

A SINGLE FANO PLANE: ZERO-DIVISORS, GAUGE STRUCTURE, AND THE MASSLESS SECTOR

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ABSTRACT. The Fano plane $\text{PG}(2, \mathbb{F}_2)$, with its $\text{GL}(3, \mathbb{F}_2)$ -symmetry, is the universal local structure of the Cayley–Dickson zero-divisor tower: every ZD pair at every level belongs to exactly one Fano-plane block carrying the same standard action (Paper 4, Fano universality corollary). This single combinatorial fact cascades into seven structural results assembled here for the first time in a unified analysis.

We prove that the normalized residues s_k of the ZD zeta function $Z(x)$ at the four Hurwitz poles encode the structural dimensions of the last normed division algebra $\mathcal{A}_3 = \mathbb{O}$: $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$ equal $\dim \mathbb{R}$, $|\text{PG}(2, \mathbb{F}_2)|$, $\dim G_2 = \dim \text{Aut}(\mathbb{O})$, $\dim \mathbb{O}$ (OP 33.3, *[Derived]* for each individual identification; *[Observation]* gap for the full G_2 -categorical proof).

We show that the Level-2 massless sector ($\mu = -2$) is the unique eigenvalue for which the Fano bipartite identity $\sum_{\mu} (2+\mu)m(\mu) = 2|\text{ZD}^*(\mathcal{A}_4)| = 168 = |\text{GL}(3, \mathbb{F}_2)|$ implies zero contribution to the gravitational spectral sum, addressing OP 24.5 with a new *[Derived]* route; the derivation of $\mu = -2$ itself from lo-flat Fano structure remains *[Conjecture, IIB-II-strong]*.

The spectral invariant $\mathcal{S} = 168 = |\text{GL}(3, \mathbb{F}_2)|$ equals the order of the Fano plane automorphism group, identified here as a new structural observation. $Z(x)$ is identified as the mathematical bridge between the two faces of the Fano bifrontality (Papers 32–33): its numerator encodes the annihilation face; its Z_{Weil} factor encodes the Hurwitz multiplication face via Schubert stratification.

G_2 appears at the $\mathbb{H}$ -pole of $Z(x)$ because $G_2 = \text{Aut}(\mathbb{O})$ simultaneously contains the gauge groups of levels $n = 2$ and $n = 3$ via $S_4 \subset \text{PSL}(2, 7) \subset \text{SU}(3) \subset G_2 \subset \text{SO}(7)$. New observations: $s_k = m(\mu)$ for $k = 1, 2$ bridges the analytic residue structure of $Z(x)$ and the adjacency spectrum of G_{84} ; Level-3 contributes exactly $|\text{ZD}^*(\mathcal{A}_4)| = 84$ to the bosonic spectral sum; $\mathcal{S} = 7 \cdot 24 = |\text{Fano}| \cdot |S_4|$ is a new factorization; and $m(-2) = 14 = \dim G_2$ is numerical only — not a G_2 -representation identity (Paper 21).

Epistemic key. *[Derived]* complete proof. *[Observation]* algebraically established pattern, no deeper categorical proof. *[Conjecture, IIB-II-strong]* mechanism proposed, gap identified. *[Speculation, IIB-II-strong]* structurally motivated. *[Open]* obstacle stated precisely.

1. INTRODUCTION: ONE OBJECT, ONE TOWER

1.1. The Fano universality corollary.

Theorem 1.1 (*[Derived]*, Paper 4 [4] Fano universality corollary). *The Fano plane $\text{PG}(2, \mathbb{F}_2)$ with its $\text{GL}(3, \mathbb{F}_2)$ -symmetry is the universal local structure of the zero-divisor tower: every ZD pair at every level belongs to exactly one Fano-plane block, and the local $\text{GL}(3, \mathbb{F}_2)$ -action is always the standard Fano action. The counting formula $|\text{ZD}(n)| = 336 \cdot \binom{n-1}{3}_2$ is the statement that $\text{ZD}(n)$ consists of exactly $\binom{n-1}{3}_2$ Fano-plane blocks, one per 3-dimensional subspace of $\mathbb{F}_2^{n-1}$, so $\text{ZD}(n)$ is a rank-336 bundle over the Grassmannian $G(3, n-1; \mathbb{F}_2)$.*

1.2. Seven cascades.

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Cascade	Label	§
Weil bridge: $Z(x)$ as face-connector	<i>[Observation]</i>	2
Schubert strata $\leftrightarrow$ Hurwitz forces	<i>[Observation]</i>	2
G_2 -theoretic residues $\{s_k\} = \{1, 7, 14, 8\}$ (OP 33.3)	<i>[Derived]</i>	3
G_2 as cross-level unifier (OP 34.1)	<i>[Observation]</i>	4
Massless Level-2 from $\mu = -2$ (OP 24.5)	<i>[Derived]</i>	5
$\mathcal{S} = \text{Aut}(\text{Fano}) $ new identity	<i>[Observation]</i>	5
OP 29.1 cascade mapped	<i>[Observation]</i>	6
$s_k = m(\mu)$, $k = 1, 2$: bridge $Z(x) \leftrightarrow G_{84}$ spec.	<i>[Observation]</i>	3
Level-3 contributes $ \text{ZD}^*(\mathcal{A}_4) = 84$ to spectral sum	<i>[Observation]</i>	5

Results from Papers 3, 4, 10, 11, 22-B, 24, 26, 28, 29 are used as inputs and cited without reproof. All new derivations appear with step-by-step proofs.

2. THE WEIL BRIDGE: $Z(x)$ AS FACE-CONNECTOR

2.1. Fano bijection and Schubert stratification.

Proposition 2.1 (*[Derived]*, Paper 4 [4]). *For $n \geq 5$, the Fano bijection $f: \text{ZD}^*(n) \rightarrow G(3, n-1; \mathbb{F}_2)$, $f(z) = \text{span}\{\pi(a_i), \pi(a_j), \pi(b_i)\}$, is $\text{Aut}(\mathcal{A}_n)$ -equivariant with uniform fibres $|f^{-1}(W)| = 336$ for every $W \in G(3, n-1; \mathbb{F}_2)$.*

The Schubert stratification of $G(3, n-1; \mathbb{F}_2)$ by $\dim(\pi_{10}(f(z))) \in \{0, 1, 2, 3\}$ stratifies $\text{ZD}(n)$ into four orbit types:

$\dim(\pi_{10})$	Orbit size	Frobenius eigenvalue	Hurwitz algebra
0	2	$1 = \dim \mathbb{R}$	$\mathbb{R}$
1	14	$2 = \dim \mathbb{C}$	$\mathbb{C}$
2	84	$4 = \dim \mathbb{H}$	$\mathbb{H}$
3	336	$8 = \dim \mathbb{O}$	$\mathbb{O}$

Observation 2.2 (*[Observation]*). The Schubert strata of $G(3, n-1; \mathbb{F}_2)$ correspond bijectively to the four Hurwitz algebras $\mathcal{A}_0, \dots, \mathcal{A}_3$, with Frobenius eigenvalue equal to the algebra dimension. Under IIB-II *[Speculation, IIB-II-strong]* [1], each stratum carries the ZD pairs whose local symmetry is controlled by the corresponding Hurwitz algebra.

2.2. The Weil identity and its two faces.

Proposition 2.3 (*[Derived]*, Paper 4 [4]). *As formal rational functions:*

$$Z(x) = 336x^4 \cdot Z_{\text{Weil}}(\text{PG}(3, \mathbb{F}_2), x), \quad Z_{\text{Weil}} = \frac{1}{(1-x)(1-2x)(1-4x)(1-8x)}.$$

The functional equation $Z(x) = 64x^4 Z(1/8x)$ is Poincaré duality of $G(3, n-1; \mathbb{F}_2)$.

Observation 2.4 (*[Observation]*). $Z(x)$ is the mathematical bridge between the two faces of the Fano bifrontality (Papers 32–33 [12, 13]), stated explicitly here for the first time:

- *Annihilation face* (Paper 32): the numerator $336x^4$ counts the ZD pairs of $\mathcal{A}_4$. The factor $336 = 2|\mathrm{PSL}(2, 7)| = 2|\mathrm{GL}(3, \mathbb{F}_2)|$ encodes the ZD structure of $\mathcal{A}_4$ [2, 3], with $\mathrm{GL}(3, \mathbb{F}_2)$ controlling the 84 active ZD elements via the Fano bijection.
- *Multiplication face* (Hurwitz tower, Paper 33): the factor $Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$ encodes the Frobenius structure of the Hurwitz tower via the Schubert stratification of Observation 2.2.

Observation 2.5 ([Observation]). The functional equation pairs $(\mathbb{C}, \mathbb{H})$ (tight: $|R_1/R_2| = 8 = \dim \mathbb{O}$, Paper 33 [13] Obs. 2.4) and $(\mathbb{R}, \mathbb{O})$ (loose: $|R_0/R_3| = 512$). Under IIB-II-strong, $(\mathbb{C}, \mathbb{H})$ is the electroweak pair, encoded as the tight pair of $Z(x)$.

3. THE G_2 -THEORETIC RESIDUE PATTERN: OP 33.3

3.1. Residues and normalization. Recall from Paper 33 [13]: the residues of $Z(x)$ at $x_k = 2^{-k}$ are $R_0 = 16, R_1 = -7, R_2 = 7/8, R_3 = -1/32$ with normalized form $\sigma_k = R_k \cdot (2^k)^4 = 16(-1)^k s_k$ and $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$.

Lemma 3.1 ([Derived]). $R_k = 336 \cdot 2^{-4k} / \prod_{j \neq k} (1 - 2^{j-k})$ for $k = 0, 1, 2, 3$. Hence $s_k = 336 / (16 \cdot |\prod_{j \neq k} (1 - 2^{j-k})|)$.

3.2. The $\mathcal{A}_3$ -memory theorem.

Theorem 3.2 ([Derived]). Each normalized residue s_k encodes a structural dimension of the last normed division algebra $\mathcal{A}_3 = \mathbb{O}$:

- (i) $s_0 = 1 = \dim(\mathcal{A}_0) = \dim \mathbb{R}$.
- (ii) $s_1 = 7 = |\mathrm{PG}(2, \mathbb{F}_2)| = \dim \mathrm{Im}(\mathbb{O})$. This follows from $336 = 48 \cdot 7$ where $48 = 6 \cdot \dim \mathbb{O}$, with the factor 7 being the Fano point count ([3, 4]).
- (iii) $s_2 = 14 = \dim G_2 = \dim \mathrm{Aut}(\mathbb{O})$. This follows from $s_2 = 2s_1 = 14$ together with $\dim(\mathrm{SO}(7)/G_2) = 21 - 14 = 7 = |\mathrm{Im}(\mathbb{O})|$ ([13, 14] Rem. 2.7).
- (iv) $s_3 = 8 = \dim(\mathcal{A}_3) = \dim \mathbb{O}$. From Lemma 3.1: $s_3 = 336 / (16 \cdot |\prod_{j=0,1,2} (1 - 2^{j-3})|) = 8$.

The three non-trivial values s_1, s_2, s_3 are all structural dimensions of $\mathbb{O}$: $\dim \mathrm{Im}(\mathbb{O})$, $\dim \mathrm{Aut}(\mathbb{O})$, $\dim \mathbb{O}$.

Remark 3.3 ([Observation]). The sequence $(s_0, s_1, s_2, s_3) = (1, 7, 14, 8)$ encodes the G_2 -module dimensions: $s_0 = \dim(\mathbf{1}_{G_2})$ (trivial), $s_1 = \dim(V_7)$ (fundamental), $s_2 = \dim(\mathfrak{g}_2)$ (adjoint), $s_3 = \dim(\mathbf{1} \oplus V_7) = \dim \mathbb{O}$. A categorical proof that the Weil factorization $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}$ necessarily extracts these G_2 -representation dimensions via the $\mathrm{PSL}(2, 7) \subset G_2$ embedding would close OP 33.3 completely.

Observation 3.4 ([Observation]). For $k = 1, 2$, the normalized residue s_k equals the spectral multiplicity $m(\mu)$ at the corresponding negative level:

$$s_1 = 7 = m(\mu = -4), \quad s_2 = 14 = m(\mu = -2).$$

Both values are structural dimensions of $\mathbb{O}$ (Theorem 3.2): $s_1 = |\mathrm{Im}(\mathbb{O})| = \dim(\mathrm{irr}_7)$ and $s_2 = \dim G_2 = \dim(\mathrm{irr}_6 \oplus \mathrm{irr}_8)$. The pattern does not extend to $k = 0, 3$ ($s_3 = 8 \neq m(0) = 42$). This bridges two independent algebraic objects: the analytic structure of $Z(x)$ (Paper 3 [3]) and the adjacency spectrum of G_{84} (Paper 11 [6]). The deeper reason is the content of OP 34.1.

4. G_2 AS CROSS-LEVEL UNIFIER

Observation 4.1 ([Observation]). The cross-level appearance $s_2 = \dim G_2 = 14$ at the $\mathbb{H}$ -pole ($k = 2$) rather than $\dim \mathrm{Aut}(\mathbb{H}) = 3$ has a structural explanation: G_2 is the first exceptional Lie group simultaneously containing $\mathrm{SU}(2)$ (gauge group at $n = 2$) and $\mathrm{SU}(3)_C$ (gauge group at $n = 3$), via the embedding chain [5, 16]:

$$S_4 \subset \mathrm{PSL}(2, 7) \subset \mathrm{SU}(3) \subset G_2 \subset \mathrm{SO}(7).$$

The $\mathbb{H}$ -residue “anticipates” G_2 because G_2 unifies the gauge structures of the two adjacent Hurwitz levels $\mathcal{A}_2$ and $\mathcal{A}_3$.

Remark 4.2. Moreover, the transition $\mathcal{A}_3 \rightarrow \mathcal{A}_4$ does not create $\mathrm{PSL}(2, 7)$: it relocates $\mathrm{PSL}(2, 7)$ from its continuous G_2 -embedding to the discrete symmetry $\mathrm{Aut}(\mathcal{A}_4) = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ of the ZD structure [5]. The residue $s_2 = \dim G_2$ records the last level at which $\mathrm{PSL}(2, 7)$ resided inside a continuous Lie group.

Remark 4.3 (Relation to Furey’s programme). The identification of $\mathrm{SU}(2)$ and $\mathrm{SU}(3)_C$ with levels $n = 2$ and $n = 3$ of the Cayley–Dickson tower is in the spirit of Furey’s programme [15], which derives the gauge *symmetry groups* $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_L \times \mathrm{SU}(3)_C$ from the algebraic structure of $\mathcal{A}_3 = \mathbb{O}$. The present paper does not derive gauge groups: it identifies a structural correspondence between Schubert strata and Hurwitz algebras under IIB-II [1] ([*Speculation, IIB-II-strong*]), and notes that $G_2 = \mathrm{Aut}(\mathbb{O})$ unifies the gauge group levels $n = 2, 3$ as its continuous subgroups $\mathrm{SU}(2)$ and $\mathrm{SU}(3)_C \subset G_2$. Paper 33 [13] addresses the distinct question of gauge coupling *constants*.

Remark 4.4 ([*Derived-Negative*], Paper 21 [7]). The coincidence $m(-2) = 14 = \dim G_2$ is *numerical only*. Paper 21 established by GAP 4.15.1: $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)} \cong \mathrm{irr}_3 \oplus \mathrm{irr}_{3^*} \oplus \mathrm{irr}_8$ ($3+3+8 = 14$), while $\mathrm{Level-2} = \mathrm{irr}_6 \oplus \mathrm{irr}_8$ ($6+8 = 14$). The two 14-dimensional $\mathrm{PSL}(2, 7)$ -modules are *not isomorphic*: the tachyonic representation irr_7 present in $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)}$ (via the irr_7 embedding of $\mathrm{PSL}(2, 7) \subset G_2$) is absent from $\mathrm{Level-2}$ [7]. The coincidence of dimensions does not imply a G_2 -representation identity; the content of OP 34.1 is to explain it analytically.

Remark 4.5 ([*Observation*]). Paper 4 [4] establishes: $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7)$ is a maximal subgroup of $G_2(\mathbb{F}_2)' = \mathrm{PSU}(3, 3)$ (the derived subgroup of G_2 over $\mathbb{F}_2$), with index 36. Furthermore, $336 = |\mathrm{SL}(3, 2):2| \leq G_2(\mathbb{F}_2)$: the ZD count at $n = 4$ equals the order of a specific subgroup of G_2 over the *finite* field $\mathbb{F}_2$. The 36 cosets of $\mathrm{GL}(3, \mathbb{F}_2)$ in $\mathrm{PSU}(3, 3)$ represent 36 “octonion-compatible Fano-plane structures” (Paper 4, §8). This finite-field embedding $\mathrm{GL}(3, \mathbb{F}_2) \subset G_2(\mathbb{F}_2)$ is the algebro-geometric counterpart of the continuous embedding $\mathrm{PSL}(2, 7) \subset \mathrm{SU}(3) \subset G_2$ (Observation 4.1). It offers a concrete path toward OP 34.1: the residue of $Z(x)$ at the $\mathbb{H}$ -pole ($k = 2$) carries $s_2 = \dim G_2 = 14$ because $G_2(\mathbb{F}_2)$ contains $\mathrm{GL}(3, \mathbb{F}_2)$ as a maximal subgroup with the count $336 = 2|\mathrm{GL}(3, \mathbb{F}_2)|$ as the orbit size.

Definition 4.6 ([*Open*]). **OP 34.1.** Prove algebraically that $s_2 = \dim G_2$ at the $\mathbb{H}$ -pole follows from $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}$ and the embedding $\mathrm{PSL}(2, 7) \subset G_2$. Required: Schubert cell structure of $\mathrm{PG}(3, \mathbb{F}_2)$, the chain $\mathrm{PSL}(2, 7) \subset \mathrm{SU}(3) \subset G_2$, and a G_2 -equivariant interpretation of the Frobenius eigenvalue $4 = \dim \mathcal{A}_2$ at the $\mathbb{H}$ -Schubert stratum.

5. THE SPECTRAL ORIGIN OF $\mu = -2$: OP 24.5

5.1. Setup.

Proposition 5.1 ([*Derived*], Papers 11, 29 [6, 11]). *The adjacency spectrum of $\mathrm{ZD}^*(\mathcal{A}_4)$ (the 84-vertex ZD pair graph G_{84}) is $\mu \in \{-4, -2, 0, +2, +4\}$ with multiplicities $\{7, 14, 42, 14, 7\}$. The graph has 7 connected components, each a 4-regular bipartite graph on 12 vertices ($6+6$), so $m(\mu) = m(-\mu)$ for all μ .*

The mass formula for S_{84} in the Black Circle background (Paper 11 [6]) is $m_i^2 = 16\lambda C_0^2(2 + \mu_i)$. Masslessness requires $\mu = -2$, the *unique* spectral value with $(2 + \mu) = 0$.

5.2. The Fano bipartite identity.

Theorem 5.2 ([*Derived*]).

$$\sum_{\mu} (2 + \mu) m(\mu) = 2 |\mathrm{ZD}^*(\mathcal{A}_4)| = 2 \cdot 84 = 168 = |\mathrm{GL}(3, \mathbb{F}_2)|.$$

Proof. Decompose: $\sum_{\mu}(2+\mu)m(\mu) = 2\sum_{\mu}m(\mu) + \sum_{\mu}\mu m(\mu)$. The first sum: $2 \cdot 84 = 168$. The second sum: $\sum_{\mu}\mu m(\mu) = \text{Tr}(A) = 0$, since G_{84} is a simple graph (zero diagonal). Substituting: $\sum_{\mu}(2+\mu)m(\mu) = 168$. The equality $168 = |\text{GL}(3, \mathbb{F}_2)|$ follows from $|\text{ZD}^*(\mathcal{A}_4)| = 84$ ([2, 3]) and $168 = 2 \cdot 84 = |\text{GL}(3, \mathbb{F}_2)|$ ([3, 4]). ✓ $\square$

Remark 5.3. The argument uses $\text{Tr}(A) = 0$ for any simple graph. Bipartiteness (Paper 29 [11] Prop. 5.1) gives the stronger $m(\mu) = m(-\mu)$, confirming the symmetric spectrum of Proposition 5.1, but is not needed for Theorem 5.2 itself.

Remark 5.4 ([Observation]). The per-level decomposition of the spectral sum reveals a further Fano interpretation:

Level	μ	$m(\mu)$	$(2+\mu)m(\mu)$	Fano reading
1	-4	7	-14	$-2 \cdot \text{Fano} $
2	-2	14	0	massless
3	0	42	+84	$ \text{ZD}^*(\mathcal{A}_4) $
4	+2	14	+56	$4 \cdot \dim G_2$
5	+4	7	+42	$6 \cdot \text{Fano} $

Level-3 ($\mu = 0$) contributes exactly $|\text{ZD}^*(\mathcal{A}_4)| = 84$: $(2+0) \cdot 42 = 84$. Under IIB-II [*Speculation, IIB-II-strong*], Level-3 is the *ZD-counting level*: its gravitational contribution to the bosonic spectral sum equals the active ZD vertex count.

Observation 5.5 ([Observation]). **The spectral invariant equals the Fano automorphism count.** $\mathcal{S} := \sum_{\mu}(2+\mu)m(\mu) = 168 = |\text{GL}(3, \mathbb{F}_2)| = |\text{Aut}(\text{PG}(2, \mathbb{F}_2))|$. The spectral invariant of the quantum field theory S_{84} equals the order of the automorphism group of the Fano plane. This follows from $|\text{ZD}^*(\mathcal{A}_4)| = 84$ [2] and $2 \cdot 84 = 168 = |\text{GL}(3, \mathbb{F}_2)|$ [3, 4]: the graph G_{84} is 4-regular with 168 edges ([4]), and $336 = 2 \cdot 168 = 2|\text{GL}(3, \mathbb{F}_2)|$ is the Weil numerator.

Remark 5.6 ([Observation]). The spectral invariant $\mathcal{S} = 168$ admits a product factorization:

$$\mathcal{S} = 168 = 7 \cdot 24 = |\text{PG}(2, \mathbb{F}_2)| \cdot |S_4|,$$

where $S_4 \subset \text{PSL}(2, 7)$ is the maximal subgroup of order 24 (point-stabiliser of $\text{PSL}(2, 7)$ acting on the 7 Fano points). The spectral invariant is the product of the Fano point count and the Fano point stabiliser order.

Corollary 5.7 ([Derived]). *The Level-2 sector ($\mu = -2$, multiplicity 14) contributes exactly zero to the spectral sum: $(2+\mu_2) \cdot m(-2) = 0 \cdot 14 = 0$. This is the unique eigenvalue in the spectrum $\{-4, -2, 0, +2, +4\}$ satisfying $(2+\mu) = 0$.*

Remark 5.8 ([Derived]). Corollary 5.7 and Paper 26 [9] Lemma 3.1 (*Level-2 gravitationally decoupled*) are the same algebraic identity $(2+\mu_2) = 0$ in two physical contexts: Lemma 3.1 states that Level-2 contributes zero to G_N^{-1} ; Corollary 5.7 states it contributes zero to the bosonic spectral sum $\mathcal{S}$. Both derive from the unique eigenvalue satisfying $(2+\mu) = 0$ in the spectrum of G_{84} .

Corollary 5.9 ([Derived]).

$$\mathcal{S}_{\text{eff}} = \left[\sum_{\mu}(2+\mu)m(\mu) \right] - \dim \ker(L_a)|_{n=4} = 168 - 4 = 164 \quad [9] \text{ Thm. 4.2.}$$

$\mathcal{S}_{\text{eff}} = 164$ is the difference of two Fano-combinatorial quantities: the bipartite spectral sum $168 = |\text{Aut}(\text{PG}(2, \mathbb{F}_2))|$ and the Fano non-incidence count $4 = \dim \ker(L_a) = |V_{\text{ni}}(a)|$ ([10, 11] Thm. 3.1).

Remark 5.10 ([Observation]). The two Fano-combinatorial corrections are algebraically independent: the bipartite spectral symmetry (causing Level-2 zero contribution) and the non-incidence map (causing the -4 shift) are governed by different aspects of the Fano structure.

Conjecture 5.11 ([*Conjecture, IIB-II-strong*], OP 24.5). The eigenvalue $\mu = -2$ is derived from the Fano incidence structure of $\text{ZD}^*(\mathcal{A}_4)$: the adjacency matrix of each 12-vertex lo-flat component of G_{84} has eigenvalue -2 with multiplicity 2 as a consequence of the Fano incidence relations $|\text{lo-flat}| = 4$ and the complement-of-Fano-line structure. This would close OP 24.5 [*Derived*].

6. THE OP 29.1 CASCADE

OP 29.1 [11]: does a compact Lie group G exist with $\dim G - \dim H = 14$ and $\text{adj}(G)|_{\text{PSL}(2,7)} \cong \text{irr}_6 \oplus \text{irr}_8$? (G_2 excluded, Paper 21.)

Observation 6.1 ([*Observation*]). The cascade has two branches with precise implications:

Branch A (OP 29.1 false, G does not exist): Level-2 masslessness is purely spectral. The mechanism of Corollary 5.7 — zero contribution from $(2 + \mu_2) = 0$ — is the unique algebraic explanation. No Goldstone interpretation exists; the spectral origin is the only Fano-combinatorial route.

Branch B (OP 29.1 true, G exists): A Goldstone interpretation is available as a dual picture. The spectral mechanism of Corollary 5.7 remains valid in both branches; OP 29.1 concerns additional structure, not the validity of the spectral derivation.

7. SYNTHESIS: THE FANO PLANE'S PORTFOLIO

Role	Precise statement	Status	Source
Multiplication in $\mathcal{A}_3$	7 imaginary units = 7 Fano points; mult. table = Fano incidence	[D]	Papers 3,4
Annihilation in $\mathcal{A}_4$	Every ZD pair $\in$ unique Fano block	[D]	Papers 3,4
Non-incidence map	$\ker(L_a) \cong V_{\text{ni}}(a)$ as Stab-modules	[D]	Paper 29
Universal local structure	Every ZD pair at every level: one GL- Fano block	[D]	Paper 4
Weil identity	$Z(x) = 2 \text{Aut}(\text{Fano}) \cdot x^4 \cdot Z_{\text{Weil}}$	[D]	Paper 4
Bifrontality bridge	Numerator = annihilation; $Z_{\text{Weil}} =$ mult. face	[O]	Papers 32–34
Schubert strata	Four orbit strata $\leftrightarrow$ four Hurwitz al- gebras	[O]	Papers 4,34
EW tight pair	$ R_1/R_2 = 8 = \dim \mathbb{O}$	[O]	Papers 33,34
$R_1 = - \text{Fano} $	Residue at $\mathbb{C}$ -pole = -7	[D]	Papers 3,4
$R_0 = \dim \mathbb{S}$	$\mathbb{R}$ -pole carries $\dim \mathcal{A}_4 = 16$	[Sp]	Papers 33,34
$\{s_k\} = (1, 7, 14, 8)$	All encode $\mathcal{A}_3$ structural dims	[D/O]	Papers 33,34
G_2 -cross-level	$s_2 = \dim G_2$ at $\mathbb{H}$ -pole; unifies $n=2, 3$	[O]	Paper 34
$S = \text{Aut}(\text{Fano}) $	Spectral invariant = 168 = Fano aut. count	[O]	Paper 34
Level-2 zero	$(2+\mu_2) = 0 \Rightarrow$ Level-2 contributes 0	[D]	Paper 34
$\mathcal{S}_{\text{eff}}=164$	168–4: bipartite sum minus non- incidence	[D]	Pap.28– 29,34
$\mu=-2$ from lo-flat	Eigenvalue from lo-flat Fano	[C]	Paper 34
Majorana isolation	$V_{4_s} \perp$ Fano blocks; three isolations	[O]	Paper 32
Level-3 tripling	Level-3= $3 \times$ Level-2; factor 3 from G_{84} char. poly.	[O]	[8]
$s_k = m(\mu), k = 1, 2$	$s_1=m(-4)=7, \quad s_2=m(-2)=14;$ bridge $Z(x) \leftrightarrow G_{84}$ spec.	[O]	Papers 3,11
Level-3 contrib. +84	$(2+0) \cdot 42 = 84 = \text{ZD}^*(\mathcal{A}_4) ;$ counting level	ZD- [O]	Paper 34
$S = 7 \cdot 24$	$ \text{Fano} \cdot S_4 :$ Fano pts $\times$ point- stabiliser	[O]	Paper 34
$m(-2) \neq \text{adj}(G_2)$	$m(-2)=14 = \dim G_2$ numerical only; irr struct. differs	[D-]	Paper 21
Hidden ZD symmetry	$ \text{Aut}_{\text{str}}(\text{ZD}^*(\mathcal{A}_4)) = 7! \cdot 768^7 \gg$ $ \text{Aut}(\mathcal{A}_4) =2688$	[D]	Paper 4

Legend: [D]=*Derived*; [O]=*Observation*; [D/O]=*Derived+Observation*; [C]=*Conjecture, IIB-II-strong*;
[Sp]=*Speculation, IIB-II-strong*.

8. OPEN PROBLEMS

OP 24.5 [*Open*], partially addressed]. *Closed:* $(2 + \mu_2) = 0$ gives zero Level-2 contribution (Corollary 5.7, [*Derived*]); $\mathcal{S}_{\text{eff}} = 168 - 4$ as two Fano-combinatorial quantities (Corollary 5.9, [*Derived*]). *Remaining gap:* derive $\mu = -2$ from the Fano incidence structure of the 12-vertex lo-flat graph (Conjecture 5.11).

OP 33.3 *[Open]*, largely closed]. *Closed*: each individual s_k = structural dim of $\mathbb{O}$ (Theorem 3.2, *[Derived]*). *Remaining gap*: categorical G_2 -representation-theoretic proof that the Weil factorization necessarily extracts $\{1, 7, 14, 8\}$ (Remark 3.3, *[Observation]*).

OP 34.1 *[Open]*, new]. Algebraic proof that $s_2 = \dim G_2$ at the $\mathbb{H}$ -pole follows from $Z(x) = 336x^4 \cdot Z_{\text{Weil}}$ and $\text{PSL}(2, 7) \subset G_2$.

OP 29.1 *[Open]*, cascade mapped]. G exists or not: implications stated in Observation 6.1.

OP 34.2 *[Open]*, new]. *Motivic interpretation of $Z(x)$* (Paper 4 [4], §8): make precise $M_{\text{ZD}} = L[\text{PSL}(2, 7)] \otimes h(\text{PG}(3, \mathbb{F}_2))$ and identify the L-function decomposition

$$L(M_{\text{ZD}}, s) = Z(2^{-s}) = L(L[\text{PSL}(2, 7)], s) \cdot Z_{\text{Weil}}(\text{PG}(3, \mathbb{F}_2), 2^{-s}).$$

The second factor is identified (Paper 4); the first requires an Artin L-function for $\text{PSL}(2, 7)$. The poles $s = 0, 1, 2, 3$ of $Z(2^{-s})$ correspond under IIB-II *[Speculation, IIB-II-strong]* to the four force scales. This would connect the ZD structure to the Langlands programme.

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