THREE ALGEBRAIC CLOSURES IN THE CAYLEY–DICKSON ZERO-DIVISOR SERIES

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Abstract. We close two open problems from the series on zero-divisors in Cayley–Dickson algebras. **(A)** We prove that no compact Lie group G satisfies $\mathrm{adj}(G)|_{\mathrm{PSL}(2,7)} \cong \mathrm{irr}_6 \oplus \mathrm{irr}_8$ (OP 29.1, *[Derived-Negative]*), definitively ruling out a Goldstone interpretation of the Level-2 masslessness of the Black Circle. **(B)** We compute $H^2(\mathrm{PSL}(2,7), (\mathbb{Z}/2\mathbb{Z})^4) \cong (\mathbb{Z}/2\mathbb{Z})^2$ as a $\mathbb{F}_2$-vector space (OP 35.1, *[Derived]*, conditional on the action matrices), confirming that the extension $1 \to (\mathbb{Z}/2\mathbb{Z})^4 \to G_{84} \to \mathrm{PSL}(2,7) \to 1$ is non-split and establishing the cohomological framework for the identification of the octonion cocycle c_{oct}. **(C)** We derive algebraically that $s_2 = \dim G_2 = 14$ at the $\mathbb{H}$-pole of $Z(x)$ from the Weil factorisation and the embedding $\mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2$ (OP 34.1, *[Derived]* for the numerical identity; *[Open]* for the categorical G_2-equivariant statement).

Contents

1. Introduction

This paper closes three open problems arising in Papers 29–35 of the series [1, 3]. The series studies zero-divisors in the sedenion algebra $\mathcal{A}_4$, their automorphism group $G_{84} = \mathrm{Aut}(\mathrm{ZD}^*(A_4))$ of order 2688, and the induced physical framework IIB.

OP 29.1 (Paper 29, Conjecture 5.6) asks whether any compact Lie group G can give the Level-2 sector of the Black Circle a mass via a Goldstone mechanism. Theorem 3.1 below answers negatively: no such G exists. The Level-2 masslessness (and the "spectral sombrero" pattern) is purely spectral in origin, arising from $Z(G_{84}) = C_2$ [8].

OP 35.1 (Paper 35, Conjecture 6.1) concerns the octonionic 2-cocycle c_{oct} and its relation to the non-split extension class $[G_{84}]$ in $H^2(\mathrm{PSL}(2,7), (\mathbb{Z}/2\mathbb{Z})^4)$. We compute the full cohomology group (Theorem 4.2) and establish the framework for the cocycle identification.

OP 34.1 (Paper 34) asks for an algebraic derivation that the normalised residue $s_2 = 14$ at the $\mathbb{H}$-pole of $Z(x)$ equals $\dim G_2 = \dim \mathrm{Aut}(\mathbb{O})$, using the Weil factorisation [4] and the chain

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$\mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2$ [2]. Theorem 5.1 gives this derivation; the categorical G_2-equivariant step remains *[Open]* (Remark 5.3).

GAP computations. All computations use GAP 4.15.1 [13] on an AMD Threadripper 2950X. The HAP package [14] provides the free resolution used in Section 4. Scripts are deposited as supplementary files on Zenodo.

2. Background and notation

We recall the objects fixed throughout the series.

- $\mathcal{A}_4$ denotes the sedenion algebra (Cayley–Dickson level $n = 4$), with imaginary basis $e_1, \ldots, e_{15}$ [1].
- $\mathrm{ZD}^*(A_4) = \mathrm{ZD}^*(\mathcal{A}_4)$ is the graph of 84 signed zero-divisor units [1].
- $G_{84} = \mathrm{Aut}(\mathrm{ZD}^*(A_4))$ has order $2688 = 2^7 \cdot 3 \cdot 7$ and structure $(\mathbb{Z}/2\mathbb{Z})^4 \cdot \mathrm{GL}(3, \mathbb{F}_2)$ [8].
- $\mathrm{PSL}(2,7) \cong \mathrm{GL}(3, \mathbb{F}_2)$ acts on the 7 Fano points of $\mathrm{PG}(2, \mathbb{F}_2)$; the irreducibles over $\mathbb{C}$ have dimensions $1, 3, 3, 6, 7, 8$. We write irr_d for the irreducible of dimension d.
- The Level-k mass spectrum of S_{84} has eigenvalue $\mu \in \{-4, -2, 0, +2, +4\}$ with multiplicities $\{7, 14, 42, 14, 7\}$ (Paper 11 [6]). Level-2 has $\mu_2 = -2$ and $(2 + \mu_2) = 0$.
- $N = (\mathbb{Z}/2\mathbb{Z})^4 \trianglelefteq G_{84}$ is the normal subgroup of sign-flip automorphisms with $\sigma = \mathrm{id}$.
- $c_{\mathrm{oct}} \in H^1(\mathrm{Oct}, (\mathbb{Z}/2\mathbb{Z})^3)$ denotes the octonionic 2-cocycle introduced in Paper 8 [5].

The spectral sombrero is the property that the eigenvalue $\mu = -2$ produces the Level-2 zero-mass sector; Conjecture 5.6 of Paper 29 attributed this to a Goldstone mechanism. Theorem 3.1 below refutes that conjecture.

3. No Lie group for Level-2 (OP 29.1)

Theorem 3.1 (*[Derived-Negative]*, GAP 4.15.1)**.** *No compact Lie group G satisfies* $\mathrm{adj}(G)|_{\mathrm{PSL}(2,7)} \cong \mathrm{irr}_6 \oplus \mathrm{irr}_8$.

Proof. The condition forces $\dim G = 6+8 = 14$. We treat the simple and semisimple cases separately.

Step 1: Simple Lie groups of dimension 14. The classical series give dimensions $n(n+2)$ (A_n), $n(2n+1)$ (B_n, C_n), $n(2n-1)$ (D_n), none of which equals 14 for any n. Among the exceptional algebras, G_2 has dimension 14. Paper 21 [7] established (by GAP computation) that $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_3^* \oplus \mathrm{irr}_8 \neq \mathrm{irr}_6 \oplus \mathrm{irr}_8$. GAP verification:

```
adj(G2)|_PSL character = [14,-2,-1,2,0,0]
target                 = [14, 2,-1,0,0,0]
Match: false
```

Hence no simple Lie group of dimension 14 works.

Step 2: Semisimple factorisation. The only partition of 14 into dimensions of simple Lie algebras is $14 = 3 + 3 + 8 = \dim A_1 + \dim A_1 + \dim A_2$. Thus $G \cong \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ (up to finite cover).

Step 3: Order obstruction for $\mathrm{SU}(2)$. $\mathrm{PSL}(2,7)$ is simple of order 168. Every group homomorphism $\mathrm{PSL}(2,7) \to \mathrm{SU}(2)$ is trivial, because the image is a finite subgroup of $\mathrm{SU}(2)$, whose orders are at most $|\tilde{A}_5| = 120 < 168$ (binary icosahedral group). Since $\mathrm{PSL}(2,7)$ is simple, any nontrivial homomorphism would be injective, which is impossible. GAP verification: $|2.A_5| = 120 < 168$ (`true`).

Conclusion. The two $\mathrm{SU}(2)$ factors contribute $\mathrm{adj}(\mathrm{SU}(2))|_{\mathrm{PSL}(2,7)} = 3 \cdot \mathrm{irr}_1$ (trivial action). The $\mathrm{SU}(3)$ factor, via $\mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2$, contributes $\mathrm{adj}(\mathrm{SU}(3))|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_8$ (Paper 21 [7]). Thus $\mathrm{adj}(G)|_{\mathrm{PSL}(2,7)} = 6 \cdot \mathrm{irr}_1 \oplus \mathrm{irr}_8 \neq \mathrm{irr}_6 \oplus \mathrm{irr}_8$. GAP verification:

```
adj(SU(2)^2 x SU(3))|_PSL = [14,6,5,6,7,7]
target                    = [14,2,-1,0,0,0]
Match: false
```

Tori contribute 0 to the adjoint and do not affect the argument. This exhausts all compact Lie groups of dimension 14. □

Corollary 3.2 (*[Derived-Negative]*). *The spectral sombrero (Level-2 masslessness in S_{84}) has no Goldstone interpretation via any compact Lie group symmetry. Conjecture 5.6 of Paper 29* [9] *is* [Derived-Negative].

Remark 3.3. The sombrero is spectral: it arises from $Z(G_{84}) = C_2$ (Paper 22-B [8]), via the bipartite structure of $\mathrm{ZD}^*(A_4)$ (Paper 29, Proposition 5.1), which forces $m(\mu) = m(-\mu)$ and singles out $\mu = -2$ as the unique eigenvalue with $(2 + \mu_2) = 0$. No spontaneous symmetry breaking is involved.

4. Cohomology of $\mathrm{Aut}(\mathcal{A}_4)$ (OP 35.1)

4.1. **Setup.** Let $N = (\mathbb{Z}/2\mathbb{Z})^4 \trianglelefteq G_{84}$ be the sign-flip subgroup (elements with $\sigma = \mathrm{id}$ in the notation of Paper 29, Proposition 2.1). By Paper 22-B [8] and Paper 29 [9]:

- N is normal, elementary abelian of order 16.
- $Q := G_{84}/N \cong \mathrm{PSL}(2, 7)$, the quotient being simple of order 168.
- $Z(G_{84}) = C_2$ (Paper 22-B, Remark 9.1).

The action of Q on N is by conjugation. Let $A_1, A_2 \in \mathrm{GL}(4, \mathbb{F}_2)$ be the action matrices for the two generators of Q (computed in the companion GAP session).

4.2. Non-splitting (Part A).

Proposition 4.1 (*[Derived]*). *The extension $1 \to N \to G_{84} \to Q \to 1$ does not split.*

Proof. A split extension with faithful Q-action on N satisfies $Z(N \rtimes Q) = 1$. But $Z(G_{84}) = C_2 \neq 1$ (Paper 22-B [8]), and the action of Q on N is faithful because $V_{\mathrm{ni}}(a)$ is irreducible as a $\mathrm{Stab}(a)$-module (Paper 29, Theorem 4.1 [9]). Hence $[G_{84}] \neq 0$ in $H^2(Q, N)$. □

4.3. Cohomology computation (Part B).

Theorem 4.2 (*[Derived]*, conditional on action matrices). *$H^2(\mathrm{PSL}(2, 7), (\mathbb{Z}/2\mathbb{Z})^4) \cong (\mathbb{Z}/2\mathbb{Z})^2$ as a $\mathbb{F}_2$-vector space. In particular $|H^2| = 4$: there are three non-trivial extension classes, of which $[G_{84}]$ is one.*

Proof. We compute via the HAP free resolution $R_\bullet \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}Q$, obtained by `ResolutionFiniteGroup(Q, 3)` in GAP with the HAP package [14]. The ranks are:

$$\dim R_0 = 1, \quad \dim R_1 = 5, \quad \dim R_2 = 13, \quad \dim R_3 = 24.$$

The cochain complex $\mathrm{Hom}_Q(R_\bullet, N)$ has modules $C^n = N^{\dim R_n} \cong (\mathbb{F}_2)^{4 \dim R_n}$:

$$(\mathbb{F}_2)^4 \xrightarrow{\delta^0} (\mathbb{F}_2)^{20} \xrightarrow{\delta^1} (\mathbb{F}_2)^{52} \xrightarrow{\delta^2} (\mathbb{F}_2)^{96}.$$

The coboundary matrices δ^n are constructed from the HAP boundary maps $R!.\mathrm{boundary}(n, j)$ and the action matrices A_1, A_2 (see GAP script `OP_35_1_H2_final.g`, deposited on Zenodo).

Verification of the chain condition: $\delta^1 \circ \delta^0 = 0$ and $\delta^2 \circ \delta^1 = 0$ (GAP: `D2*D1=0: true`, `D3*D2=0: true`).

Ranks:

$$\mathrm{rank}(\delta^0) = 3, \quad \mathrm{rank}(\delta^1) = 16, \quad \mathrm{rank}(\delta^2) = 34.$$

The cohomology groups are:

$$\begin{aligned}
\dim H^0(Q,N) &= \dim C^0 - \operatorname{rank}(\delta^0) = 4-3 = 1,\\
\dim H^1(Q,N) &= (\dim C^1 - \operatorname{rank}(\delta^1)) - \operatorname{rank}(\delta^0) = (20-16)-3 = 1,\\
\dim H^2(Q,N) &= (\dim C^2 - \operatorname{rank}(\delta^2)) - \operatorname{rank}(\delta^1) = (52-34)-16 = 2.
\end{aligned}$$

□

Remark 4.3. The module N is *reducible* over $\mathbb{F}_2$: $H^0(Q,N)=1$ implies that the global sign-flip $\pi \in N$ (generating $Z(G_{84}) = C_2$) is fixed by the conjugation action. Hence $N \cong \mathbb{F}_2 \oplus V_3$ as $\mathbb{F}_2[Q]$-modules, where V_3 is a 3-dimensional non-trivial summand. Correspondingly, $H^2(Q,N) = H^2(Q,\mathbb{F}_2) \oplus H^2(Q,V_3)$, with one factor from the Schur multiplier $H_2(\mathrm{PSL}(2,7),\mathbb{Z}) = \mathbb{Z}/2$ and one from V_3.

Remark 4.4. The 2-cocycle of the G_{84} extension, restricted to the two Q-generators q_1, q_2, evaluates as

$$f(q_1,q_1) = [1,1,1,0], \quad f(q_2,q_1) = [1,0,0,1] \quad \in (\mathbb{F}_2)^4.$$

Both are non-zero, consistent with $[G_{84}] \neq 0$.

Corollary 4.5 (*[Open]*)**.** *OP 35.1 (Paper 35, Conjecture 6.1): the identification* $[G_{84}] = \text{lo-bit}(c_{\mathrm{oct}})$ *in* $H^2(Q,N)$ *requires comparing the 2-cocycle of Remark 4.4 with the lo-bit projection of* c_{oct} *from Paper 8* [5]*. Since* $\dim H^2 = 2$ *(three non-trivial classes), the identification is not automatic and remains* [Open].

5. The G_2 cross-level identity (OP 34.1)

The ZD zeta function $Z(x)$ has a simple pole at $x = \frac{1}{4}$, the reciprocal of the Frobenius eigenvalue $2^2 = 4 = \dim \mathbb{H}$ of $\mathrm{PG}(3,\mathbb{F}_2)$. Paper 34 [11] observes that the associated normalised residue satisfies $s_2 = 14 = \dim G_2 = \dim \mathrm{Aut}(\mathbb{O})$ (Thm. 3.2, *[Derived]*). OP 34.1 asks for an algebraic derivation of this identification from the Weil factorisation and the embedding chain $\mathrm{PSL}(2,7) \subset \mathrm{SU}(3) \subset G_2$. Theorem 5.1 gives such a derivation. The computation $s_2 = 14$ and the equality $14 = \dim G_2$ are both *[Derived]*. The algebraic reason *why* the Weil mechanism at the $\mathbb{H}$-pole selects $\dim \mathrm{Aut}(\mathcal{A}_3)$ — rather than another 14-dimensional structure — contains one residual gap, flagged *[Open]* in Remark 5.3.

Theorem 5.1 (*[Derived]* (with *[Open]* gap in part (iii)))**.** *Let* $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(x)$ *be the ZD zeta function in its Weil factorisation* (Paper 4, [4])*. The normalised residue at the* $\mathbb{H}$*-pole satisfies* $s_2 = 14 = \dim G_2 = \dim \mathrm{Aut}(\mathbb{O})$*. Concretely:*

- *(i)* [Derived] $s_2 = 14$*, derived from the residue* $R_2 = \frac{7}{8}$ *and the normalisation formula of Paper 33* [10].
- *(ii)* [Derived] $14 = \dim G_2$*, by standard Lie theory:* $G_2 = \mathrm{Aut}(\mathbb{O})$ *has rank* 2 *and* 12 *roots, giving* $\dim G_2 = 14$ [standard].
- *(iii)* [Observation] *The number* $14 = 2 \cdot |\mathrm{PG}(2,\mathbb{F}_2)|$ *arises in the Weil residue via the Fano factor* $|\mathrm{PG}(2,\mathbb{F}_2)| = |\mathrm{Im}(\mathbb{O})| = 7$ *and the level-*1 *normalisation weight* $\dim \mathcal{A}_1 = 2$*. The PSL-adjoint decomposition* $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_3^* \oplus \mathrm{irr}_8$ (Paper 21, [7]) *makes* $\dim G_2 = 14$ *explicit, but the* G_2*-equivariant map identifying the Weil weight space at* x_2 *with* $\mathrm{adj}(G_2)$ *is* [Open] *(Remark 5.3).*

Proof. **Part (i): computation of s_2.** By the Weil factorisation [4],

$$Z(x) = 336x^4 \cdot \prod_{k=0}^{3} (1-2^k x)^{-1}.$$

This has four simple poles at $x_k = 2^{-k}$, with Frobenius eigenvalues $2^k = \dim \mathcal{A}_k$ for $k = 0, 1, 2, 3$. By Paper 33 Prop. 2.3 [10],

$$R_2 := \operatorname{Res}_{x=1/4} Z(x) = \tfrac{7}{8}.$$

By Paper 33, Obs. 2.6 [10], the normalised residues satisfy

$$R_k \cdot \dim(\mathcal{A}_k)^4 \;=\; 16 \cdot (-1)^k \cdot s_k. \tag{1}$$

At $k = 2$, with $\dim(\mathcal{A}_2) = \dim \mathbb{H} = 4$:

$$s_2 = \frac{R_2 \cdot 4^4}{16} = \frac{\frac{7}{8} \cdot 256}{16} = \frac{7 \cdot 32}{16} = 14. \qquad \checkmark$$

Part (ii): $14 = \dim G_2$. The compact exceptional Lie group $G_2 = \mathrm{Aut}(\mathbb{O})$ has root system G_2 of rank 2 with 12 roots, so $\dim G_2 = 2 + 12 = 14$ *[standard]*.

Part (iii): tracing the Fano factor 7. By Paper 33 Prop. 2.4 [10], the residues satisfy the symmetry $R_k = -2^{9-6k} R_{3-k}$. At $k = 2$:

$$R_2 = -2^{-3} \cdot R_1 = -\tfrac{1}{8} \cdot (-7) = \tfrac{7}{8},$$

where $R_1 = -7 = -|\mathrm{PG}(2, \mathbb{F}_2)|$ *(Paper 33 Prop. 2.3; Paper 4 Weil structure* [10, 4]*)*. Substituting into (1):

$$s_2 = \frac{|\mathrm{PG}(2, \mathbb{F}_2)| \cdot \dim(\mathcal{A}_2)^4}{8 \cdot 16} = |\mathrm{PG}(2, \mathbb{F}_2)| \cdot \dim \mathcal{A}_1 = 7 \cdot 2 = 14, \tag{2}$$

since $4^4/(8 \cdot 16) = 256/128 = 2 = \dim \mathcal{A}_1$. The Fano number $7 = |\mathrm{PG}(2, \mathbb{F}_2)| = |\mathrm{Im}(\mathbb{O})|$ *[standard]* is simultaneously the combinatorial size of the Fano plane (governing $\mathbb{O}$-multiplication, Paper 4 [4]) and the dimension of the imaginary subspace $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ on which $G_2 = \mathrm{Aut}(\mathbb{O})$ acts. The identity $14 = 7 \cdot 2 = |\mathrm{Im}(\mathbb{O})| \cdot \dim \mathcal{A}_1$ is thus consistent with $\dim G_2 = 14$, and the embedding $\mathrm{PSL}(2, 7) \subset \mathrm{SU}(3) \subset G_2$ [2] provides the structural link between the Fano symmetry group and G_2.

To see the dimension split explicitly: since $\mathrm{SU}(3)$ is a maximal subgroup of G_2 with $\dim \mathrm{SU}(3) = 8$ and $G_2/\mathrm{SU}(3) \cong S^6$ *[standard]*, we have $\dim G_2 = 8 + 6 = 14$. The adjoint of G_2 restricts *(Paper 21,* [7]*)* to

$$\mathrm{adj}(G_2)\big|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_3^* \oplus \mathrm{irr}_8, \qquad 3 + 3 + 8 = 14,$$

where $\mathrm{irr}_8 = \mathrm{adj}(\mathrm{SU}(3))|_{\mathrm{PSL}(2,7)}$ *(Paper 36, §3)* and $\mathrm{irr}_3 \oplus \mathrm{irr}_3^*$ is the tangent space of the fibre $G_2/\mathrm{SU}(3) \cong S^6$.

A G_2-equivariant isomorphism between the 14-dimensional Weil weight space at x_2 and $\mathrm{adj}(G_2)$ would complete the proof that $s_2 = \dim G_2$ for algebraic rather than numerical reasons. This step is *[Open]*; see Remark 5.3. □

Corollary 5.2 (*[Derived]* for s_2; *[Open]* categorically)**.** *OP 33.3 is closed at* [Derived] *level for the index $k = 2$: the Weil factorisation $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(x)$, the normalisation* (1) *of Paper 33* [10], *and standard Lie theory together force $s_2 = \dim G_2 = \dim \mathrm{Aut}(\mathbb{O}) = 14$. The categorical proof that the Weil mechanism* necessarily *extracts* $\dim \mathrm{Aut}(\mathcal{A}_3)$ *at the $\mathbb{H}$-pole — as opposed to verifying that two quantities that are both equal to* 14 *happen to coincide — remains* [Open].

Remark 5.3 (*[Open]*)**.** *(OP 33.3 residual; OP 34.1 residual.)* A fully categorical proof of $s_2 = \dim G_2$ would require a G_2-equivariant isomorphism

$$W_2 \xrightarrow{\sim} \mathrm{adj}(G_2),$$

where W_2 is the weight-4 cohomological factor of $Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2))$ at the Frobenius eigenvalue $2^2 = 4 = \dim \mathcal{A}_2$. Two obstructions must be addressed before claiming such an isomorphism:

- The Level-2 spectral module of G_{84}, with PSL-decomposition $\mathrm{irr}_6 \oplus \mathrm{irr}_8$, is *not* isomorphic to $\mathrm{adj}(G_2)|_{\mathrm{PSL}(2,7)} = \mathrm{irr}_3 \oplus \mathrm{irr}_3^* \oplus \mathrm{irr}_8$ as a $\mathrm{PSL}(2,7)$-module (*[Derived-Negative]*, Paper 21 [7]). The equivariant map must therefore operate at the level of G_2-geometry — the $G_2/\mathrm{SU}(3)$ fibration — and not via the G_{84}-spectral decomposition.
- The formula $s_2 = |\mathrm{Im}(\mathbb{O})| \cdot \dim \mathcal{A}_1 = 7 \cdot 2$ (2) decomposes 14 differently from the Lie-theoretic split $\dim G_2 = 8 + 6 = \dim \mathrm{SU}(3) + \dim S^6$; an equivariant proof must reconcile these two factorizations of 14.

6. Open problems

OP 35.1 (residual). Identify $[G_{84}] \in H^2(\mathrm{PSL}(2,7), N)$ explicitly as a 2-cocycle and compare with the lo-bit projection of c_{oct} (Paper 8 [5]). With $\dim H^2 = 2$, a finite check over four classes suffices.

OP 34.1 (partially closed by Theorem 5.1). See Section 5.

OP 33.3 (partially closed by Corollary 5.2). G_2-equivariant isomorphism $W_2 \xrightarrow{\sim} \mathrm{adj}(G_2)$ (see Remark 5.3).

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