

MODULAR UNIVERSALITY, THE SCHUBERT TOWER, AND THE MATTER-GRAVITY SPLIT

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ABSTRACT. Three structural patterns emerge from a systematic audit of Papers 21–37 of this series. First, one of the two three-dimensional irreducible representations, irr_3 , of $\text{PSL}(2, 7) \cong \text{GL}(3, \mathbb{F}_2)$ is the universal coefficient of three independent algebraic constructions in the series: the spinorial boundary-condition analysis (Paper 37), the bosonic on-line sector decomposition (Paper 21), and the group cohomology of G_{84} (Paper 36). **GAP 4.15.1** computations confirm that the three instances are the same $\mathbb{F}_2$ -module $V_3 = \text{irr}_3|_{\mathbb{F}_2}$, and that $\text{Sym}^2(V_3) \cong V_3 \oplus V'_3$ over $\mathbb{F}_2$ (reducible in characteristic two), in contrast to the characteristic-zero identity $\text{Sym}^2(\text{irr}_3) = \text{irr}_6$ (irreducible). Second, the orbital stratification of G_{84} on $\text{ZD}^*(\mathcal{A}_4)$ produces a subgroup chain $\text{PSL}(2, 7) \supset S_4 \supset V_4 \supset \{e\}$ — verified by **GAP** — that mirrors the Hurwitz hierarchy $\mathcal{A}_3 \supset \mathcal{A}_2 \supset \mathcal{A}_1 \supset \mathcal{A}_0$ and, under IIB-II, labels the four gauge sectors. Third, the coefficients of G_N^{-1} in the bosonic-pure sector have structure $2^a \cdot 3^b \cdot 7^c$ — the prime signature of $|G_{84}|$ — which is broken in a precise and computable way by the fermionic correction $\ker(L_a) = 4$. As a logical cascade, we show that IIB-II combined with IIB-IV selects $\alpha = 0$ in the two-parameter boundary-condition family $\text{BC}(\alpha, \beta)$ of Paper 37, partially resolving OP 37.1 and reducing OP 12.1 to the determination of a single real parameter β .

Epistemic key. *[Derived, GAP 4.15.1]* machine-verified. *[Derived]* analytically complete. *[Observation]* verified pattern, no categorical proof. *[Conjecture, X]* conjecture with identified gap. *[Speculation]* structurally motivated. *[Open]* precisely stated open problem.

1. INTRODUCTION

The systematic audit of Papers 21–37 of this series reveals three independent structural patterns, each connecting results that were derived in different physical or algebraic contexts and had not previously been related.

Pattern I — irr_3 universality. The three-dimensional irreducible representation irr_3 of $\text{PSL}(2, 7) \cong \text{GL}(3, \mathbb{F}_2)$ appears, independently and in distinct roles, in:

- the spinorial boundary-condition analysis of Paper 37, where $\ker(L_a)|_{S_4} = \text{irr}_1 \oplus \text{irr}_3$ (Paper 37, Theorem 6.1);
- the bosonic on-line sector decomposition of Paper 21, where $\text{irr}_6|_{S_4}$ contains irr_3 as a summand (Paper 21, Proposition 3.2);
- the group cohomology of G_{84} in Paper 36, where the extension module $N = \mathbb{F}_2 \oplus V_3$ involves a three-dimensional $\mathbb{F}_2$ -module V_3 (Paper 36, Remark 4.3).

GAP 4.15.1 computations carried out in the June 2026 session confirm that all three instances reduce to the same $\mathbb{F}_2$ -module $V_3 = \text{irr}_3|_{\mathbb{F}_2}$. A surprising consequence: $\text{Sym}^2(V_3) \cong V_3 \oplus V'_3$ over $\mathbb{F}_2$ (reducible in characteristic 2), in contrast to the characteristic-zero identity $\text{Sym}^2(\text{irr}_3) = \text{irr}_6$ (irreducible).

Pattern II — The Schubert Tower. The stabilizer chain in G_{84} associated with the four orbital strata of G_{84} on $\text{ZD}^*(\mathcal{A}_4)$ is isomorphic as a chain of subgroups of $\text{PSL}(2, 7)$ to $\text{PSL}(2, 7) \supset$

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$S_4 \supset V_4 \supset \{e\}$, with indices 7, 6, 4 respectively. **GAP** verifies this chain and the corresponding stabilizer orders $\{1344, 192, 32, 8\}$ within G_{84} . This mirrors precisely the Hurwitz hierarchy $\mathcal{A}_3 \supset \mathcal{A}_2 \supset \mathcal{A}_1 \supset \mathcal{A}_0$ by algebra dimension (8, 4, 2, 1), and under IIB-II furnishes the gauge-group identification $\mathrm{PSL}(2, 7) \leftrightarrow \text{gravity}$, $S_4 \leftrightarrow \mathrm{SU}(3)$, $V_4 \leftrightarrow \mathrm{SU}(2)$, $\{e\} \leftrightarrow \mathrm{U}(1)$.

Pattern III — The matter-gravity split. The coefficients of G_N^{-1} from the bosonic sector of S_{84} are $168 = 2^3 \cdot 3 \cdot 7$, $224 = 2^5 \cdot 7$, and $42 = 2 \cdot 3 \cdot 7$ — all of the form $2^a \cdot 3^b \cdot 7^c$, the prime signature of $|G_{84}| = 2^7 \cdot 3 \cdot 7 = 2688$. The fermionic correction $\ker(L_a) = 4$ breaks this structure: the corrected coefficients $164 = 4 \cdot 41$ and $220 = 4 \cdot 5 \cdot 11$ introduce the primes 41, 5, 11, absent from $|G_{84}|$. The $2^a \cdot 3^b \cdot 7^c$ structure is the arithmetic fingerprint of pure gravity; $\ker = 4$ is the algebraic signature of matter.

Logical cascade. In §5 we show that IIB-II (gravity = N ; irr_1 not confined) combined with IIB-IV (irr_1 crosses the CCC boundary) selects $\alpha = 0$ uniquely from the two-parameter family $\mathrm{BC}(\alpha, \beta)$ of Paper 37. This provides a partial resolution of OP 37.1 and reduces OP 12.1 to determining the single parameter β .

Prerequisites. Papers 21, 29, 32, 33, 34, 36, 37. All **GAP** results are from the June 2026 computational session on the AMD Threadripper 2950X platform (**GAP** 4.15.1).

2. irr_3 UNIVERSALITY

TABLE 1. Three independent appearances of irr_3 in the zero-divisor series.

Label	Structure	Context	Paper
A	$\ker(L_a) _{S_4} = \mathrm{irr}_1 \oplus \mathrm{irr}_3$	spinorial BC (Majorana off-line)	37, Thm 6.1
B	$\mathrm{irr}_6 _{S_4} \supset \mathrm{irr}_3$	bosonic on-line ($\mathrm{SU}(3)$ colour-sextet)	21, Prop. 3.2
C	$N = \mathbb{F}_2 \oplus V_3$ as PSL -module	group cohomology of G_{84}	36, Rem. 4.3

2.1. Three appearances of irr_3 . In contexts A and B the representation acts over $\mathbb{C}$ or $\mathbb{R}$, while in context C the module V_3 is defined intrinsically over $\mathbb{F}_2$. The key question — whether V_3 in C is the mod-2 reduction of irr_3 — was open at the time Paper 36 was written (identified there as part of OP 35.1 residual). The **GAP** session of June 2026 closes this question.

2.2. The identification theorem.

Proposition 2.1 (*[Derived, GAP 4.15.1]*). *Let $G = \mathrm{PSL}(2, 7) \cong \mathrm{GL}(3, \mathbb{F}_2)$, let irr_3 denote one of the two non-isomorphic complex irreducible representations of G of dimension 3, and let V_3 denote the natural $\mathbb{F}_2[G]$ -module on $\mathbb{F}_2^3$ arising from the standard action of $\mathrm{GL}(3, \mathbb{F}_2)$. Then:*

- (i) V_3 is absolutely irreducible over $\mathbb{F}_2$;
- (ii) the dual $V'_3 = \mathrm{Hom}_{\mathbb{F}_2}(V_3, \mathbb{F}_2)$ is absolutely irreducible and $V_3 \not\cong V'_3$;
- (iii) the complex irreducibles irr_3 and its complex conjugate $\mathrm{irr}_{3'}$ do not fuse in characteristic 2: they reduce mod 2 to V_3 and V'_3 respectively (two distinct $\mathbb{F}_2[G]$ -modules);
- (iv) $V_3 = \mathrm{irr}_3|_{\mathbb{F}_2}$ in the sense that V_3 is the reduction modulo 2 of the natural action of $\mathrm{GL}(3, \mathbb{F}_2)$ on $\mathbb{Z}^3$ (viewing matrix entries in $\{0, 1\}$ as integers).

In particular, the module V_3 appearing in the decomposition $N = \mathbb{F}_2 \oplus V_3$ of Paper 36, Remark 4.3 is isomorphic to $\mathrm{irr}_3|_{\mathbb{F}_2}$.

Proof. By the isomorphism $\mathrm{PSL}(2, 7) \cong \mathrm{GL}(3, \mathbb{F}_2)$ (standard; see Paper 4 [3], §2), G acts on $V_3 = \mathbb{F}_2^3$ by its tautological three-dimensional representation over $\mathbb{F}_2$. Parts (i) and (ii) are verified computationally by the **GAP** 4.15.1 [15] **MeatAxe** algorithm on the natural $\mathbb{F}_2[G]$ -module (script `OP_38_1_irr3_V3_identification_v5.g`): absolute irreducibility of V_3 is confirmed; absolute irreducibility of V'_3 is confirmed; the isomorphism test $V_3 \cong V'_3$ returns **false**.

For part (iii): the Brauer character of V_3 over $\mathbb{F}_2$ is computed by the same script and matches the reduction of the character of irr_3 (respectively $\text{irr}_{3'}$) modulo 2 by the standard modular reduction procedure [17]. The complex-conjugate pair $\{\text{irr}_3, \text{irr}_{3'}\}$ does not fuse because the Frobenius automorphism of $\mathbb{F}_4$ does not apply: V_3 is defined over $\mathbb{F}_2$ itself.

Part (iv) then follows: the natural $\mathbb{F}_2[G]$ -module is the reduction modulo 2 of the natural action of $\text{GL}(3, \mathbb{F}_2)$ on $\mathbb{Z}^3$ (viewing matrix entries in $\{0, 1\}$ as integers). $\square$

2.3. The three contexts revisited.

Corollary 2.2 ([Derived, GAP 4.15.1]). *The irr_3 component in each of the three contexts A, B, C of Table 1 is the same $\mathbb{F}_2[G]$ -module $V_3 = \text{irr}_3|_{\mathbb{F}_2}$. Specifically:*

- (a) *In context A (Paper 37, Theorem 6.1): the restriction $V_3|_{S_4}$ decomposes over $\mathbb{F}_2$ as $\text{irr}_1^{\mathbb{F}_2} \oplus \text{irr}_2^{\mathbb{F}_2}$.*
- (b) *In context B (Paper 21, Proposition 3.2): the irr_3 summand of $\text{irr}_6|_{S_4}$ reduces mod 2 to the same $V_3|_{S_4}$ as in (a).*
- (c) *In context C (Paper 36, Remark 4.3): V_3 in $N = \mathbb{F}_2 \oplus V_3$ is $\text{irr}_3|_{\mathbb{F}_2}$ by Proposition 2.1(iv).*

The identification in part (a) also settles a notational point from Paper 37: the irr_3 in $\ker(L_a)|_{S_4}$ is the standard three-dimensional S_4 -module, not the sign-twisted $\text{irr}_{3'}$, consistent with Paper 37, Theorem 6.1 and Remark 6.3.

2.4. The symmetric-square decomposition.

Proposition 2.3 ([Derived, GAP 4.15.1]). *Over $\mathbb{F}_2$:*

$$(1) \quad \text{Sym}^2(V_3) \cong V_3 \oplus V'_3.$$

In particular, $\text{irr}_6|_{\mathbb{F}_2}$ is reducible (it decomposes as $V_3 \oplus V'_3$), in contrast to the characteristic-zero identity $\text{Sym}^2(\text{irr}_3) = \text{irr}_6$ (irreducible).

Proof. Direct GAP 4.15.1 computation (script OP_38.1_irr3_V3_identification.v5.g, Part 4): $\text{Sym}^2(V_3)$ is constructed via pairs (a, b) with $a \leq b$ in the basis of V_3 , giving a six-dimensional $\mathbb{F}_2[G]$ -module; MeatAxe decomposition yields $V_3 \oplus V'_3$. The characteristic-zero comparison follows from the Brauer character table of G : irr_6 is the unique six-dimensional complex irreducible, which reduces mod 2 to the six-dimensional module $V_3 \oplus V'_3$ (reducible). $\square$

Corollary 2.4 ([Derived, GAP 4.15.1]). *The dual pair $\{V_3, V'_3\} = \{\text{irr}_3|_{\mathbb{F}_2}, \text{irr}_{3'}|_{\mathbb{F}_2}\}$ constitutes the complete $\mathbb{F}_2$ -image of the characteristic-zero pair $\{\text{irr}_3, \text{irr}_{3'}\}$, and their direct sum $V_3 \oplus V'_3 = \text{irr}_6|_{\mathbb{F}_2}$ is the six-dimensional $\mathbb{F}_2[G]$ -module.*

Remark 2.5 ([Observation]). The sign-twisted representation $\text{irr}_{3'} = V'_3$ is *absent* from contexts A and B: in A, $\ker(L_a)|_{S_4} = \text{irr}_1 \oplus \text{irr}_3$ with no $\text{irr}_{3'}$ (Paper 37, Theorem 6.1 and OP 37.2 [Derived]); in B, $\text{irr}_{3'}$ appears only in the *fermionic* off-line sector $\text{irr}_8|_{S_4} = \text{irr}_2 \oplus \text{irr}_3 \oplus \text{irr}_{3'}$ (Paper 21, §4). V'_3 is present only in context C (as the second summand of N) and in the symmetric square of V_3 (Proposition 2.3). This asymmetry distinguishes the spinorial off-line sector and the bosonic on-line sector from the fermionic off-line sector irr_8 , and may be relevant for the physical interpretation of the matter-gravity split in §4.

Remark 2.6 ([Observation], new). The identity $\text{Sym}^2(V_3) = V_3 \oplus V'_3$ has a suggestive physical reading under IIB-II: if V_3 labels the gauge (colour-triplette) sector and V'_3 labels the gravity-adjacent dual sector, then the symmetric tensor product of two gauge quanta decomposes into one gauge and one dual-gauge mode. This is a characteristic-2 analogue of the colour-sextet decomposition $\mathbf{3} \otimes_s \mathbf{3} = \mathbf{6}$ in $\text{SU}(3)$, broken to $\mathbf{3} \oplus \bar{\mathbf{3}}$ by the mod-2 reduction. We flag this as a new observation; the physical interpretation requires a systematic analysis beyond this paper.

3. THE SCHUBERT TOWER

3.1. The subgroup chain.

Proposition 3.1 (*[Derived, GAP 4.15.1]*). *The subgroup chain*

$$(2) \quad \mathrm{PSL}(2, 7) \supset S_4 \supset V_4 \supset \{e\}$$

satisfies:

- (i) $S_4 = \mathrm{Stab}_{\mathrm{PSL}(2,7)}(\ell_a)$ is the stabiliser of a Fano line $\ell_a \in \mathrm{PG}(2, \mathbb{F}_2)$ in $\mathrm{PSL}(2, 7) \cong \mathrm{GL}(3, \mathbb{F}_2)$, of order $|S_4| = 24$;
- (ii) $V_4 = \ker(S_4 \rightarrow S_3) \trianglelefteq S_4$ is the unique Klein four-subgroup normal in S_4 , of order $|V_4| = 4$ (where $S_4 \rightarrow S_3 \cong S_4/V_4$ is the action of S_4 on the three points of the Fano line ℓ_a);
- (iii) the indices are $[\mathrm{PSL}(2, 7) : S_4] = 7$, $[S_4 : V_4] = 6$, $[V_4 : \{e\}] = 4$;
- (iv) the action matrices of $\mathrm{PSL}(2, 7)$ on $N = (\mathbb{Z}/2\mathbb{Z})^4$ satisfy $A^{|G|} = \mathrm{id}$, and the product relation $A_1 \cdot A_2 = \mathrm{ActionMat}(q_1 q_2)$ confirms the group homomorphism $\mathrm{PSL}(2, 7) \rightarrow \mathrm{GL}(4, \mathbb{F}_2)$.

Proof. Parts (i)–(iii): standard facts about $\mathrm{GL}(3, \mathbb{F}_2)$ acting on $\mathrm{PG}(2, \mathbb{F}_2)$ [16]. The Fano plane has 7 lines; each line stabiliser has index 7 in $\mathrm{PSL}(2, 7)$, hence order $168/7 = 24 = |S_4|$. The kernel of the action of S_4 on the 3-element complementary set of the Fano line is V_4 (Klein four-group), giving $[S_4 : V_4] = 6$ and $|V_4| = 4$. These are confirmed by `GAP 4.15.1` (script `OP_38_3_Schubert_tower_stabilizers_v2.g`, Parts 1–2).

Part (iv): the action matrices of the generators of $\mathrm{PSL}(2, 7)$ on $N = (\mathbb{Z}/2\mathbb{Z})^4$ are computed in `GAP 4.15.1` (script `OP_38_3_Schubert_tower_stabilizers_v4.g`, Parts 3–4): each action matrix satisfies $A^{|G|} = \mathrm{id}$, and the product relation $A_1 \cdot A_2 = \mathrm{ActionMat}(q_1 q_2)$ is verified, confirming the homomorphism property. An explicit realisation of G_{84} as a permutation group on 84 points is pending a technical issue in the `SemidirectProduct` `GAP` API; the subgroup chain (i)–(iii) and the stabilizer orders of Table 2 are established by parts (i)–(iii) and the orbit-stabilizer theorem from $|G_{84}| = 2688$. $\square$

TABLE 2. Orbital strata of G_{84} on $\mathrm{ZD}^*(\mathcal{A}_4)$ and the Schubert tower correspondence. The column “Schubert level” indicates the subgroup in (2) associated with each stratum; the stated $|\mathrm{Stab}|$ values are derived from $|G_{84}| = 2688$ and the orbit-stabilizer theorem; the identification of their PSL -level is an *[Observation]*.

Stratum	Orbit size	$ \mathrm{Stab} \text{ in } G_{84} $	Schubert level	Hurwitz algebra
1 (Fano poles)	2	1344	$\mathrm{PSL}(2, 7)$	$\mathcal{A}_3 = \mathbb{O}$, $\dim = 8$
2 (Level-2)	14	192	S_4	$\mathcal{A}_2 = \mathbb{H}$, $\dim = 4$
3 (ZD^*)	84	32	V_4	$\mathcal{A}_1 = \mathbb{C}$, $\dim = 2$
4 (ZD direct)	336	8	$\{e\}$	$\mathcal{A}_0 = \mathbb{R}$, $\dim = 1$

3.2. Correspondence with orbital strata. The stabilizer orders are derived from $|G_{84}| = 2688$ by the orbit-stabilizer theorem: $2688/2 = 1344$, $2688/14 = 192$, $2688/84 = 32$, $2688/336 = 8$.

Observation 3.2 (*[Observation]*). The subgroup chain $\mathrm{PSL}(2, 7) \supset S_4 \supset V_4 \supset \{e\}$ that governs the orbital stratification of G_{84} (Table 2) is parallel, stratum by stratum, to the Hurwitz hierarchy of Cayley–Dickson algebras $\mathcal{A}_3 \supset \mathcal{A}_2 \supset \mathcal{A}_1 \supset \mathcal{A}_0$ by the correspondence:

$$(3) \quad \mathrm{PSL}(2, 7) \leftrightarrow \mathcal{A}_3, \quad S_4 \leftrightarrow \mathcal{A}_2, \quad V_4 \leftrightarrow \mathcal{A}_1, \quad \{e\} \leftrightarrow \mathcal{A}_0.$$

The Hurwitz dimensions (8, 4, 2, 1) and the stratum indices (7, 6, 4) of (2) are independent numerical invariants of the same algebraic tower.

This observation was not stated in this form in any previous paper of the series; it is a new structural connection emerging from the systematic audit carried out for this paper.

3.3. The matter-force identification under IIB-II.

Observation 3.3 (*[Observation]*, *[Conjecture, IIB-II]*). Under the IIB-II identification of irr_1 as the gravitational singlet (Paper 37, Remark 6.2), the four strata of the Schubert tower correspond to the fundamental gauge sectors of the Standard Model plus gravity:

$$(4) \quad \text{PSL}(2, 7) \leftrightarrow \text{gravity}, \quad S_4 \leftrightarrow \text{SU}(3), \quad V_4 \leftrightarrow \text{SU}(2), \quad \{e\} \leftrightarrow \text{U}(1).$$

The orbit sizes (2, 14, 84, 336) then count the degrees of freedom at each gauge level. The assignment $S_4 \leftrightarrow \text{SU}(3)$ is consistent with the $\text{SU}(3)$ -colour interpretation of $\text{irr}_6|_{S_4}$ in Paper 21, Proposition 3.2. The assignment $V_4 \leftrightarrow \text{SU}(2)$ is consistent with the Klein-four structure as the elementary non-abelian two-group (lowest non-trivial instance in the Hurwitz hierarchy at $n = 2$).

Remark 3.4 (*[Observation]*, new). The successive indices 7, 6, 4 of the Schubert chain are not powers of 2 but factor as: $7 = |\text{lines}(\text{PG}(2, \mathbb{F}_2))|$ (number of lines of the Fano plane), $6 = |S_3| = |\text{Aut}(V_4)|$ (automorphisms of the Klein four-group), $4 = |V_4|$ (order of the group itself). Each index thus equals a canonical group-theoretic invariant of the corresponding Cayley–Dickson transition: $7 = |\text{PG}(2, \mathbb{F}_2)|$ parametrises the $\mathcal{A}_3 \rightarrow \mathcal{A}_2$ step; $6 = |\text{Aut}(V_4)|$ the $\mathcal{A}_2 \rightarrow \mathcal{A}_1$ step; $4 = |V_4|$ the $\mathcal{A}_1 \rightarrow \mathcal{A}_0$ step. This arithmetic is a new observation.

4. THE MATTER-GRAVITY SPLIT

Paper 26, Theorem 4.2 (conditional on the Sakharov–Adler mechanism and Papers 19, 29) gives the one-loop induced Newton constant:

$$(5) \quad G_N^{-1} = \frac{16\lambda N(C_0)}{12\pi} \left[164 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 220 \ln 2 - 42 \ln 3 \right].$$

This is the sum of a bosonic-pure contribution (coefficients 168, 224, 42) and a fermionic correction (from $\ker(L_a) = 4$, Paper 29, Theorem 3.1 and Corollary 3.2):

$$(6) \quad G_N^{-1} = \underbrace{\frac{16\lambda N(C_0)}{12\pi} \left[168 \ln \frac{\Lambda^2}{16\lambda N(C_0)} - 224 \ln 2 - 42 \ln 3 \right]}_{\text{bosonic-pure: structure } 2^a \cdot 3^b \cdot 7^c} + \underbrace{\frac{16\lambda N(C_0)}{12\pi} \left[-4 \ln \frac{\Lambda^2}{16\lambda N(C_0)} + 4 \ln 2 \right]}_{\text{fermionic correction: } \ker(L_a) = 4}.$$

Observation 4.1 (*[Observation]*). The coefficients of the bosonic-pure sector of (6) are:

Coefficient	Value	Factorisation	Algebraic origin
$\ln(\Lambda^2/\dots)$ (bosonic)	168	$2^3 \cdot 3 \cdot 7$	$\mathcal{S} = \text{GL}(3, \mathbb{F}_2) $, Paper 26 Prop. 3.3
$\ln 2$ (bosonic)	224	$2^5 \cdot 7$	supertrace $\text{Tr}(A^2)/2$, Paper 28 §5
$\ln 3$	42	$2 \cdot 3 \cdot 7$	Level-1 contribution, Paper 26 §3
$\ln(\Lambda^2/\dots)$ (total)	164	$4 \cdot 41$	NOT of the form $2^a \cdot 3^b \cdot 7^c$
$\ln 2$ (total)	220	$4 \cdot 5 \cdot 11$	NOT of the form $2^a \cdot 3^b \cdot 7^c$

Every coefficient in the bosonic-pure sector belongs to the multiplicative semigroup generated by $\{2, 3, 7\}$ — the prime divisors of $|G_{84}| = 2688$. The fermionic correction introduces the primes 41, 5, and 11, which do not divide $|G_{84}|$.

Remark 4.2 (*[Speculation]*). We interpret the $2^a \cdot 3^b \cdot 7^c$ structure as the *arithmetic fingerprint of pure gravity*: the spectral action of S_{84} with the four Majorana modes $\ker(L_a)$ switched off would produce a Newton constant whose logarithmic coefficients are entirely governed by $|G_{84}|$. The four

Majorana modes — the algebraic matter content, via Paper 29, Theorem 3.1 — break this structure in a precise and measurable way. This is a quantitative version of IIB-II-strong: schematically,

$$(7) \quad G_N^{-1} = \underbrace{[\text{term with structure } 2^a \cdot 3^b \cdot 7^c]}_{\text{“pure gravity”}} - \underbrace{[\ker(L_a) \cdot (\ln \Lambda^2 - \ln 2)]}_{\text{“matter signature”}},$$

(exact form: Paper 28 [8], Theorem 5.1), where $\ker(L_a) = 4 = 2^2$ is the only factor of matter in the formula, and the primes 41, 5, and 11 are “accidental” residues with no algebraic interpretation within the G_{84} -tower. Connection to the full IIB-II-strong programme [*Conjecture, IIB-II-strong*].

Remark 4.3 ([*Open*]). Whether the $2^a \cdot 3^b \cdot 7^c$ property of the bosonic-pure coefficients is a *theorem* — derivable from $|G_{84}| = 2^7 \cdot 3 \cdot 7$ and the action of $\text{PSL}(2, 7)$ on S_{84} — or merely an arithmetic coincidence is not established. A proof would require showing that the Sakharov–Adler spectral sum $\mathcal{S} = \sum_i d_i(2 + \mu_i)$ and the supertrace constants are expressible solely in terms of $|G_{84}|$ and its prime factorisation.

5. IIB-II + IIB-IV $\Rightarrow \alpha = 0$

Paper 37, Corollary 6.4 establishes that the $\text{PSL}(2, 7)$ -invariant boundary conditions for the Level-3 spinorial modes 4_s at the Black Circle radius $r = r_s$ form a two-parameter family $\text{BC}(\alpha, \beta)$. The Dirichlet condition $\chi(r_s) = 0$ corresponds to $\alpha = \beta \neq 0$. The parameter α controls confinement of the irr_1 gravitational singlet; β controls confinement of the irr_3 gauge triplet (Paper 37, Remark 6.2 and the proof of Corollary 6.4).

We now present a logical argument showing that the conjunction $\text{IIB-II} \wedge \text{IIB-IV}$ forces $\alpha = 0$.

Conjecture 5.1 ([*Conjecture, IIB-II + IIB-IV*]). Assuming IIB-II (gravity = N ; the irr_1 gravitational singlet is not confined by algebraic boundary conditions) and IIB-IV (the irr_1 singlet of Level-2 can cross the Conformal Cyclic Cosmology boundary; Paper 21, Remark 3.7; Paper 32, Cor. 5.3), the boundary condition for the Level-3 spinorial sector is

$$(8) \quad \text{BC}(0, \beta): \quad \beta \cdot P_{\text{irr}_3} \hat{F}|_{r_s} = 0, \quad \beta \neq 0,$$

i.e. $\alpha = 0$ and only irr_3 is confined.

Logical argument. Step 1 (from IIB-II). Under IIB-II, the irr_1 component of $\ker(L_a)|_{S_4}$ is identified as the gravitational singlet (Paper 37, Remark 6.2). IIB-II asserts that gravitational degrees of freedom are not confined by the algebraic boundary of the Black Circle but propagate freely. In terms of $\text{BC}(\alpha, \beta)$ (Paper 37, Corollary 6.4): free propagation of $\text{irr}_1 \Rightarrow \alpha = 0$, i.e. $P_{\text{irr}_1} \hat{F}|_{r_s} \neq 0$ is not forced to zero. In particular, $\alpha = \beta$ (full Dirichlet, IIB-I) is *incompatible* with IIB-II.

Step 2 (from IIB-IV). IIB-IV asserts that the irr_1 singlet of Level-2 is the unique CCC-crossing candidate (Paper 21, Remark 3.7; Paper 32, Cor. 5.3: it satisfies the three necessary conditions: massless, complementary-series boundary, S_4 -invariant). The irr_1 component of $\ker(L_a)|_{S_4}$ (Paper 37, Theorem 6.1) is the same type of S_4 -singlet as the Level-2 CCC candidate. IIB-IV therefore requires the boundary condition to be transmissive for irr_1 : $\alpha = 0$.

Step 3 (conjunction). IIB-II alone gives $\alpha \neq \beta$ (no full Dirichlet); IIB-IV alone gives $\alpha = 0$. Their conjunction gives $\alpha = 0$ uniquely. The full Dirichlet condition IIB-I ($\alpha = \beta$) is incompatible with $\text{IIB-II} \wedge \text{IIB-IV}$.

The residual freedom is $\beta \neq 0$: the irr_3 gauge triplet is still confined, but its confinement parameter β is not fixed by this argument. $\square$

Corollary 5.2 ([*Conjecture, IIB-II + IIB-IV*]). Under *Conjecture 5.1*:

- (i) *OP 37.1 (IIB-I vs. IIB-II tension) is partially resolved: IIB-I is eliminated under the conjunction $\text{IIB-II} \wedge \text{IIB-IV}$; the surviving boundary condition is $\text{BC}(0, \beta)$.*
- (ii) *OP 12.1 (selector from the two-parameter family) reduces to a one-parameter problem: determine β , the confinement amplitude of the irr_3 gauge triplet at $r = r_s$.*

(iii) *The cascade is:*

$$OP\ 37.1 \xrightarrow{IIB-II+IV} \alpha = 0 \xrightarrow{BC(0,\beta)} OP\ 12.1 \text{ reduced to } OP\ 37.4 \text{ (determine } \beta).$$

Remark 5.3 (*[Conjecture, IIB-II + IIB-IV]*). The exclusion of IIB-I under $IIB-II \wedge IIB-IV$ does not eliminate IIB-I from the series: it shows that the three conjectures IIB-I, IIB-II, IIB-IV form an inconsistent triple. One of them must be weakened or restricted in scope. The most natural weakening is to restrict IIB-I to the bosonic sector (Level-1 through Level-3 bosonic modes) and IIB-II to the spinorial sector (4_s modes), in which case no contradiction arises. This is an additional open question not formalised as a separate OP here.

6. OPEN PROBLEMS

OP 35.1 residual (V_3 -component) — *[Open]*. The identification $V_3 = \text{irr}_3|_{\mathbb{F}_2}$ (Proposition 2.1) closes the sub-question “which $\mathbb{F}_2$ -module is V_3 ?” of OP 35.1 as *[Derived, GAP 4.15.1]*. The remaining open question is to identify the V_3 -projection of the cocycle $[G_{84}] \in H^2(\text{PSL}(2, 7), N)$ as a specific class in $H^2(\text{PSL}(2, 7), V_3) \cong (\mathbb{Z}/2\mathbb{Z})^r$ (for some r to be computed) and to compare it with the low-order-bit extraction from $c_0(q_i, q_j)$ computed in Paper 36 [13], §4. The $\mathbb{F}_2$ -component of $[G_{84}]$ was identified with the low-order bit of c_0 by the June 2026 GAP computation of the present paper (script `OP_38_2_G84_cocycle_vs_c_oct_v3.g`, *[Derived, GAP 4.15.1]*).

OP B1 (ZD Dirac operator on $\ker$) — *[Open]*. Establish that the ZD Dirac operator D (Paper 23 [6]) restricted to the spinorial sector 4_s satisfies $D|_{\ker(L_a)} = D|_{V_{\text{ni}}(a)}$ as operators on $\ker(L_a) \cong V_{\text{ni}}(a)$ (Paper 29 [9], Theorem 3.1), and compute the spectrum of $D|_{\ker}$ in terms of the Fano non-incidence geometry. This would provide the algebraic foundation for the four Majorana spinors as eigenmodes of the ZD Dirac operator, directly from the Fano combinatorics. A proof sketch is available but requires formal verification of the γ -matrix closure $\gamma_p: \ker \rightarrow \ker$ for $p \notin \ell_a$.

OP 37.4 (energy minimisation of β) — *[Open]*. Among the one-parameter family $BC(0, \beta)$ selected by Conjecture 5.1, determine the value of β that minimises the vacuum energy of the Grassmann–Fock space $\bigwedge^\bullet(\mathbb{C}^4)$ [4] under the Dirac operator on the de Sitter interior with self-adjoint extension $BC(0, \beta)$. Computation via ζ -function regularisation on the de Sitter interior (see Paper 37 [14], OP 37.4 for detailed formulation). Resolution of OP 37.4 would close OP 12.1 under Conjecture 5.1.

OP 34.1 residual (G_2 -equivariant isomorphism) — *[Open]*. Establish the G_2 -equivariant isomorphism $W_2 \cong \text{adj}(G_2)$, where W_2 is the weight space at the quaternionic pole of the ZD zeta function $Z(x)$ and $\text{adj}(G_2)$ is the 14-dimensional adjoint representation of $G_2 = \text{Aut}(\mathbb{O})$. Paper 36 [13], §5 established $s_2 = \dim G_2 = 14$ numerically (*[Derived]*); the categorical isomorphism remains open (Paper 36, Remark 5.3). This is the remaining gap in OP 33.3 (opened in Paper 33 [11], partially closed in Paper 34 [12] and Paper 36 [13]).

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