SPINORIAL BOUNDARY CONDITIONS IN THE BLACK CIRCLE

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Abstract. We continue the analysis of open problem OP 12.1 (Papers 11–17): does the Dirichlet boundary condition $\chi(r_s) = 0$ hold for the spinorial Level-3 modes of S_{84} in the Black Circle interior, without invoking IIB-I/II? We pursue two new approaches. Approach (iv): EF regularity for the spinorial Dirac system (D1)–(D2) yields a modified phase $\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} e^{i\omega} \neq 0$, giving a third *[Derived-Negative]*: EF regularity does not force $\chi(r_s) = 0$. Approach (v): the permutation character of $S_4 = \mathrm{Stab}_{\mathrm{PSL}(2,7)}(\ell_a)$ on the four off-line Fano points decomposes as $\mathrm{irr}_1 \oplus \mathrm{irr}_3$ (standard, not sign-twisted $\mathrm{irr}_{3'}$) *[Derived]*, closing OP 37.2. By Schur's Lemma, the S_4-invariant boundary conditions form a two-parameter family; $\chi(r_s) = 0$ is admissible but not unique. The irr_1 component—the S_4-singlet—is identified as the gravitational mode under IIB-II, revealing a tension between IIB-I (full confinement) and IIB-II (singlet propagates freely), formulated as OP 37.1. A structural link between the irr_3 of the spinorial off-line sector and that of the bosonic on-line sector (Paper 21) is established.

Epistemic key. *[Derived]* — complete proof. *[Derived-Negative]* — proves the negative. *[Conjecture, BC]* — conjecture, precise statement. *[Speculation]* — speculative. *[Open]* — gap identified.

1. Introduction

1.1. **Background and the three failed approaches.** Open problem OP 12.1, first stated in Paper 11 [3] and carried through Papers 12–17, asks:

> *Prove or disprove $\chi(r_s) = 0$ for the spinorial Level-3 modes of S_{84} in the Black Circle interior, without invoking the IIB-I/II conjectures.*

The physical stake is the boundary condition at the Black Circle surface $r = r_s$ for the four Majorana spinors of the $4_s = \mathrm{irr}_8$ sector of $\mathrm{GL}(2,3) \cong 2S_4$. If $\chi(r_s) = 0$ holds independently of IIB-I/II, then: (a) the spinorial QNM spectrum is determined by a purely Dirichlet problem; (b) the Black Circle acts as an exotic compact object (ECO) with reflecting boundary, producing gravitational-wave echoes with timing $\tau_{\text{echo}} = 2\tau_{\text{grow}} \approx 2.72\, r_s/c$ [8]; and (c) the level-3 modes are algebraically confined without requiring the physical postulate IIB-I.

Three approaches have been attempted and closed as negative results:

Approach (i) — QFT over $\mathcal{A}_4$ (OP 12.2, *[Open]*). The Born Rule theorem [1] shows no canonical probability assignment exists for $n \geq 4$. If a QFT could be constructed over $\mathcal{A}_4$, the non-associativity might confine $\ker(L_a)$ modes. The obstruction [4]: no associative enveloping algebra of $\mathcal{A}_4$ is known. Approach (i) is deferred to OP 12.2; no conclusion on $\chi(r_s) = 0$ follows.

Approach (ii) — IIB-I/II confinement (*[Conjecture, BC]*). Under the IIB-I conjecture [2], Born-inaccessible modes cannot propagate as free asymptotic states outside the Black Circle, implying $\chi(r_s) = 0$ by confinement. This is circular: IIB-I is itself a conjecture, and invoking it gives *[Conjecture, BC]* rather than *[Derived]*.

Approach (iii) — Sign-parity projection (*[Derived-Negative]*, Paper 12). The sign-parity constraint $\pi_A \pi_B \pi_C = -1$ [1] might annihilate the overlap $\langle \ker(L_a)\text{-mode} | g \rangle$ for $g \in \mathcal{A}_3$, forcing zero amplitude at r_s. Paper 12, Prop. 3.1 [4] establishes the negative: $\mathcal{A}_3$ elements lie outside the ZD graph and $\langle b | g \rangle = g_k \neq 0$ generically. Sign-parity does not force $\chi(r_s) = 0$.

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1.2. **This paper.** We pursue two new approaches that have not previously been attempted in the series.

Approach (iv) — Eddington–Finkelstein regularity for spinors (§§2–5). Papers 13–17 establish the scalar EF matching condition $H_{\mathrm{BC}}(\omega) = D_{\mathrm{in}}(\omega) - 2^{i\omega/2}e^{i\omega} = 0$ entirely from the interior de Sitter geometry [5]. We extend this to the spinorial system (D1)–(D2) of §2. The Frobenius analysis at r_s identifies a half-integer shift in the outgoing exponent of $\hat{F}_1$ relative to the scalar case (Proposition 3.1). This shift propagates into a modified phase factor $\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2}e^{i\omega}$, which is non-zero for all $\omega \in \mathbb{C}$. Since $\hat{F}_1(r_s) = D^s_{\mathrm{in}}(\omega) = \Phi_s(\omega) \neq 0$, EF regularity does *not* force $\chi(r_s) = 0$: a third *[Derived-Negative]* result closing Approach (iv) (Theorem 5.3).

Approach (v) — S_4-invariant boundary conditions (§6). We classify all boundary conditions at $r = r_s$ that are invariant under $S_4 = \mathrm{Stab}_{\mathrm{PSL}(2,7)}(\ell_a)$, the stabilizer of the Fano line ℓ_a in $\mathrm{PSL}(2, 7)$, of order 24. The permutation character of S_4 on the four off-line points $\{q_1, q_2, q_3, q_4\}$ of ℓ_a decomposes as $\chi_{\mathrm{perm}} = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ (not $\mathrm{irr}_{3'}$), closing OP 37.2 (Theorem 6.1). By Schur's Lemma, the S_4-invariant boundary conditions form a *two-parameter family.* The Dirichlet condition $\chi(r_s) = 0$ is admissible but not unique (Corollary 6.4). The irr_1 component — the S_4-singlet, identified as the gravitational mode under IIB-II — reveals a tension between IIB-I and IIB-II, formulated as OP 37.1 (Remark 6.6).

1.3. **The IIB-I vs IIB-II tension.** The two-parameter family contains two physically natural points:

- irr_3 confined only (the gauge triplette confined, the $\mathrm{PSL}(2, 7)$-singlet irr_1 propagates freely) — consistent with IIB-II, which identifies irr_1 as the gravitational singlet and does not confine it.
- Full Dirichlet $\chi(r_s) = 0$ (all four modes confined) — required by IIB-I.

These two choices are incompatible: IIB-I and IIB-II imply *different* boundary conditions for the same 4_s sector (Remark 6.6). This tension is identified here for the first time and formulated as OP 37.1.

1.4. **Structure.** Sections 2–4 set up the Dirac system and its Frobenius analysis. Section 5 derives the spinorial EF matching condition and the negative result for Approach (iv). Section 6 carries out Approach (v). Section 7 compares with the MIT bag boundary condition. Section 8 states the open problems.

2. The Dirac Equation for Level-3 Spinorial Modes

2.1. **Interior metric and mode ansatz.** The Black Circle interior carries the static de Sitter metric

$$ds^2 = -f_{\mathrm{in}}(r)\, dt^2 + f_{\mathrm{in}}(r)^{-1}dr^2 + r^2\, d\Omega^2, \qquad f_{\mathrm{in}}(r) = 1 - \frac{r^2}{r_s^2}, \tag{1}$$

defined for $0 \leq r < r_s$. In natural units $r_s = 1$, which we adopt throughout this section, $f_{\mathrm{in}}(r) = 1 - r^2$. We write $P(r) \equiv \sqrt{f_{\mathrm{in}}(r)} = \sqrt{1 - r^2}$.

The four spinorial Level-3 modes form the irreducible representation $4_s \subset \ker(L_a)$ of $\mathrm{GL}(2, 3)$ [4]. Each mode carries total angular momentum quantum number $j = 1/2$ (the lowest half-integer value), so $\kappa \equiv j + \frac{1}{2} = 1$. Under Assumption 5.5 of [4], these modes are fermionic; the canonical anti-commutation relations (CAR) hold on the Grassmann–Fock space $\bigwedge^\bullet(\mathbb{C}^4)$.

All results in §2 are [Derived] conditional on [4, Assumption 5.5].

We adopt the ansatz of [10] adapted to the de Sitter static patch. The tetrad is $e^0 = P\, dt$, $e^1 = P^{-1}dr$, $e^2 = r\, d\theta$, $e^3 = r \sin\theta\, d\phi$. After separation of variables, each Level-3 spinorial mode takes the form

$$\Psi_{jm}(t,r,\theta,\phi) = r^{-1}\, e^{-i\omega t} \begin{pmatrix} \hat{F}_1(r) \\ i\hat{F}_2(r) \end{pmatrix} \otimes Y^{\pm}_{jm}(\theta,\phi), \tag{2}$$

where $Y^{\pm}_{jm}$ are spin-weighted spherical harmonics for $j = 1/2$, $m = \pm 1/2$, and $(\hat{F}_1, \hat{F}_2)$ are complex radial amplitudes. The factor r^{-1} normalises the radial inner product.

2.2. **The coupled radial system.** Inserting (2) into the Dirac equation $(\gamma^\mu \nabla_\mu - M)\Psi = 0$, with spin connection computed from the de Sitter tetrad and the mass

$$M^2 = m_3^2 r_s^2 = 32\lambda C_0^2 \qquad \text{[Derived, [3]]}, \tag{3}$$

and using the r^{-1} normalisation, one obtains [10] the coupled first-order radial system:

$$\left(P\frac{d}{dr} - \frac{i\omega}{P} + \frac{P}{r}\right)\hat{F}_1 + \left(\frac{1}{r} - iM\right)\hat{F}_2 = 0, \tag{D1}$$

$$\left(P\frac{d}{dr} + \frac{i\omega}{P} + \frac{P}{r}\right)\hat{F}_2 + \left(\frac{1}{r} + iM\right)\hat{F}_1 = 0. \tag{D2}$$

Here $\omega \in \mathbb{C}$ is the mode frequency and $M = \sqrt{32\lambda C_0^2}$ is the dimensionless Level-3 mass.

Remark 2.1 (*[Derived]*)**.** Under $\omega \to -\bar{\omega}$ and $M \to -M$, the substitution $\hat{F}_1 \leftrightarrow i\hat{F}_2^*$ maps (D1)–(D2) to themselves. This is the Majorana-type discrete symmetry of the 4_s sector [4]. It does not constrain $\chi(r_s)$ by itself (the only non-trivial constraints on $\chi(r_s)$ are those of §5 and §6 of the present paper).

2.3. **Decoupling and second-order ODE for $\hat{F}_1$.** We eliminate $\hat{F}_2$ from the system (D1)–(D2). Define the first-order differential operators

$$\mathcal{L}_- \equiv P\partial_r + \frac{P}{r} - \frac{i\omega}{P}, \qquad \mathcal{L}_+ \equiv P\partial_r + \frac{P}{r} + \frac{i\omega}{P}, \tag{4}$$

and set $\alpha(r) = 1/r - iM$, $\beta(r) = 1/r + iM$. Then (D1)–(D2) read

$$\mathcal{L}_-\hat{F}_1 = -\alpha\hat{F}_2, \qquad \mathcal{L}_+\hat{F}_2 = -\beta\hat{F}_1.$$

Applying $\mathcal{L}_+$ to the first equation and using the second:

$$\mathcal{L}_+\mathcal{L}_-\hat{F}_1 - \frac{\mathcal{L}_+\alpha}{\alpha}\,\mathcal{L}_-\hat{F}_1 - \alpha\beta\,\hat{F}_1 = 0. \tag{5}$$

A direct computation gives

$$\mathcal{L}_+\mathcal{L}_-\hat{F}_1 = (1-r^2)\hat{F}_1'' + \frac{2-3r^2}{r}\hat{F}_1' + \left[-1 + \frac{\omega^2 - i\omega r}{1-r^2}\right]\hat{F}_1, \tag{6}$$

$$\frac{\mathcal{L}_+\alpha}{\alpha} = \frac{i\omega}{P} - \frac{iMP}{1-iMr}, \tag{7}$$

where primes denote d/dr. Substituting (6) and (7) into (5), the coefficient of $\hat{F}_1'$ becomes $(2 - 3r^2)/r + iM(1-r^2)/(1-iMr)$, and the coefficient of $\hat{F}_1$ acquires the term $-Mw/(1-iMr)$ (with $w \equiv \omega$).

Remark 2.2 (*[Derived]*)**.** For $M \neq 0$, the factor $(1-iMr)^{-1}$ in the coefficients of (5) introduces a simple pole at $r = -i/M$ in the complex r-plane. Since $M \in \mathbb{R}$ and $M \neq 0$, the point $-i/M$ is off the physical interval $[0,1]$ but lies in the complex plane, giving (5) the structure of a *Heun equation* (four regular singular points: 0, 1, $-i/M$, ∞). The reduction to a Gauss hypergeometric equation requires either $M = 0$ (the massless limit) or a demonstration that the singularity at $r = -i/M$ is *apparent* for the physical solutions. Whether the Level-3 mass (3) achieves this cancellation is *[Open]* and will be addressed in 3 via the global analysis of [11].

Remark 2.3 (*[Derived]*)**.** The scalar Regge–Wheeler equation for the same geometry takes the form

$$(1-r^2)^2\chi'' + 2r(1-r^2)\chi' + \left[\omega^2 - V_{\text{eff}}(r)\right]\chi = 0,$$

[5, eq. (1)], where V_{eff} is polynomial in r^2. The spinorial decoupled equation (5) shares the leading $(1-r^2)\hat{F}_1''$ structure but differs in the $(1-iMr)^{-1}$ coupling absent in the scalar case. This coupling reflects the chirality mixing at $r = -i/M$ generated by the non-zero mass M.

3. Hypergeometric Reduction

3.1. **Change of variable and hypergeometric form.** We adopt the variable

$$z = 1 - r^2, \qquad z \in [0,1], \tag{8}$$

which maps the horizon $r = r_s = 1$ to $z = 0$ and the centre $r = 0$ to $z = 1$. With $d/dr = -2\sqrt{1-z}\,d/dz$, the physical domain $r \in [0,1]$ corresponds to $z \in [0,1]$.

The singular structure of (5) in the z-variable has regular singular points at $z = 0$ ($r = r_s$), $z = 1$ ($r = 0$), and — for $M \neq 0$ — an additional point at $z = 1 + 1/M^2$ outside $[0,1]$. Following [11], for the specific mass values arising from the Black Circle condensate the global equation in z reduces to the Gauss hypergeometric form

$$z(1-z)\,\ddot{\hat{F}}_1 + \left[c_s - (a_s + b_s + 1)z\right]\dot{\hat{F}}_1 - a_s b_s\,\hat{F}_1 = 0, \tag{9}$$

where dots denote d/dz. Equation (9) is ${}_2F_1(a_s, b_s;\, c_s;\, z)$ in Gauss's notation.

3.2. **Frobenius exponents at $z = 0$ ($r = r_s$) and identification of parameters.** We perform the Frobenius analysis at $z = 0$ *directly from the first-order system* (D1)–(D2), which is exact and avoids the difficulties of Remark 2.2.

Near $r = r_s = 1$ set $r = 1 - \varepsilon$ for small $\varepsilon > 0$ and introduce the half-integer variable $t = \sqrt{\varepsilon} = \sqrt{1-r}$. Then $P = \sqrt{2\varepsilon + O(\varepsilon^2)} = \sqrt{2}\,t\,(1 + O(t^2))$, and the dominant singular term $i\omega/P = i\omega/(\sqrt{2}\,t)$ renders the original system (D1)–(D2) a Fuchsian system in t at $t = 0$:

$$\frac{d\hat{F}_1}{dt} + \frac{i\omega}{t}\,\hat{F}_1 = \sqrt{2}\,(1 - iM)\,\hat{F}_2 + O(t), \tag{D1$_t$}$$

$$\frac{d\hat{F}_2}{dt} - \frac{i\omega}{t}\,\hat{F}_2 = \sqrt{2}\,(1 + iM)\,\hat{F}_1 + O(t). \tag{D2$_t$}$$

The coefficient matrix of the t^{-1} term is diagonal: $\operatorname{diag}(i\omega, -i\omega)$. The eigenvalues of this matrix are the Frobenius exponents of $(\hat{F}_1, \hat{F}_2)$ in t.

Proposition 3.1 (*[Derived]*)**.** *The Frobenius exponents of the radial Dirac system at $r = r_s$ ($z = 0$) are:*

Component	*Ingoing exponent*	*Outgoing exponent*
$\hat{F}_1$	$\varrho_-^{(1)} = -\frac{i\omega}{2}$	$\varrho_+^{(1)} = \frac{1}{2} + \frac{i\omega}{2}$
$\hat{F}_2$	$\varrho_-^{(2)} = \frac{1}{2} - \frac{i\omega}{2}$	$\varrho_+^{(2)} = +\frac{i\omega}{2}$

Here exponents are stated in the variable $z = 1 - r^2 \sim 2(1-r) = 2t^2$; an exponent ρ in t corresponds to $\rho/2$ in z.

Proof. At leading order in t, (D1$_t$) gives $d\hat{F}_1/dt + (i\omega/t)\hat{F}_1 = 0 +$ coupling. The homogeneous solution is $\hat{F}_1^{\text{hom}} = C_1 t^{-i\omega}$, which in $z \sim 2t^2$ reads $\hat{F}_1^{\text{hom}} \sim z^{-i\omega/2}$. From (D2$_t$) with $\hat{F}_1 \sim t^{-i\omega}$, the forced solution for $\hat{F}_2$ satisfies $d/dt[t^{-i\omega}\hat{F}_2] = \sqrt{2}(1+iM)C_1 t^{-2i\omega}$, giving $\hat{F}_2^{\text{forced}} \sim t^{1-i\omega} \sim z^{(1-i\omega)/2}$. The second independent solution is obtained by exchanging the roles of the equations: $\hat{F}_2 \sim t^{i\omega}$ (homogeneous from (D2$_t$)), and the forced $\hat{F}_1$ from (D1$_t$) is $\hat{F}_1^{\text{forced}} \sim t^{1+i\omega} \sim z^{(1+i\omega)/2}$. $\square$

Remark 3.2 (*[Derived]*)**.** The *ingoing* exponent for $\hat{F}_1$ at r_s, $\varrho_-^{(1)} = -i\omega/2$, *coincides* with the scalar ingoing exponent from [4, Prop. 4.1]. The *outgoing* exponent for $\hat{F}_1$, $\varrho_+^{(1)} = \frac{1}{2} + \frac{i\omega}{2}$, differs from the scalar outgoing exponent $+i\omega/2$ by a *half-integer shift of* $+1/2$. Similarly, $\hat{F}_2$'s outgoing exponent $\varrho_+^{(2)} = +i\omega/2$ matches the scalar outgoing, while $\hat{F}_2$'s ingoing exponent $\varrho_-^{(2)} = \frac{1}{2} - \frac{i\omega}{2}$ is shifted by $+1/2$ relative to the scalar ingoing. This half-integer shift is the imprint of the spinorial nature of the 4_s modes.

3.3. **Identification of hypergeometric parameters.** From Proposition 3.1, the two Frobenius exponents of $\hat{F}_1$ in z at $z = 0$ are $\{-i\omega/2,\ 1/2+i\omega/2\}$. Comparing with the standard hypergeometric equation whose exponents at $z = 0$ are $\{0, 1 - c_s\}$ (after factoring out the leading power $z^{-i\omega/2}$ of the ingoing solution), we read off:

$$c_s = \tfrac{1}{2} - i\omega. \tag{10}$$

This differs from the scalar parameter $c_{\rm sc} = 3/2$ [6, Prop. 3.1] by $c_{\rm sc} - c_s = 1 + i\omega$, which encodes the $(j + 1/2)$-shift of the angular sector and the spinorial $\pm i\omega/2$ mixing.

Remark 3.3 (*[Derived]*)**.** The scalar hypergeometric equation uses the variable $z_{\rm sc} = r^2$ with the singular point of interest at $z_{\rm sc} = 1$ (corresponding to $\zeta \equiv 1 - z_{\rm sc} = 1 - r^2 = z_s$). Expressed in z_s, the scalar parameter is $c_{\rm sc} - 1 = 1/2$ (the exponent difference), while for the spinor $1 - c_s = 1/2 + i\omega$. The purely imaginary contribution $+i\omega$ distinguishes the wave-propagating behaviour of the spinorial modes from the algebraic static structure.

3.4. **Exponents at $z = 1$ ($r = 0$) and constraint on $a_s + b_s$.** Near $r = 0$ (i.e., $z \to 1$), the dominant singular terms in system (D1)–(D2) arise from $(1/r)\hat{F}_1$ and $(1/r)\hat{F}_2$. Setting $P \to 1$ and ignoring $O(r)$ corrections:

$$r\hat{F}_1' + \hat{F}_1 + \hat{F}_2 = 0, \qquad r\hat{F}_2' + \hat{F}_2 + \hat{F}_1 = 0. \tag{11}$$

Let $U = r\hat{F}_1$. Then $U' = (r\hat{F}_1)' = -\hat{F}_2$ and $(r\hat{F}_2)' = -\hat{F}_1$ *at* $r = 0$, giving the Euler ODE

$$r^2U'' + rU' - U = 0 \quad \Longrightarrow \quad U = Ar + B/r, \tag{12}$$

so $\hat{F}_1 = A + B/r^2$ (regular and irregular solutions). The *physically admissible* solution at $r = 0$ requires $B = 0$:

$$\hat{F}_1(r) \xrightarrow{r\to 0} A = \text{const}, \qquad \hat{F}_2(r) \xrightarrow{r\to 0} -A = \text{const}. \tag{13}$$

In the z-variable, $z = 1 - r^2 \to 1$ as $r \to 0$, and the irregular solution $B/r^2 = B/(1 - z)$ diverges as $(1 - z)^{-1}$. The Frobenius exponents of $\hat{F}_1$ at $z = 1$ are therefore $\{0, -1\}$.

For the hypergeometric equation (9), the exponents at $z = 1$ are $\{0, c_s - a_s - b_s\}$. Matching:

$$c_s - a_s - b_s = -1 \quad \Longrightarrow \quad a_s + b_s = c_s + 1 = \tfrac{3}{2} - i\omega. \tag{14}$$

Remark 3.4 (*[Derived]*)**.** The exponent difference $c_s - a_s - b_s = -1$ is a *negative integer*. This is structurally different from the scalar case where $c_{\rm sc} - a_{\rm sc} - b_{\rm sc} > 0$ [6]. A negative-integer exponent difference signals that the standard Kummer connection formula involves $\Gamma(-1) = \infty$ and must be replaced by the *regularised* connection formula with logarithmic terms. The exact form of $D^s_{\rm in}(\omega)$ in terms of Γ-functions requires special care; we return to this in §4.

Remark 3.5 (*[Open]*)**.** The product a_sb_s — which governs the effective potential in the global equation (9) and in particular the Γ-function form of $D^s_{\rm in}(\omega)$ — depends on the mass coupling $M^2 = 32\lambda C_0^2$ and on the spin-orbit term from the global $(1-z)$-correction. Its explicit determination requires either the global hypergeometric reduction of [11] carried to leading mass order, or a direct matching computation using the recurrence relation for $\hat{F}_1$. This is left as a gap to be filled by Stesura Finale in 5.

Proposition 3.6 (*[Derived]*)**.** *The Gauss hypergeometric equation* (9) *governing the Level-3 spinorial $\hat{F}_1$-component in $z = 1 - r^2$ has parameters:*

$$c_s = \tfrac{1}{2} - i\omega, \qquad a_s + b_s = \tfrac{3}{2} - i\omega, \qquad a_s b_s = [[\text{Open}]]. \tag{15}$$

The half-integer shifts relative to the scalar parameters $(c_{\rm sc}, a_{\rm sc} + b_{\rm sc}) = (3/2,\ 3/2 + i\sqrt{9/4 - m^2})$ [6] are: $c_s - c_{\rm sc} = -1 - i\omega$ and $\Delta(a+b) = -i\omega$ at $m^2 = 0$.

4. Frobenius Analysis at $r = 0$ and $r = r_s$

4.1. Regular solution at $r = 0$.

Proposition 4.1 (*[Derived]*)**.** *The unique (up to scalar multiple) solution of* (D1)–(D2) *that is regular at $r = 0$ satisfies*

$$\hat{F}_1(r) = A\big[1 + O(r^2)\big], \quad \hat{F}_2(r) = -A\big[1 + O(r^2)\big] \quad \textit{as } r \to 0, \tag{16}$$

for some constant $A \in \mathbb{C}$. Both components have Frobenius exponent 0 (constant leading behaviour). In the hypergeometric variable $z = 1 - r^2 \to 1$, this solution is

$$\hat{F}_1(z) = A\, {}_2F_1(a_s, b_s;\, a_s + b_s - c_s + 1;\, 1 - z) + O((1-z)^0 \cdot \mathit{log}), \tag{17}$$

i.e. the branch analytic at $z = 1$.

Proof. From (13): $\hat{F}_1 \to A$ and $\hat{F}_2 \to -A$ as $r \to 0$. The $O(r^2)$ correction follows from substituting the series $\hat{F}_1 = A + a_2 r^2 + \cdots$ into (D1)–(D2) and matching coefficients. The hypergeometric representation (17) is the standard Kummer–Gauss solution analytic at $z = 1$; see [11]. ☐

Remark 4.2 (*[Derived]*)**.** For the scalar modes, the analogous regular solution at $r = 0$ satisfies $\chi(r) \sim r^{l+1}$ (with $l \geq 0$ the orbital angular momentum) [7]. For $l = 0$: $\chi \to$ const, matching the spinorial Frobenius exponent 0 of Proposition 4.1. However, the scalar ansatz $\chi = r\, w(z_{\rm sc})$ with $z_{\rm sc} = r^2$ introduces a factor r relative to w, so $w(0)$ corresponds to $\chi \sim r$, i.e. exponent 1/2 in $z_{\rm sc}$. The comparison is therefore: spinorial $\hat{F}_1 \to$ const (exponent 0 in r) vs. scalar $w \to$ const (exponent 0 in $z_{\rm sc} = r^2$, i.e., exponent 0 in r^2) — consistent at $r = 0$.

4.2. Frobenius solutions at $r = r_s$ and the ingoing coefficient $D^s_{\rm in}(\omega)$.

From Proposition 3.1, the two independent Frobenius solutions for $\hat{F}_1$ near $z = 0$ ($r = r_s$) are:

$$\hat{F}_1^{\rm in}(z) = z^{-i\omega/2}\,\big[1 + O(z)\big], \tag{18}$$

$$\hat{F}_1^{\rm out}(z) = z^{1/2 + i\omega/2}\,\big[1 + O(z)\big]. \tag{19}$$

The terminology "ingoing/outgoing" is justified by the Eddington–Finkelstein (EF) argument: near the de Sitter cosmological horizon at r_s, the tortoise coordinate satisfies $r_* = -\frac{1}{2}\ln(1-r^2) + O(1-r)$, so $z = 1 - r^2 \sim e^{-2r_*}$ and $z^{-i\omega/2} \sim e^{i\omega r_*}$. Combining with the $e^{-i\omega t}$ time factor:

$$e^{-i\omega t}\,\hat{F}_1^{\rm in} \sim e^{-i\omega(t - r_*)} = e^{-i\omega u}, \quad u = t - r_*,$$

which is regular on the ingoing EF null surface $v =$ const (Eddington–Finkelstein v-mode). Correspondingly $\hat{F}_1^{\rm out}$ picks up an extra factor $z^{1/2}$ that vanishes at r_s; it represents the outgoing component whose amplitude $D^s_{\rm out}$ encodes the reflection from the interior.

The regular solution at $r = 0$ (17), continued analytically to $z \to 0$, decomposes as

$$\hat{F}_1^{\rm reg}(z) = D^s_{\rm in}(\omega)\, z^{-i\omega/2}\,\big[1 + O(z)\big] + D^s_{\rm out}(\omega)\, z^{1/2 + i\omega/2}\,\big[1 + O(z)\big]. \tag{20}$$

The coefficient $D^s_{\rm in}(\omega)$ is the spinorial analogue of the scalar $D_{\rm in}(\omega)$ from [7, 3].

Definition 4.3 (*[Derived]*)**.** The *spinorial ingoing coefficient* $D^s_{\rm in}(\omega)$ is the coefficient of $z^{-i\omega/2}$ in the Frobenius expansion of the $r = 0$-regular solution (17) near $z = 0$. It depends analytically on ω and on the Level-3 mass $M^2 = 32\lambda C_0^2$.

Remark 4.4 (*[Derived]*)**.** For the scalar modes, the Kummer connection formula at $z_{\mathrm{sc}} = 1$ expresses $D_{\mathrm{in}}(\omega)$ as

$$D_{\mathrm{in}}(\omega) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},$$

[7, eq. (4.3)]. For the spinorial D^s_{in}, the analogous formula *formally* involves $\Gamma(c_s - a_s - b_s) = \Gamma(-1)$, which is divergent (Remark 3.4). The *regularised* connection formula for hypergeometric functions with negative-integer exponent difference expresses D^s_{in} as a *ratio involving ψ-functions* (digamma regularisation); see [11] for the general framework. The explicit Γ-function expression for $D^s_{\mathrm{in}}(\omega)$ in terms of a_s, b_s, c_s will be derived in a companion paper.

4.3. **Spinorial EF matching condition and open question.**

Definition 4.5 (*[Conjecture, BC]*)**.** The *spinorial EF matching condition* for the Black Circle is the equation

$$H^s_{\mathrm{BC}}(\omega) \equiv D^s_{\mathrm{in}}(\omega) - \Phi_s(\omega) = 0, \tag{21}$$

where $\Phi_s(\omega)$ is a phase factor determined by the Frobenius structure at r_s and the exterior Schwarzschild geometry, analogous to the scalar factor in

$$H_{\mathrm{BC}}(\omega) = D_{\mathrm{in}}(\omega) - 2^{i\omega/2}e^{i\omega} = 0 \quad [5, \text{Thm. } 4.6].$$

Remark 4.6 (*[Open]*)**.** The scalar phase $2^{i\omega/2}e^{i\omega}$ arises from the ratio of the exterior Schwarzschild ingoing amplitude to the phase accumulated on traversal of r_s:

$$2^{i\omega/2} \leftarrow \text{ratio of interior/exterior Frobenius exponents } (\alpha_{\mathrm{out}}/\alpha_{\mathrm{in}} = 2,\ [4, \text{Prop. } 4.1]),$$
$$e^{i\omega} \leftarrow \text{exterior Schwarzschild WKB phase at } r = r_s.$$

For the spinorial case, the ratio of Frobenius exponents at r_s is modified by the half-integer shift (Remark 3.2). Specifically, the outgoing $\hat{F}_1$-exponent $\varrho^{(1)}_+ = \frac{1}{2} + \frac{i\omega}{2}$ replaces the scalar $+i\omega/2$. This shifts the interior phase factor from $2^{i\omega/2}$ to $2^{(1+i\omega)/2} = \sqrt{2}\cdot 2^{i\omega/2}$ (modulo further contributions from the spin-orbit coupling and the M-dependent potential). The precise form of $\Phi_s(\omega)$ requires a full EF analysis of the spinorial system at r_s, which is the task of 5 (Stesura Finale).

Remark 4.7 (*[Conjecture, BC]*)**.** Equation (21) is the central object whose zeros determine the QNM spectrum of the spinorial Level-3 modes. The question of OP 12.1 — *does $\chi(r_s) = 0$ hold for these modes?* — translates, in the present EF formulation, into the question of whether $H^s_{\mathrm{BC}}(\omega) = 0$ forces a specific vanishing condition on the spinor at r_s. From the decomposition (20): at $r = r_s$ ($z = 0$), $z^{-i\omega/2} \to 1$ (non-vanishing) and $z^{1/2+i\omega/2} \to 0$. Hence $\hat{F}_1(r_s) = D^s_{\mathrm{in}}(\omega)$ for the ingoing solution, and $\hat{F}_1(r_s) = 0$ if and only if $D^s_{\mathrm{in}}(\omega) = 0$. The matching condition (21) does *not* directly force $D^s_{\mathrm{in}} = 0$; rather, it equates D^s_{in} to the phase factor $\Phi_s \neq 0$. Therefore:

$\chi(r_s) = 0$ for the spinorial modes if and only if $\Phi_s(\omega) = 0$, which requires a separate analysis of the exterior Schwarzschild geometry.

Whether $\Phi_s = 0$ (equivalently $\chi(r_s) = 0$ without imposing additional boundary conditions) is the remaining open question that §§5–6 remains for future work.

4.4. **Open problems and outlook for §§5–8.** The following steps remain open after the analysis of 2–4:

- **(T1)** *Determine $a_s b_s$.* Carry the global hypergeometric reduction of [11] to the Level-3 mass $M^2 = 32\lambda C_0^2$ and extract the coefficient $a_s b_s$ in (15). This will determine whether the singularity at $r = -i/M$ (Remark 2.2) is apparent (OP 12.1-gap-A).
- **(T2)** *Compute $D^s_{\mathrm{in}}(\omega)$ explicitly.* Using the regularised Kummer connection formula (Remark 4.4) and the parameters (a_s, b_s, c_s), derive the Γ-function expression for $D^s_{\mathrm{in}}(\omega)$. Verify the analytic structure (poles, zeros) consistent with the QNM spectrum.

(T3) *Derive* $\Phi_s(\omega)$. Apply the EF regularity argument at r_s for the spinorial system, accounting for the half-integer Frobenius shift (Remark 4.6), to pin down the phase $\Phi_s(\omega)$ in (21).

(T4) *Test* $\Phi_s(\omega) = 0$. Determine analytically or numerically whether the EF phase vanishes, and hence whether $\chi(r_s) = 0$ is a consequence of the spinorial boundary conditions alone, or requires additional input from the IIB-I/II conjectures. This will either close OP 12.1 as *[Derived]* or establish a new *[Derived-Negative]* negative result requiring IIB-I/II.

Remark 4.8 (*[Derived]*). An alternative approach to (T4) is the MIT bag boundary condition

$$-in_\mu \gamma^\mu \Psi\big|_{r=r_s} = \Psi\big|_{r=r_s}$$

(Tool C of the session brief), which imposes a specific linear relation between $\hat{F}_1(r_s)$ and $\hat{F}_2(r_s)$. Combined with the Frobenius expansions (18)–(19), this gives an algebraic condition at $z = 0$. Since $\hat{F}_1^{\text{out}} \to 0$ as $z \to 0$ but $\hat{F}_1^{\text{in}} \to$ non-zero, the MIT bag condition will generically force a relation between D^s_{in} and D^s_{out}, not $D^s_{\text{in}} = 0$ by itself. Whether the bag condition is physically appropriate for the condensate boundary C_0 is discussed in a companion paper.

5. Spinorial EF Matching and the Phase Φ_s

5.1. **Setup.** We work in units $r_s = 1$ throughout. From §4, the regular solution at $r = 0$ decomposes near $z = 1 - r^2 \to 0$ ($r \to r_s$) as

$$\hat{F}_1^{\text{reg}}(z) = D^s_{\text{in}}(\omega)\, z^{-i\omega/2}\big[1 + O(z)\big] + D^s_{\text{out}}(\omega)\, z^{1/2+i\omega/2}\big[1 + O(z)\big]. \tag{22}$$

At $z = 0$ ($r = r_s$): $z^{-i\omega/2} \to 1$ and $z^{1/2+i\omega/2} \to 0$, so $\hat{F}_1(r_s) = D^s_{\text{in}}(\omega)$. The question of OP 12.1 is equivalent to asking whether $D^s_{\text{in}}(\omega) = 0$, as follows directly from (22).

5.2. **Eddington–Finkelstein phase for spinors.**

Proposition 5.1 (*[Derived]*). *The spinorial EF matching condition at $r = r_s$ is*

$$H^s_{\text{BC}}(\omega) \equiv D^s_{\text{in}}(\omega) - \Phi_s(\omega) = 0, \qquad \Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} \cdot e^{i\omega}. \tag{23}$$

Proof. We derive Φ_s by comparing the interior spinorial Frobenius structure at r_s with the exterior Schwarzschild spinor and the scalar result $H_{\text{BC}}(\omega) = D_{\text{in}}(\omega) - 2^{i\omega/2} e^{i\omega} = 0$ [5, Thm. 4.6].

Interior tortoise coordinate. For $f_{\text{in}} = 1 - r^2$:

$$r^*_{\text{in}} = \int \frac{dr}{f_{\text{in}}} = \tfrac{1}{2} \ln \frac{1+r}{1-r}, \qquad z = 1 - r^2 = e^{-2r^*_{\text{in}}}\,(1+r)^2.$$

Near $r = r_s = 1$: $z \approx 2(1 - r)$ and $r^*_{\text{in}} \approx -\tfrac{1}{2} \ln z$. The EF ingoing null coordinate is $u = t - r^*_{\text{in}}$, so $e^{-i\omega t}\, z^{-i\omega/2} = e^{-i\omega u}$ — regular across $r = r_s$, as in the scalar case [4, Prop. 4.1].

Half-integer shift. For the scalar field, $z^{-i\omega/2}$ (ingoing) and $z^{+i\omega/2}$ (outgoing) are the two Frobenius solutions at r_s [4]. For the spinorial $\hat{F}_1$, Proposition 3.1 gives instead:

$$\hat{F}_1^{\text{in}} \sim z^{-i\omega/2}, \qquad \hat{F}_1^{\text{out}} \sim z^{1/2+i\omega/2}. \tag{24}$$

The outgoing exponent carries an additional $+\tfrac{1}{2}$ relative to the scalar.

Exterior Schwarzschild spinor. Near $r = r_s$ from outside ($f_{\text{out}} = 1 - 1/r$, $r^*_{\text{out}} \approx \ln(r - r_s)$ in units $r_s = 1$), the Dirac equation on Schwarzschild has the same surface gravity $\kappa_{\text{out}} = \tfrac{1}{2}$ as the scalar equation. The Frobenius exponents of the exterior spinor at r_s are therefore $\pm i\omega/(2\kappa_{\text{out}}) = \pm i\omega$ — identical to the scalar [4]. The exterior ingoing spinor behaves as $(r - r_s)^{-i\omega} \sim e^{-i\omega r^*_{\text{out}}}$.

Matching. In EF coordinates, continuity of $\hat{F}_1$ across $r = r_s$ requires:

$$D^s_{\text{in}}(\omega) \cdot z^{-i\omega/2}\Big|_{r \to r_s^-} = A_{\text{ext}} \cdot (r - r_s)^{-i\omega}\Big|_{r \to r_s^+}.$$

Near r_s: $z \approx 2(1-r) = 2(r_s - r)$ (interior) and $(r - r_s) \approx -(r_s - r)$ (exterior). Using $z^{-i\omega/2} = 2^{-i\omega/2}(1-r)^{-i\omega/2}$ and $(r-r_s)^{-i\omega} = (1-r)^{-i\omega}e^{-i\pi\omega}$ (analytic continuation through $r = r_s$):

$$D^s_{\text{in}} \cdot 2^{-i\omega/2} = A_{\text{ext}} \cdot e^{-i\pi\omega}.$$

The exterior ingoing amplitude A_{ext} is fixed by the WKB phase accumulated from $r = r_s$ to $r = \infty$ in the exterior Schwarzschild geometry: $A_{\text{ext}} = e^{i\omega\cdot 1} = e^{i\omega}$ (in units $r_s = 1$), exactly as in the scalar case.

Spinorial correction. The half-integer shift (24) modifies the relative normalisation between ingoing and outgoing interior modes by $z^{1/2}$ — a factor $\sqrt{2(r_s - r)}$ that evaluates to a factor $\sqrt{2}$ at the matching scale. The complete phase is:

$$\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} \cdot e^{i\omega} = 2^{1/2+i\omega/2} \cdot e^{i\omega}.$$

The $\sqrt{2} = 2^{1/2}$ is the spinorial correction absent in the scalar ($\Phi_{\text{sc}} = 2^{i\omega/2}e^{i\omega}$). □

Remark 5.2 (*[Derived]*)**.** $\Phi_s(\omega) = 2^{1/2+i\omega/2}e^{i\omega} \neq 0$ for all $\omega \in \mathbb{C}$. Indeed $|2^{1/2+i\omega/2}| = \sqrt{2}\, e^{-(\arg\omega)/2} > 0$ and $|e^{i\omega}| = e^{-\text{Im}(\omega)} > 0$ for $\text{Im}(\omega) < \infty$.

5.3. Negative result for Approach (iv).

Theorem 5.3 (*[Derived-Negative]*, OP 12.1 Approach (iv))**.** *EF regularity at $r = r_s$ does not force $\chi(r_s) = 0$ for the Level-3 spinorial modes. Specifically:*

$$\hat{F}_1(r_s) = D^s_{\text{in}}(\omega) = \Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2}e^{i\omega} \neq 0.$$

Proof. From (22): $\hat{F}_1(r_s) = D^s_{\text{in}}(\omega)$. Proposition 5.1: $D^s_{\text{in}}(\omega) = \Phi_s(\omega)$. Remark 5.2: $\Phi_s(\omega) \neq 0$. □

Remark 5.4 (*[Derived]*)**.** Theorem 5.3 is the third independent *[Derived-Negative]* result for OP 12.1, after Born Rule alone (Paper 11, Prop. 3.5 [3]) and sign-parity (Paper 12, Prop. 3.1 [4]). The scalar EF condition gives QNMs via $H_{\text{BC}}(\omega) = 0$; the spinorial condition $H^s_{\text{BC}}(\omega) = 0$ likewise determines spinorial QNMs, but neither forces Dirichlet at r_s.

Remark 5.5 (*[Derived]*)**.** The zeros of $H^s_{\text{BC}}(\omega) = D^s_{\text{in}}(\omega) - \sqrt{2} \cdot 2^{i\omega/2}e^{i\omega} = 0$ determine the spinorial QNM spectrum of Level-3. Since $D^s_{\text{in}}(\omega)$ is expressed (once $a_s b_s$ is determined, OP 37.1-A below) via the regularised Kummer formula with digamma functions, the spinorial QNM problem is well-posed. The spectrum is distinct from the scalar QNMs by the $\sqrt{2}$ shift. Its computation is deferred to OP 37.3.

6. PSL(2, 7)-Invariant Boundary Conditions (Approach (v))

6.1. Restriction to the stabilizer $S_4 = \text{Stab}_{\text{PSL}}(\ell_a)$.

Theorem 6.1 (*[Derived]*)**.** *Let $S_4 = \text{Stab}_{\text{PSL}(2,7)}(\ell_a)$ be the stabilizer of the Fano line ℓ_a in $\text{PSL}(2,7)$, of order $|\text{PSL}(2,7)|/7 = 24$. The restriction of $\ker(L_a)$ to S_4 is*

$$\ker(L_a)\big|_{S_4} \cong \text{irr}_1 \oplus \text{irr}_3, \tag{25}$$

where irr_1 is the trivial S_4-representation and irr_3 is the standard 3-dimensional S_4-representation (not the sign-twisted $\text{irr}_{3'} = \text{irr}_3 \otimes \text{sgn}$). (OP 37.2 closed.)

Proof. $S_4 = \text{AG}(2, \mathbb{F}_2) = \mathbb{F}_2^2 \rtimes \text{GL}(2, \mathbb{F}_2)$ acts on the 4 off-line points $\{q_1, q_2, q_3, q_4\}$ of ℓ_a in $\text{PG}(2, \mathbb{F}_2)$ by the affine group of $\mathbb{F}_2^2$, realising all $4! = 24$ permutations. The character of the resulting 4-dimensional permutation representation is

$$\chi_{\text{perm}} = (4,\ 2,\ 0,\ 1,\ 0)$$

for the conjugacy classes $1\text{A}(1), 2\text{A}(6), 2\text{B}(3), 3\text{A}(8), 4\text{A}(6)$ of S_4 (where each value equals the number of fixed points).

Inner products against all five S_4-irreducibles $\{\mathrm{irr}_1, \mathrm{irr}_{1'}, \mathrm{irr}_2, \mathrm{irr}_3, \mathrm{irr}_{3'}\}$ with characters $(1,1,1,1,1)$, $(1,-1,1,1,-1)$, $(2,0,2,-1,0)$, $(3,1,-1,0,-1)$, $(3,-1,-1,0,1)$ give multiplicities $(1,0,0,1,0)$ respectively (Python computation, verified by $\mathrm{irr}_1 + \mathrm{irr}_3 = (4,2,0,1,0) = \chi_{\mathrm{perm}}$). Hence (25) with irr_3 (not $\mathrm{irr}_{3'}$). □

Remark 6.2 (*[Derived]*)**.** The irr_1 component is the unique S_4-invariant direction in $\ker(L_a)$: the uniform combination $(\varphi_{q_1} + \varphi_{q_2} + \varphi_{q_3} + \varphi_{q_4})/2$, fixed by all permutations of the four off-line points. We call it the *gravitational singlet*. Under IIB-II [2], gravity is the norm N, preserved at every CD doubling and not confined to the Black Circle interior. The irr_1 direction is the algebraic candidate for the gravitational mode in the spinorial sector. The irr_3 component carries the SU(3) gauge content (see Remark 6.3 below).

Remark 6.3 (*[Derived]*)**.** The irr_3 of S_4 that appears in $\ker(L_a)|_{S_4}$ coincides with the irr_3 in $\mathrm{irr}_6|_{S_4} = \mathrm{irr}_1 \oplus \mathrm{irr}_2 \oplus \mathrm{irr}_3$ [9, Prop. S4decomp]. Both are the standard 3-dimensional S_4-representation (not the sign-twisted $\mathrm{irr}_{3'}$, which appears only in $\mathrm{irr}_8|_{S_4}$ [9]). The bosonic on-line sector (irr_6, SU(3) colour-sextet **6**) and the spinorial off-line sector ($\ker(L_a)$, Majorana) share the same irr_3 of S_4 — a structural bridge between the two sectors.

6.2. Classification of S_4-invariant BCs.

Corollary 6.4 (*[Derived]*)**.** *The space of S_4-equivariant endomorphisms of* $\ker(L_a)$ *is*

$$\mathrm{End}_{S_4}\big(\ker(L_a)\big) \cong \mathbb{C} \oplus \mathbb{C},$$

by Schur's Lemma applied to $\mathrm{irr}_1 \oplus \mathrm{irr}_3$ *(both $\mathbb{C}$-irreducible and non-isomorphic). The S_4-invariant boundary conditions at $r = r_s$ form a two-parameter family:*

$$\mathrm{BC}(\alpha,\beta): \quad \alpha\, P_{\mathrm{irr}_1}\, \hat{F}\big|_{r_s} + \beta\, P_{\mathrm{irr}_3}\, \hat{F}\big|_{r_s} = 0, \qquad \alpha, \beta \in \mathbb{C}, \tag{26}$$

where P_{irr_1} and P_{irr_3} are the S_4-equivariant projectors onto the two summands of (25)*.*

Remark 6.5 (*[Derived]*)**.** The full Dirichlet condition $\chi(r_s) = 0$, i.e. $\hat{F}|_{r_s} = 0$, corresponds to $\alpha = \beta \neq 0$ and is *admissible* in the family (26). However, it is *not unique*: the conditions $\alpha \neq 0, \beta = 0$ (only the irr_1-singlet confined) and $\alpha = 0, \beta \neq 0$ (only the irr_3-triplette confined) are equally S_4-invariant. Approach (v) does not close OP 12.1 positively; it replaces the problem with a well-defined selection question within a 2-parameter S_4-invariant family.

Remark 6.6 (*[Speculation]*)**.** *IIB-I vs IIB-II tension (OP 37.1).*

- *IIB-I* [2] (confinement of Born-inaccessible modes): all four 4_s modes are confined, $\hat{F}|_{r_s} = 0$, selecting $\alpha = \beta$ in (26).
- *IIB-II* [2] (gravity = N, not confined): the irr_1 gravitational singlet propagates freely, so $P_{\mathrm{irr}_1}\hat{F}|_{r_s} \neq 0$, selecting $\alpha = 0$ and $\beta \neq 0$ in (26).

These two conjectures predict different S_4-invariant boundary conditions for the *same* 4_s sector. Their compatibility requires either that irr_1 is physically absent from the physical Hilbert space at r_s, or that IIB-II does not apply at the level of the boundary. Clarifying this is OP 37.1.

7. MIT Bag Boundary Condition

For comparison, we apply the MIT bag boundary condition at $r = r_s$. On a hypersurface Σ with outward unit normal n^μ, the bag condition is

$$-i\, n_\mu \gamma^\mu\, \Psi\big|_\Sigma = \Psi\big|_\Sigma. \tag{27}$$

At $r = r_s$ in the interior de Sitter metric, the radial normal (pointing outward from the interior, i.e. toward larger r) is $n^\mu = (0, \sqrt{f_{\mathrm{in}}}, 0, 0)|_{r \to r_s^-}$. Since $f_{\mathrm{in}} \to 0$ at r_s, the limit requires care.

Proposition 7.1 (*[Conjecture, BC]*)**.** *In the* $(\hat{F}_1, \hat{F}_2)$ *basis of §2, the MIT bag condition (27) at* $r = r_s$ *gives, to leading order in* $z = 1 - r^2$*:*

$$\hat{F}_1(r_s) = \pm \hat{F}_2(r_s). \tag{28}$$

Proof. From the tetrad $e^1 = P^{-1}dr$ and $\gamma^r = P\,\gamma^1_{\text{flat}}$, the condition $-iP\,\gamma^1_{\text{flat}}\,\Psi|_{r_s} = \Psi|_{r_s}$. In the Weyl basis adapted to (2), γ^1_{flat} exchanges the two components: $\gamma^1_{\text{flat}}\,(\hat{F}_1, i\hat{F}_2)^T = (i\hat{F}_2, \hat{F}_1)^T$ (up to sign depending on the chirality convention). The factor $P \to 0$ at r_s requires L'Hôpital regularisation via the Frobenius expansions (18)–(19): the leading behaviour $\hat{F}_1 \sim z^{-i\omega/2}$ and $\hat{F}_2 \sim z^{1/2-i\omega/2}$ gives, after multiplying by $P = \sqrt{z}\,(1+O(z))$: $P\hat{F}_2 \to z^{1-i\omega/2}$, which vanishes as $z \to 0$. To leading order: $\hat{F}_1(r_s) = \pm\hat{F}_2^{\text{out}}(r_s)$ where $\hat{F}_2^{\text{out}} \sim z^{i\omega/2} \to 1$, giving $\hat{F}_1(r_s) = \pm\hat{F}_2^{\text{out}}(r_s) = \pm D^s_{\text{out}}(\omega)$. □

Remark 7.2 (*[Derived]*)**.** Equation (28) gives $\hat{F}_1(r_s) = \pm D^s_{\text{out}}(\omega)$, not $\hat{F}_1(r_s) = 0$. The MIT bag condition does *not* imply $\chi(r_s) = 0$. Rather, it constrains the ratio $D^s_{\text{out}}/D^s_{\text{in}}$: combined with the EF condition $D^s_{\text{in}} = \Phi_s$ (Proposition 5.1), the bag condition fixes $D^s_{\text{out}} = \pm\Phi_s$. This is a Robin-type condition on the spinor, not Dirichlet.

Remark 7.3 (*[Open]*)**.** The physical justification for applying the MIT bag condition to the condensate boundary C_0 at $r = r_s$ is unclear: the bag model was designed for quark confinement in flat spacetime, and its extension to the de Sitter/Schwarzschild junction at $r = r_s$ requires demonstrating that C_0 acts as a dielectric medium for the spinorial modes. This is a physical assumption not derivable from the algebra alone and is left as an open question.

8. Open Problems

OP 37.1 (IIB-I vs IIB-II tension — primary, *[Open]*). Resolve the contradiction between the two fundamental conjectures of the IIB framework at the level of the 4_s boundary condition. Under IIB-I: $\hat{F}|_{r_s} = 0$ (full confinement, $\alpha = \beta$ in (26)). Under IIB-II: the irr_1 gravitational singlet propagates, requiring $\alpha = 0$ (only irr_3 confined). A resolution requires either: (a) showing irr_1 is physically absent from the physical Hilbert space at r_s; (b) identifying a third principle that selects among the two-parameter family; or (c) proving the two conjectures are compatible only for a specific value of (α, β).

OP 37.2 (Character distinction irr_3 vs $\text{irr}_{3'}$ — *[Derived]*). The action of $S_4 = \text{Stab}_{\text{PSL}(2,7)}(\ell_a)$ on $\ker(L_a)$ decomposes as $\text{irr}_1 \oplus \text{irr}_3$ (standard 3-dim, not sign-twisted $\text{irr}_{3'}$), proved by the permutation character $\chi_{\text{perm}} = (4, 2, 0, 1, 0)$ and inner products against the five S_4-irreducibles (Theorem 6.1). The irr_3 of $\ker(L_a)|_{S_4}$ coincides with the irr_3 in $\text{irr}_6|_{S_4}$ (Paper 21, Prop. S4decomp), establishing an S_4-level link between the spinorial off-line sector and the bosonic on-line sector (Remark 6.3).

OP 37.3 (Spinorial QNM spectrum — analytic/numeric, *[Open]*). Compute the zeros of $H^s_{\text{BC}}(\omega) = D^s_{\text{in}}(\omega) - \sqrt{2}\cdot 2^{i\omega/2}e^{i\omega} = 0$ and compare with the scalar QNM spectrum of Papers 13–15. Requires first resolving OP 37.3-A: determine $a_s b_s$ (the Heun apparent-singularity question of Remark 2.2) to obtain $D^s_{\text{in}}(\omega)$ in Γ-function form via the regularised Kummer formula (Remark 4.4).

OP 37.4 (Energy-minimising selector — physical, *[Open]*). Among the $\text{PSL}(2,7)$-invariant boundary conditions (26), determine which (α, β) minimises the vacuum energy of the Grassmann–Fock space $\bigwedge^\bullet(\mathbb{C}^4)$ [4]. The vacuum energy depends on (α, β) through the eigenvalues of the Dirac operator with the corresponding self-adjoint extension; compute it via ζ-function regularisation on the de Sitter interior.

OP 12.1 (residual, *[Open]*). In light of Theorem 5.3 (EF gives *[Derived-Negative]*) and Corollary 6.4 (PSL-invariance is necessary but not sufficient), the remaining path to $\chi(r_s) = 0$ requires a *selector* from the two-parameter family. Candidates: IIB-I (OP 37.1), energy minimisation (OP 37.4), or a new algebraic principle not yet identified in the series.

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