THE MAJORANA DIRAC OPERATOR AND THE FANO–CAYLEY–DICKSON DOUBLING

GIOVANNI LEVRATTI

Abstract. We study the ZD Dirac operator $\mathcal{D} = \sum_{p\in\bar{\ell}_a} \gamma_p \partial_p$ restricted to the Majorana sector 4_s of the sedenion scalar field theory S_{84}, where $\gamma_p = L^{\mathcal{A}_3}_{e_p}$ are the Clifford generators (left-multiplication in the octonions) and $\bar{\ell}_a$ is the set of Fano points non-incident to the line ℓ_a associated with a ZD-active element a. We prove that $\mathcal{D}\big|_{4_s}$ does *not* map $\ker(L_a) = V_{\rm ni}(a)$ to itself *[Derived-Negative, GAP 4.15.1]*: every Clifford generator γ_p with $p \in \bar{\ell}_a$ maps $V_{\rm ni}(a)$ into the complementary on-line subspace ℓ_a. The correct closure holds at second order: $\mathcal{D}^2\big|_{4_s}$ maps $\ker(L_a)$ to itself *[Derived, GAP 4.15.1]*. The algebraic mechanism is the Cayley–Dickson doubling $\mathcal{A}_2 \to \mathcal{A}_3$: the decomposition $\mathrm{Im}(\mathcal{A}_3) = \ell_a \oplus V_{\rm ni}(a)$ identifies $\ell_a = \mathrm{Im}(\mathbb{H})$ (imaginary quaternions, pre-doubling sector) and $V_{\rm ni}(a) = \mathbb{H}\cdot e_4$ (post-doubling sector, isomorphic to $\ker(L_a)$). The complete bipartition table – $V_{\rm ni} \times V_{\rm ni} \to \ell_a$, $\ell_a \times V_{\rm ni} \to V_{\rm ni}$, $V_{\rm ni} \times \ell_a \to V_{\rm ni}$, $\ell_a \times \ell_a \to \ell_a$ – is verified by GAP 4.15.1 for all seven Fano lines simultaneously. As a consequence, $\mathcal{D}^2|_{\ker(L_a)}$ is a well-defined real 4×4 operator whose spectrum provides the algebraic data for the boundary-condition parameter β in $\mathrm{BC}(0,\beta)$ (Paper 38), reducing OP 37.4 and OP 12.1 to an explicit spectral computation. Two GAP 4.15.1 scripts are deposited as supplementary files.

Epistemic key. *[Derived]* complete proof. *[Derived, GAP 4.15.1]* machine-verified. *[Derived-Negative]* proves the negative. *[Open]* precisely stated gap.

1. Introduction

The ZD Dirac operator $\mathcal{D}$ was introduced in Paper 23 [5] as a differential operator coupling the 84 real scalar fields of S_{84} via the Clifford representation ρ of Paper 7 [2]. Papers 24 and 25 [6, 7] assembled the four-step fermionic chain $\rho \to \mathrm{irr}_8 = 4_s \to \mathrm{FS}(4_s) = +1 \to \mathcal{D}|_{4_s}$, establishing the Majorana sector on algebraic grounds. Paper 37 [9] identified the two-parameter family $\mathrm{BC}(\alpha,\beta)$ of S_4-invariant boundary conditions, and Paper 38 [10] fixed $\alpha = 0$ under the conjunction IIB-II + IIB-IV, reducing OP 12.1 to determining β.

The present paper addresses Open Problem B1 (opened in Paper 23): does $\mathcal{D}|_{4_s}$ close on $\ker(L_a) = V_{\rm ni}(a)$? The answer is *[Derived-Negative]*: the generator γ_p maps $V_{\rm ni}(a)$ into ℓ_a, not back into itself. The correct closure holds for $\mathcal{D}^2$, and the mechanism is the Cayley–Dickson doubling $\mathcal{A}_2 \to \mathcal{A}_3$. This is a new structural observation not noted in any previous paper of the series.

Prerequisites. Papers 7, 19, 23, 24, 25, 29, 37, 38.

2. Preliminaries

2.1. **The non-incident subspace $V_{\rm ni}(a)$ and its identification with $\ker(L_a)$.** Fix a ZD-active element $a \in \mathcal{A}_4$ and let $\ell_a \subset \mathrm{PG}(2,\mathbb{F}_2)$ be the Fano line determined by the lo_3-class of a. Following [8], define the *non-incident subspace*

$$V_{\rm ni}(a) \;:=\; \mathrm{span}_{\mathbb{R}}\{e_p : p \notin \ell_a\} \;\subset\; \mathcal{A}_3,$$

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where $\{e_p\}$ are the standard imaginary unit vectors of $\mathcal{A}_3 = \mathrm{Oct}$ with $p \in \{1, \ldots, 7\}$. Since $|\mathrm{PG}(2, \mathbb{F}_2)| = 7$ and each Fano line contains 3 points, we have $\dim V_{\mathrm{ni}}(a) = 4$.

Proposition 2.1 (*[Derived]*; [1] Thm. 5.17, [8] Lem. 2.6, Thm. 3.1)**.** *For every ZD-active element $a \in \mathcal{A}_4$:*

(i) $\dim_{\mathbb{R}} \ker(L_a) = 2^{n-2} = 4$ *at level* $n = 4$.
(ii) $\ker(L_a)$ *has an explicit basis* $\{k_1, k_2, k_3, k_4\}$ *with each* k_j *supported on the* lo_3*-classes of* $V_{\mathrm{ni}}(a)$ (GAP 4.15.1 computation; [8] Lem. 2.6).
(iii) $\ker(L_a) \cong V_{\mathrm{ni}}(a)$ *as* $\mathrm{Stab}_{G_{84}}(a)$*-modules* (GAP 4.15.1 + analytic; [8] Thm. 3.1).

In what follows we identify $\ker(L_a)$ with $V_{\mathrm{ni}}(a)$ via the isomorphism of item (iii) and write both interchangeably. The Fano line is denoted ℓ_a; the complementary off-line set is $\bar{\ell}_a = \{1, \ldots, 7\} \setminus \ell_a$, so $V_{\mathrm{ni}}(a) = \mathrm{span}\{e_p : p \in \bar{\ell}_a\}$.

2.2. **The Clifford representation ρ.** The Clifford representation of the ZD graph is defined in [2]:

Definition 2.2 (*[Derived]*; [2] Def. 4.1)**.** Let $\mathcal{A}_3 = \mathrm{Oct}$ and let $L^{\mathcal{A}_3}_{e_p}$ denote the 8×8 real matrix of left-multiplication by e_p in $\mathcal{A}_3$. Define

$$\rho : \mathcal{ZD}^*(\mathcal{A}_4) \longrightarrow \mathrm{Cl}(0, 7) \cong M_8(\mathbb{R}), \qquad \rho\big(e_p + \sigma e_{8+k}\big) := L^{\mathcal{A}_3}_{e_p}.$$

Write $\gamma_p := L^{\mathcal{A}_3}_{e_p} \in \mathrm{Cl}(0, 7)$ for $p \in \{1, \ldots, 7\}$.

The generators satisfy $\{\gamma_p, \gamma_q\} = -2\delta_{pq} I_8$ for all $p, q \in \{1, \ldots, 7\}$, in particular $\gamma_p^2 = -I_8$ *([2] Lem. 3.1)*. The map ρ depends only on the ε-index p; it is constant on each of the seven fibres $\{A \in \mathcal{ZD}^*(\mathcal{A}_4) : \varepsilon(A) = p\}$, each of cardinality $84/7 = 12$.

Remark 2.3 (Non-associativity)**.** Since $\mathcal{A}_3$ is *not* associative, $\gamma_p \circ \gamma_q \neq L^{\mathcal{A}_3}_{e_p \cdot e_q}$ in general: the Clifford product $\gamma_p \gamma_q$ is a bivector of grade 2, not a grade-1 generator. This distinction is central to the two-step mechanism proved in §3–§4.

2.3. **The ZD Dirac operator and its restriction to 4_s.**

Definition 2.4 (*[Derived]*; [5] Def. 2.1)**.** Let $\{\phi_a\}_{a \in \mathcal{ZD}^*(\mathcal{A}_4)}$ be the 84 real scalar fields of S_{84} [3]. The *ZD Dirac operator* is

$$\mathcal{D} \;:=\; \sum_{a \in \mathcal{ZD}^*(\mathcal{A}_4)} \gamma_a \frac{\partial}{\partial \phi_a} \;=\; \sum_{p=1}^{7} \gamma_p \, \widehat{\partial}_p, \qquad \widehat{\partial}_p := \sum_{\varepsilon(a) = p} \frac{\partial}{\partial \phi_a}.$$

$\mathcal{D}$ acts on $C^\infty(\mathbb{R}^{84}, \mathcal{S})$ where $\mathcal{S} \cong \mathbb{R}^8$ is the $\mathrm{Cl}(0, 7)$ spinor module.

The restriction to the Majorana (4_s) sector is determined by the Fano non-incidence structure. For the canonical representative $a = e_2 + e_{11}$ with $\ell_a = \{1, 2, 3\}$ and $\bar{\ell}_a = \{4, 5, 6, 7\}$:

$$\mathcal{D}\big|_{4_s} \;=\; \gamma_4 \partial_4 + \gamma_5 \partial_5 + \gamma_6 \partial_6 + \gamma_7 \partial_7 \;=\; \sum_{p \in \bar{\ell}_a} \gamma_p \, \partial_p, \tag{1}$$

where $\partial_p = \partial / \partial \phi_p$ and each $\gamma_p = L^{\mathcal{A}_3}_{e_p}$ carries the same off-line index $p \in \bar{\ell}_a = \{4, 5, 6, 7\}$ [5].

2.4. **The Clifford chain.** The following four-step chain, assembled in [7], underlies all subsequent analysis:

Corollary 2.5 (Fermionic chain; *[Derived]*; [7] Cor. 3.1)**.**

$$\begin{aligned} &\textit{(i)}\ \ \rho : \mathcal{ZD}^*(\mathcal{A}_4) \to \mathrm{Cl}(0,7) \quad [[2]\ \text{Def. 4.1}] \\ &\textit{(ii)}\ \ \mathrm{irr}_8 = 4_s \ = \ \mathrm{Cl}(0,7)\text{-}\textit{spinor module} \quad [[6]\ \text{Prop. 3.1}] \\ &\textit{(iii)}\ \ \mathrm{FS}(4_s) = +1 \ \Rightarrow \ \textit{four Majorana spinors} \quad [[4]] \\ &\textit{(iv)}\ \ \mathcal{D}\big|_{4_s} = \sum_{p \in \bar{\ell}_a} \gamma_p \partial_p \quad [[5]] \end{aligned}$$

Every link is [Derived] *without additional hypotheses.*

2.5. **The closure question.** The operator $\mathcal{D}|_{4_s}$ involves only the generators $\{\gamma_p : p \in \bar{\ell}_a\} = \{L^{\mathcal{A}_3}_{e_p} : p \in V_{\mathrm{ni}}(a)\}$. It is natural to ask whether $\mathcal{D}|_{4_s}$ maps $\ker(L_a) = V_{\mathrm{ni}}(a)$ to itself, i.e. whether

$$L^{\mathcal{A}_3}_{e_p}\big(V_{\mathrm{ni}}(a)\big) \ \subseteq \ V_{\mathrm{ni}}(a) \quad \forall\, p \in \bar{\ell}_a.$$

The answer is *[Derived-Negative]*: $L^{\mathcal{A}_3}_{e_p}$ maps $V_{\mathrm{ni}}(a)$ into ℓ_a, not back into $V_{\mathrm{ni}}(a)$. The correct closure holds at second order: $\mathcal{D}^2\big|_{4_s}$ maps $\ker(L_a) \to \ker(L_a)$. The algebraic mechanism and its proof occupy §§3–4.

3. The bipartition structure of $\mathrm{Im}(\mathcal{A}_3)$

3.1. **Cayley–Dickson decomposition of** $\mathrm{Im}(\mathcal{A}_3)$**.** Recall the Cayley–Dickson doubling $\mathcal{A}_2 \to \mathcal{A}_3$: $\mathcal{A}_3 = \mathcal{A}_2 \oplus \mathcal{A}_2 \cdot e_4$, where $\mathcal{A}_2 = \mathbb{H}$ is the quaternion sub-algebra. The imaginary part splits accordingly:

$$\mathrm{Im}(\mathcal{A}_3) \ = \ \underbrace{\mathrm{Im}(\mathbb{H})}_{\ell_a} \oplus \underbrace{\mathbb{H} \cdot e_4}_{V_{\mathrm{ni}}(a)}, \tag{2}$$

where, for the canonical Fano line $\ell_a = \{1, 2, 3\}$:

$$\begin{aligned} \ell_a \ &= \ \mathrm{Im}(\mathbb{H}) \ = \ \mathrm{span}\{e_1, e_2, e_3\} \quad \text{(imaginary quaternions, sub-algebra of } \mathcal{A}_3\text{)}, \\ V_{\mathrm{ni}}(a) \ &= \ \mathbb{H} \cdot e_4 \ = \ \mathrm{span}\{e_4, e_5, e_6, e_7\} \quad \text{(new directions from the CD doubling)}. \end{aligned}$$

This identification holds for all 7 Fano lines simultaneously by G_{84}-transitivity on the ZD vertex set [8].

3.2. **The complete bipartition table.**

Theorem 3.1 (*[Derived]*, GAP 4.15.1; all seven Fano lines)**.** *Let ℓ_a and $V_{\mathrm{ni}}(a)$ be as above, identified with $\mathrm{Im}(\mathbb{H})$ and $\mathbb{H} \cdot e_4$ respectively via the CD decomposition* (2)*. Under the octonion product in $\mathcal{A}_3$, the following inclusions hold for every Fano line ℓ_a:*

Left factor	*Right factor*	*Product lands in*
$V_{\mathrm{ni}}(a)$	$V_{\mathrm{ni}}(a)$	ℓ_a
ℓ_a	$V_{\mathrm{ni}}(a)$	$V_{\mathrm{ni}}(a)$
$V_{\mathrm{ni}}(a)$	ℓ_a	$V_{\mathrm{ni}}(a)$
ℓ_a	ℓ_a	ℓ_a

(GAP 4.15.1 computation; scripts deposited as supplementary files `OP_39_0_ker_closure_test.g` and `OP_39_1_bipartition_complete.g`.)

Proof. We verify the four cases explicitly.

Case 1: $V_{\rm ni}\times V_{\rm ni}\to\ell_a$. For $p,q\in\bar\ell_a=\{4,5,6,7\}$ with $p\neq q$, the octonion product $e_p\cdot e_q=\pm e_r$ is determined by the Fano XOR rule: $r=p\oplus q$ (bitwise XOR on $\mathbb{F}_2^3$). One checks directly that $p\oplus q\in\{1,2,3\}=\ell_a$ for all distinct $p,q\in\{4,5,6,7\}$:

$$\begin{aligned} &4\oplus5=1, \quad 4\oplus6=2, \quad 4\oplus7=3,\\ &5\oplus6=3, \quad 5\oplus7=2, \quad 6\oplus7=1.\end{aligned}$$

Hence $e_p\cdot e_q\in\ell_a$ for all $p\neq q$ in $\bar\ell_a$. GAP confirms this for all 7 Fano lines.

Case 4: $\ell_a\times\ell_a\to\ell_a$. $\mathbb{H}\subset\mathcal{A}_3$ is an associative sub-algebra; $\mathrm{Im}(\mathbb{H})$ is closed under the quaternion product. Explicitly: $\{e_1,e_2,e_3\}$ satisfy $e_1e_2=e_3$, $e_2e_3=e_1$, $e_3e_1=e_2$ (and antisymmetric transposes), all landing in ℓ_a.

Cases 2 and 3: $\ell_a\times V_{\rm ni}\to V_{\rm ni}$ and $V_{\rm ni}\times\ell_a\to V_{\rm ni}$. For $h\in\ell_a=\{1,2,3\}$ and $p\in\bar\ell_a=\{4,5,6,7\}$, the octonion product $e_h\cdot e_p=\pm e_{h\oplus p}$ and $e_p\cdot e_h=\mp e_{h\oplus p}$ [2], where $h\oplus p$ is bitwise XOR on $\mathbb{F}_2^3$. We verify directly that $h\oplus p\in\{4,5,6,7\}=\bar\ell_a$ for all $h\in\{1,2,3\}$, $p\in\{4,5,6,7\}$:

$$\begin{aligned} &1\oplus4=5, \quad 1\oplus5=4, \quad 1\oplus6=7, \quad 1\oplus7=6,\\ &2\oplus4=6, \quad 2\oplus5=7, \quad 2\oplus6=4, \quad 2\oplus7=5,\\ &3\oplus4=7, \quad 3\oplus5=6, \quad 3\oplus6=5, \quad 3\oplus7=4.\end{aligned}$$

All 12 values lie in $\{4,5,6,7\}=\bar\ell_a=V_{\rm ni}(a)$, establishing Cases 2 and 3 for the canonical line $\ell_a=\{1,2,3\}$. This argument uses only the Fano XOR rule $r=h\oplus p$, and in particular does not invoke associativity of $\mathcal{A}_3$. GAP verifies the analogous enumeration for all 7 Fano lines (script `OP_39_1_bipartition_complete.g`). □

Remark 3.2 (*[Derived]*)**.** Case 4 confirms that $\ell_a=\mathrm{Im}(\mathbb{H})$ is a *sub-algebra* of $\mathrm{Im}(\mathcal{A}_3)$ (isomorphic to $\mathfrak{su}(2)$ as a Lie algebra under the commutator bracket). Cases 2 and 3 confirm that $V_{\rm ni}(a)$ is a two-sided $\mathrm{Im}(\mathbb{H})$-module. Together, the four cases give a $\mathbb{Z}/2\mathbb{Z}$-graded *vector space* decomposition $\mathrm{Im}(\mathcal{A}_3)=\ell_a\oplus V_{\rm ni}(a)$ with multiplicative closure: $\ell_a\cdot\ell_a\subseteq\ell_a$ (even × even → even) and $\ell_a\cdot V_{\rm ni}+V_{\rm ni}\cdot\ell_a\subseteq V_{\rm ni}$ (even × odd → odd). Note: since $\mathcal{A}_3$ is non-associative, this is *not* a graded algebra in the standard sense; it is a graded module structure of $V_{\rm ni}(a)$ over the associative sub-algebra $\mathbb{H}$.

3.3. Direct closure is *[Derived-Negative]*.

Proposition 3.3 (*[Derived-Negative]*, GAP 4.15.1)**.** *For every $p\in\bar\ell_a=V_{\rm ni}(a)$:*

$$L^{\mathcal{A}_3}_{e_p}\big(V_{\rm ni}(a)\big)\ \subseteq\ \ell_a, \qquad L^{\mathcal{A}_3}_{e_p}\big(V_{\rm ni}(a)\big)\ \not\subseteq\ V_{\rm ni}(a).$$

Consequently, $\mathcal{D}\big|_{4_s}$ does not *map $\ker(L_a)$ to $\ker(L_a)$.*

Proof. Immediate from Case 1 of Theorem 3.1: $e_p\cdot e_q\in\ell_a$ for all $p,q\in\bar\ell_a$ with $p\neq q$, and $e_p\cdot e_p=-1\notin\mathrm{Im}(\mathcal{A}_3)$. Since $\gamma_p=L^{\mathcal{A}_3}_{e_p}$, the image $\gamma_p(e_q)=e_p\cdot e_q\in\ell_a\neq V_{\rm ni}(a)$ for all $q\in\bar\ell_a\setminus\{p\}$. GAP 4.15.1 exhaustively verifies this for all 7 Fano lines (script `OP_39_0_ker_closure_test.g`). □

Remark 3.4 (*[Derived]*)**.** Proposition 3.3 is not an obstacle but a structural signal: the ZD Dirac operator $\mathcal{D}|_{4_s}$ alternates between ℓ_a and $V_{\rm ni}(a)$ rather than preserving either summand separately. Closure is recovered at second order: $\mathcal{D}^2=\mathcal{D}\circ\mathcal{D}$ executes the two-step path $V_{\rm ni}\xrightarrow{\mathcal{D}}\ell_a\xrightarrow{\mathcal{D}}V_{\rm ni}$, as proved in §4.

4. $\mathcal{D}^2$ closes on $\ker(L_a)$

4.1. Main theorem.

Theorem 4.1 (*[Derived]*, GAP 4.15.1)**.** *Let $\mathcal{D}\big|_{4_s}=\sum_{p\in\bar\ell_a}\gamma_p\partial_p$ be the restriction of the ZD Dirac operator to the Majorana sector (1), with $\gamma_p=L^{\mathcal{A}_3}_{e_p}$. Then*

$$\mathcal{D}^2\big|_{4_s}\big(V_{\rm ni}(a)\big)\ \subseteq\ V_{\rm ni}(a).$$

That is, $V_{\mathrm{ni}}(a) = \ker(L_a)$ is an invariant subspace of $\mathcal{D}^2\big|_{4_s}$. (GAP 4.15.1; scripts deposited as supplementary files `OP_39_0_ker_closure_test.g` and `OP_39_1_bipartition_complete.g`.)

Proof. Write

$$\mathcal{D}^2\big|_{4_s} = \Big(\sum_{p\in\bar{\ell}_a} \gamma_p\partial_p\Big)^2 = \underbrace{\sum_{p\in\bar{\ell}_a} \gamma_p^2\partial_p^2}_{\text{diagonal}} + \underbrace{\sum_{\substack{p,q\in\bar{\ell}_a\\ p\neq q}} \gamma_p\gamma_q\,\partial_p\partial_q}_{\text{off-diagonal}}.$$

Diagonal terms. Since $\gamma_p^2 = -I_8$ for all p ([2] Lem. 3.1), the diagonal sum equals $-\big(\sum_p \partial_p^2\big)I_8$, which acts as $-I_8$ times the fibre Laplacian on the field space and as the identity on the spinor space $\mathcal{S}$. In particular it maps every subspace of $\mathcal{S}$, including $V_{\mathrm{ni}}(a)$, to itself.

Off-diagonal terms. Fix $p \neq q$ in $\bar{\ell}_a$ and $e_r \in V_{\mathrm{ni}}(a)$. We track the two-step action of $\gamma_p\gamma_q$ on e_r:

Step 1. $\gamma_q(e_r) = e_q\cdot e_r$. Since $p,q,r \in \bar{\ell}_a = V_{\mathrm{ni}}(a)$ and $q\neq r$, Theorem 3.1 Case 1 gives $e_q\cdot e_r \in \ell_a$.

Step 2. $\gamma_p\big(e_q\cdot e_r\big) = e_p\cdot(e_q\cdot e_r)$. Now $p\in\bar{\ell}_a = V_{\mathrm{ni}}(a)$ and $e_q\cdot e_r\in\ell_a$, so Theorem 3.1 Case 3 gives $e_p\cdot(e_q\cdot e_r)\in V_{\mathrm{ni}}(a)$.

Hence $\gamma_p\gamma_q(e_r)\in V_{\mathrm{ni}}(a)$ for every $e_r\in V_{\mathrm{ni}}(a)$ and every $p\neq q$ in $\bar{\ell}_a$. Since $V_{\mathrm{ni}}(a)$ is a real vector space and $\gamma_p\gamma_q$ is $\mathbb{R}$-linear, the off-diagonal part of $\mathcal{D}^2$ maps $V_{\mathrm{ni}}(a)$ to itself.

The case $q = r$ does not arise in Step 1, because $\gamma_q(e_q) = e_q\cdot e_q = -1 \notin \mathrm{Im}(\mathcal{A}_3)$: the operator $\gamma_q\partial_q$ annihilates the e_q-component of a spinor when acting on the imaginary sector, contributing only the scalar term absorbed into the diagonal.

Combining diagonal and off-diagonal: $\mathcal{D}^2(V_{\mathrm{ni}}(a)) \subseteq V_{\mathrm{ni}}(a)$. GAP 4.15.1 exhaustively verifies $e_p\cdot(e_q\cdot e_r)\in V_{\mathrm{ni}}(a)$ for all $p,q,r\in\bar{\ell}_a$ with $q\neq r$, for all 7 Fano lines (script `OP_39_1_bipartition_complete.g`). □

Remark 4.2 (*[Derived]*)**.** The two-step mechanism is the algebraic heart of this paper:

$$V_{\mathrm{ni}}(a) \xrightarrow{\mathcal{D}} \ell_a \xrightarrow{\mathcal{D}} V_{\mathrm{ni}}(a).$$

Each single application of $\mathcal{D}$ exchanges the two summands of the $\mathbb{Z}/2\mathbb{Z}$-graded decomposition $\mathrm{Im}(\mathcal{A}_3) = \ell_a \oplus V_{\mathrm{ni}}(a)$ (Remark 3.2). Closure is therefore a property of the *square*, not of the operator itself, exactly as Proposition 3.3 forces. The underlying algebraic mechanism is the CD doubling $\mathcal{A}_2\to\mathcal{A}_3$: $\ell_a = \mathrm{Im}(\mathbb{H})$ is the pre-doubling sector and $V_{\mathrm{ni}}(a) = \mathbb{H}\cdot e_4$ is the post-doubling sector.

4.2. Well-posedness on $V_{\mathrm{ni}}(a)$.

Corollary 4.3 (*[Derived]*)**.** *$\mathcal{D}^2\big|_{V_{\mathrm{ni}}(a)}$ is a well-defined $\mathbb{R}$-linear operator on the 4-dimensional real vector space $V_{\mathrm{ni}}(a)\cong\ker(L_a)$. Its matrix in the basis $\{e_4,e_5,e_6,e_7\}$ (canonical representative) is a real 4×4 matrix whose eigenvalues determine the spectrum of $\mathcal{D}^2$ on the Majorana sector.*

Proof. Immediate from Theorem 4.1: $V_{\mathrm{ni}}(a)$ is an invariant subspace of $\mathcal{D}^2\big|_{4_s}$, hence the restriction is a well-defined endomorphism of $V_{\mathrm{ni}}(a)$. Dimension 4 follows from Proposition 2.1(i). □

4.3. Reduction of OP 37.4 to a spectral computation.

Remark 4.4 (*[Open]*; OP 37.4 / OP 39.1)**.** Paper 38 [10] establishes $\alpha = 0$ in the two-parameter family $\mathrm{BC}(\alpha,\beta)$ of Paper 37 [9] *([10] Conj. 5.1, conditional on IIB-II + IIB-IV)*, and reduces the cascade OP 37.1 → OP 37.4 → OP 12.1 to the determination of β *([10] Cor. 5.2)*.

Corollary 4.3 provides the algebraic object whose spectrum governs β: the eigenvalues

$$\mathrm{spec}\big(\mathcal{D}^2\big|_{V_{\mathrm{ni}}(a)}\big) \subset \mathbb{R}$$

of the 4×4 real matrix of $\mathcal{D}^2|_{V_{\mathrm{ni}}(a)}$ are the candidate spectral data entering the zeta-function regularisation of the Dirac operator on the de Sitter interior with self-adjoint extension $\mathrm{BC}(0,\beta)$.

The explicit computation of $\mathrm{spec}(\mathcal{D}^2|_{V_{\mathrm{ni}}})$ and the identification of the energy-minimising β via zeta-function regularisation are *[Open]* (OP 39.1; carried from OP 37.4).

5. Consequences and cascade

5.1. Reduction of OP 37.4 to a spectral computation. Corollary 4.3 identifies the algebraic object governing the boundary-condition parameter β. Under the cascade of Paper 38 [10] (Conj. 5.1 and Cor. 5.2, conditional on IIB-II + IIB-IV):

$$\text{OP 37.1} \xrightarrow{\alpha=0} \text{OP 37.4} \xrightarrow{\mathrm{spec}(\mathcal{D}^2|_{V_{\mathrm{ni}}})} \text{OP 12.1 closed.}$$

The eigenvalues of the 4×4 real matrix of $\mathcal{D}^2\big|_{V_{\mathrm{ni}}(a)}$ in the basis $\{e_4, e_5, e_6, e_7\}$ are the candidate data entering the ζ-function regularisation of $\mathcal{D}$ on the de Sitter interior with self-adjoint extension $\mathrm{BC}(0, \beta)$ [9].

6. Open problems

OP 39.1 (spectrum of $\mathcal{D}^2|_{V_{\mathrm{ni}}}$) — *[Open]*. Compute the eigenvalues of $\mathcal{D}^2\big|_{V_{\mathrm{ni}}(a)}$ explicitly as a 4×4 real matrix in the basis $\{e_4, e_5, e_6, e_7\}$, and identify the energy-minimising value of β via ζ-function regularisation of $\mathcal{D}$ on the de Sitter interior with self-adjoint extension $\mathrm{BC}(0, \beta)$ [9]. Resolution of OP 39.1 closes OP 37.4 and, under Conj. 5.1 of Paper 38 [10], closes OP 12.1.

OP B1 residual — *[Open]*. The present paper establishes that $\mathcal{D}^2$ closes on $\ker(L_a)$. The original formulation of OP B1 (Paper 23 [5]) asked for $D|_{\ker} = D|_{V_{\mathrm{ni}}}$; this is answered *[Derived-Negative]* (Proposition 3.3) and *[Derived]* for D^2 (Theorem 4.1). The remaining gap is the explicit spectral analysis of OP 39.1.

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Independent Researcher, Modena, Italy
Email address: giovanni.levratti.research@gmail.com
URL: https://orcid.org/0009-0000-9804-3452