in Paper 10 [5], Remark (rem:full-chain). The  $\mathcal{A}_3 \rightarrow \mathcal{A}_4$  transition does not create  $\mathrm{PSL}(2, 7)$ : it transforms it from its continuous  $G_2$  embedding to its role as symmetry of  $\mathrm{ZD}^*(\mathcal{A}_4)$ .

### 3.2. The spectral transition theorem.

**Theorem 3.1 ([Derived]).** Let  $a \in \mathcal{A}_k$  be a unit ( $N(a) = 1$ ) and  $M_a := L_a^\top L_a$  its Gram operator.

(i) At  $n \leq 3$ :  $N(ab) = N(b)$  for all  $b$ , so  $L_a$  is an  $N$ -isometry and  $M_a = \mathrm{Id}_{2^k}$ ; hence

$$\mathrm{Spec}(M_a)|_{n \leq 3} = \{1^{2^k}\}.$$

$\ker(L_a) = 0$  for all nonzero  $a$ ; no dark sector; no amplification.

(ii) At  $n = 4$  ([2] Thm. 5.8; [7] Thm. 3.1):

$$\mathrm{Spec}(M_a)|_{n=4} = \{0^4, 1^8, 2^4\} \quad \text{for every ZD unit } a.$$

*Remark 3.2 ([Observation]).* The spectral transition  $\{1^{2^k}\} \rightarrow \{0^4, 1^8, 2^4\}$  at  $n = 3 \rightarrow 4$  is the algebraic signature of the gauge-to-gravity transition: at  $n \leq 3$ , every unit multiplication is a norm isometry (no dark sector, no amplification); at  $n = 4$ , multiplication can annihilate ( $V_0$ ), preserve ( $V_1$ ), or amplify ( $V_2$ ) the norm. The gravitational degree of freedom is  $V_0$ : algebraically absent at  $n \leq 3$ , algebraically present at  $n = 4$ .

### 3.3. Spectral invariants at $n \leq 3$ .

**Proposition 3.3** (*[Derived]* for (i); *[Conjecture, IIB-II-strong]* for (ii)). (i)  $\dim \ker(L_a) = 0$  at all Hurwitz levels, forcing any fermionic correction to the spectral invariant  $\mathcal{S}_k$  to zero.

(ii) *[Conjecture, IIB-II-strong]*: if the Sakharov–Adler mechanism at level  $k$  is structurally analogous to  $n = 4$  (scalar field on  $\text{Aut}(\mathcal{A}_k)$ , spectral sum over adjoint), then

$$\mathcal{S}_k = \dim \text{Aut}(\mathcal{A}_k), \quad \mathcal{S}_3 = 14, \quad \mathcal{S}_2 = 3, \quad \mathcal{S}_1 = 2.$$

*Remark 3.4.* The two claims are distinct in strength: (i) is *[Derived]* from Hurwitz (Theorem 3.1(i)); (ii) requires the conjecture that the spectral sum over  $\text{Aut}(\mathcal{A}_k)$  gives exactly  $\dim \text{Aut}(\mathcal{A}_k)$ , which is the content of the proposed mechanism and is not derived.

## 4. THE GENERALISED SAKHAROV–ADLER MECHANISM

**4.1. Review at  $n = 4$ .** At  $n = 4$ , using the Sakharov–Adler induced gravity mechanism [12, 13], Paper 26 [6] Theorem 4.2:

$$G_N^{-1} = \frac{16\lambda N(C_0)}{12\pi} [\mathcal{S}_{\text{eff}} \ln(\Lambda^2/16\lambda N(C_0)) - \text{bos.} - \text{ferm.}],$$

with  $\mathcal{S}_{\text{eff}} = 164 = |\text{GL}(3, \mathbb{F}_2)| - \dim \ker(L_a)|_{n=4}$ .

### 4.2. Proposed mechanism at $n = 1, 2, 3$ .

*Conjecture 4.1* (*[Conjecture, IIB-II-strong]*). At level  $n = k$  ( $k = 1, 2, 3$ ), a scalar field  $\Phi_k$  valued in (or coupled to)  $\text{Aut}(\mathcal{A}_k)$  generates, via the Sakharov–Adler one-loop mechanism, a gauge kinetic term:

$$g_k^{-2} \propto \mathcal{S}_k \cdot \ln(\Lambda^2/\mu_k^2),$$

with  $\mathcal{S}_k = \dim \text{Aut}(\mathcal{A}_k)$  (Proposition 3.3(ii)). The algebraic chain is:

$$\begin{aligned} n = 1: \quad \mathbb{C} &\rightarrow \Phi_1 \text{ on } U(1) \supset \text{Aut}(\mathbb{C}) && \rightarrow g_1^2 \\ n = 2: \quad \mathbb{H} &\rightarrow \Phi_2 \text{ on } SO(3) \cong \text{Aut}(\mathbb{H}) && \rightarrow g_2^2 \\ n = 3: \quad \mathbb{O} &\rightarrow \Phi_3 \text{ on } G_2 = \text{Aut}(\mathbb{O}) && \rightarrow g_3^2 \\ n = 4: \quad \mathbb{S} &\rightarrow \Phi_4 \text{ on } \text{ZD}^*(\mathcal{A}_4) && \rightarrow G_N. \end{aligned}$$

*Remark 4.2* (Gauge group vs. algebra automorphism). At  $n = 1$ :  $\text{Aut}(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$  (complex conjugation, a discrete group of order 2). The  $U(1)$  gauge group of electromagnetism acts on  $\mathbb{C}$  as phase rotations  $z \mapsto e^{i\theta}z$ , preserving  $N(z) = |z|^2$  but not the Cayley–Dickson algebra structure for  $\theta \notin \{0, \pi\}$ .  $U(1)$  is therefore NOT a subgroup of  $\text{Aut}(\mathbb{C})$ . The identification is rather:  $U(1)$  is the connected component of the group of  $N$ -isometries of  $\mathbb{C}$  that commutes with the  $\mathbb{R}$ -linear structure, i.e.  $SO(2) \cong U(1)$ . This group acts on the imaginary part  $\text{Im}(\mathbb{C}) \cong \mathbb{R}$  and does not extend to  $\text{Aut}(\mathbb{H}) = SO(3)$  as a subgroup (since  $U(1) \not\subset SO(3)$  in any compatible sense at the algebra level). The conjectured level- $k$  matter field  $\Phi_k$  runs over the Lie group  $\text{Aut}(\mathcal{A}_k)$ ; the physical gauge group is its connected cover.

### 4.3. Gauge groups stop at their native level *[Derived]*.

**Proposition 4.3** (*[Derived]*, Paper 10 [5] Rem. (full-chain); Paper 2 [2]). *Each gauge group stops at its native algebra level:*

- $SU(2) \not\subset \text{Aut}(\mathbb{O}) = G_2$  as a compatible subgroup:  $G_2$  contains  $SU(3)$  but its  $SU(2)$  subgroups do not descend from the  $n = 2$  quaternionic structure.
- $G_2 \not\subset \text{Aut}(\mathbb{S})$ :  $\mathbb{S}$  is not alternative, so the octonionic automorphism group has no embedding in  $\text{Aut}(\mathbb{S})$ .

*Remark 4.4 ([Observation]).* The stopping is encoded in  $Z(x)$ : the pole at  $x = 2^{-k}$  marks the level where the  $k$ -th force ceases to embed in the next automorphism group. The residues  $R_k$  are the analytic weights of these stopping points.

#### 4.4. AMT: exact at $n \leq 3$ , average at $n \geq 4$ .

**Proposition 4.5** (*[Derived]*). (i) At  $n \leq 3$ :  $N(av) = N(a)N(v)$  for all  $v$  (Hurwitz). Hence  $\mathbb{E}_v[N(av)] = N(a) \cdot \mathbb{E}_v[N(v)]$  point-by-point, not just on average.

(ii) At  $n \geq 4$ :  $N(av) \neq N(a)N(v)$  on ZD pairs. Paper 2 [2] (AMT, Thm. 5.3):  $\mathbb{E}_v[N(av)] = N(a)$  holds only on average over a uniform distribution on unit sphere.

### 5. NUMERICAL CANDIDATES FOR $g_1^2, g_2^2, g_3^2$

| Ratio       | Value                 | Comment              |
|-------------|-----------------------|----------------------|
| $R_0/R_1$   | $-16/7$               |                      |
| $ R_1/R_2 $ | $8 = \dim \mathbb{O}$ | <i>[Observation]</i> |
| $R_2/R_3$   | $-28$                 |                      |
| $ R_0/R_3 $ | $512 = 2^9$           |                      |

At the Planck scale, the SM gauge couplings satisfy approximately  $\alpha_1 \approx 0.017$ ,  $\alpha_2 \approx 0.034$ ,  $\alpha_3 \approx 0.024$ , giving  $g_1^2/g_2^2 \approx 0.50$ ,  $g_2^2/g_3^2 \approx 1.42$ . The residue ratios differ by orders of magnitude from these values.

**No numerical identification is claimed.** Connecting  $R_k$  to  $g_k^2$  requires specifying  $\Phi_k$  and executing the full loop integral (OP 33.1), then running from  $\Lambda$  to observable energies (OP 33.2).

*Observation 5.1 ([Observation]).*  $|R_1/R_2| = 8 = \dim \mathbb{O}$ : the EM/weak residue ratio equals the octonionic dimension. Under IIB-II-strong *[Speculation, IIB-II-strong]*, the strong coupling (level  $n = 3$ ,  $\mathbb{O}$ ) is “encoded” in the ratio of the electroweak residues.

### 6. FANO BIFRONTALITY AS THE TRANSITION MECHANISM

Paper 32 [8] established the Fano bifrontality: at  $n \leq 3$ , the Fano plane organises multiplication (one face); at  $n = 4$ , it organises annihilation (second face). The spectral transition theorem (§3) is the spectral expression of the same passage: multiplication face  $\rightarrow \text{Spec}(M_a) = \{1^{2^k}\}$ ; annihilation face  $\rightarrow \text{Spec}(M_a) = \{0^4, 1^8, 2^4\}$ .

*Conjecture 6.1 ([Conjecture, IIB-II-strong]).* Under IIB-II-strong, the bifrontality is the complete mechanism of the gauge-to-gravity transition: the Fano plane in its multiplication face supports the three gauge forces via the  $\text{Aut}(\mathcal{A}_k)$  loop; the same Fano plane in its annihilation face generates gravity via  $\text{ZD}^*(\mathcal{A}_4)$ . The single combinatorial object has two faces, each supporting one sector of fundamental interactions.

### 7. OPEN PROBLEMS

**OP 33.1** *[Open]*. Identification of the matter field  $\Phi_k$  at each level  $n = k$ : what scalar (or fermion) field on  $\text{Aut}(\mathcal{A}_k)$  executes the Sakharov–Adler loop? The physical gauge group ( $\text{U}(1)$ ,  $\text{SU}(2)$ ,  $G_2$ ) must be derived, not assumed.

**OP 33.2** *[Open]*. Derivation of the renormalization group running from  $\Lambda_{\text{Pl}}$  to observable energies  $\mu \sim M_Z$ , converting the Planck-scale residue ratios to the physical coupling ratios.

**OP 33.3** *[Open]*. Algebraic proof that  $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$ . Specifically: why does the  $\mathbb{H}$ -pole (level  $k = 2$ ) carry  $s_2 = \dim G_2 = \dim \text{Aut}(\mathbb{O})$  rather than  $\dim \text{Aut}(\mathbb{H}) = 3$ ? Is there a  $G_2$  Clebsch–Gordan or cohomological explanation?

## 8. CONCLUSIONS

The four residues  $R_k$  of  $Z(x)$  are the analytic signatures of the four Hurwitz levels in the global zero-divisor counting function. The rescaled residues  $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$  carry the dimensions of  $\mathbb{R}$ , the Fano plane,  $G_2$ , and  $\mathbb{O}$  respectively. Their alternating sum is zero by the residue theorem on  $\mathbb{P}^1$ . The identity  $s_2 = 2s_1$  is a structural fact about  $G_2 \subset \mathrm{SO}(7)$ .

The spectral transition  $\mathrm{Spec}(M_a) : \{1^{2^k}\} \rightarrow \{0^4, 1^8, 2^4\}$  at  $n = 3 \rightarrow 4$  (*[Derived]*) is the algebraic signature of the gauge-to-gravity transition in spectral language.

Under IIB-II-strong [*Speculation, IIB-II-strong*], the Sakharov–Adler mechanism at level  $k$  runs over  $\mathrm{Aut}(\mathcal{A}_k)$  with  $\mathcal{S}_k = \dim \mathrm{Aut}(\mathcal{A}_k)$  [*Conjecture, IIB-II-strong*], completing the tower from gauge forces at  $n = 1, 2, 3$  to gravity at  $n = 4$  via the Fano bifrontality.

## ACKNOWLEDGEMENTS

Computational assistance provided by Claude (Anthropic). The author takes full responsibility for any errors.

## REFERENCES

- [1] G. Levratti, *Sign-Parity Classification of Zero-Divisors in Cayley–Dickson Algebras*, v2, Zenodo (2026), [doi:10.5281/zenodo.20315751](https://doi.org/10.5281/zenodo.20315751).
- [2] G. Levratti, *The Informational Identity Boundary*, v5, Zenodo (2026), [doi:10.5281/zenodo.20611877](https://doi.org/10.5281/zenodo.20611877).
- [3] G. Levratti, *Sign Structure, Automorphism Groups, and the ZD Zeta Function*, v4, Zenodo (2026), [doi:10.5281/zenodo.20397587](https://doi.org/10.5281/zenodo.20397587).
- [4] G. Levratti, *Fano Geometry, Automorphisms, and the Weil Zeta Function*, v10, Zenodo (2026), [doi:10.5281/zenodo.20417859](https://doi.org/10.5281/zenodo.20417859).
- [5] G. Levratti, *Loop, Threshold, and Residue: The Algebraic Structure of the  $\mathcal{A}_3 \leftrightarrow \mathcal{A}_4$  Transition*, v1, Zenodo (2026), [doi:10.5281/zenodo.20507204](https://doi.org/10.5281/zenodo.20507204).
- [6] G. Levratti, *Induced Gravity from the Sedenion Scalar Field Theory  $S_{84}$* , v1, Zenodo (2026), [doi:10.5281/zenodo.20693540](https://doi.org/10.5281/zenodo.20693540).
- [7] G. Levratti, *The Missing Four: Fano Non-Incidence in the Sedenion Zero-Divisor Algebra*, v1, Zenodo (2026), [doi:10.5281/zenodo.20720965](https://doi.org/10.5281/zenodo.20720965).
- [8] G. Levratti, *The Fano Bifrontality: Gravity and Matter at the Cayley–Dickson Boundary*, v2, Zenodo (2026), [doi:10.5281/zenodo.20746352](https://doi.org/10.5281/zenodo.20746352).
- [9] C. Furey, Standard Model physics from an algebra? *Ph.D. thesis, University of Waterloo* (2015); see also *Eur. Phys. J. C* **78** (2018), 375, [doi:10.1140/epjc/s10052-018-5844-7](https://doi.org/10.1140/epjc/s10052-018-5844-7).
- [10] C. Furey, Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra, *Phys. Lett. B* **785** (2018), 84, [doi:10.1016/j.physletb.2018.08.032](https://doi.org/10.1016/j.physletb.2018.08.032).
- [11] J. C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002), 145–205, [doi:10.1090/S0273-0979-01-00934-X](https://doi.org/10.1090/S0273-0979-01-00934-X).
- [12] A. D. Sakharov, Vacuum quantum fluctuations in curved space and the theory of gravitation, *Sov. Phys. Dokl.* **12** (1968), 1040.
- [13] V. P. Frolov and D. V. Fursaev, Thermal fields, entropy, and black holes, *Class. Quant. Grav.* **15** (1998), 2041, [doi:10.1088/0264-9381/15/8/001](https://doi.org/10.1088/0264-9381/15/8/001).
- [14] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer (1972).
- [15] T. A. Springer and F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional Groups*, Springer (2000).

INDEPENDENT RESEARCHER, MODENA, ITALY

Email address: [giovanni.levratti.research@gmail.com](mailto:giovanni.levratti.research@gmail.com)