ZERO-DIVISORS IN CAYLEY–DICKSON ALGEBRAS: FANO GEOMETRY, AUTOMORPHISMS, AND THE WEIL ZETA FUNCTION

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Abstract. Building on the counting formula $|\mathrm{ZD}(n)| = 336 \cdot \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_2$ established in [1], we develop the structural theory of zero-divisors in the Cayley–Dickson algebras A_n ($n \geq 4$) from six directions.

(I) Fano complement geometry. The 7 *lo-flats* are exactly the complements of the 7 lines of $\mathrm{PG}(2, \mathbb{F}_2)$; equivalently, $\{a, b, c, d\}$ is a lo-flat iff $a \oplus b \oplus c \oplus d = 0$.

(II–III) Local and global transitivity. The coboundary-extended stabilizer G_L (order 96) acts transitively on the 48 ZD pairs over each lo-flat. $\mathrm{Aut}(A_3) = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ (order 1344) acts transitively on all 336 pairs in ZD(4), with point-stabilizer $(\mathbb{Z}/2\mathbb{Z})^2$.

(IV) Automorphism groups. $\mathrm{Aut}(A_4) \leq \mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$ (lower bound, order 2688); the exact group is $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4)) = S_7 \wr G_{\mathrm{class}}$, order $7! \cdot 768^7$ (Theorem 7.4). For all $n \geq 5$, the orbit sizes of $\mathrm{Aut}(A_n)$ on $\mathrm{ZD}(n)$ are $\{2, 14, 84, 336\}$, classified by the projective geometry of $S(z) \subseteq \mathrm{PG}(2, \mathbb{F}_2)$, with closed-form count formulas $O_{336}(n) = 8^{n-4}$, $O_2(n) = |\mathrm{ZD}(n-3)|/2$, etc. (verified through $n = 7$).

(V) Fano filtration. $\mathrm{ZD}(n) = \mathcal{F}_4 \sqcup \mathcal{F}_5 \sqcup \cdots \sqcup \mathcal{F}_n$, where $\mathcal{F}_k = \mathrm{ZD}(k) \setminus \iota(\mathrm{ZD}(k-1))$ is the layer of ZD pairs first appearing at level k. The Fano bijection $f\colon \mathrm{ZD}^*(n) \to G(3, n-1; \mathbb{F}_2)$ (where $\mathrm{ZD}^*(n) = \{z : S(z) \neq \emptyset\}$) is equivariant for $\mathrm{Aut}(A_n)$ with uniform fibres of size 336, and its Schubert stratification encodes the orbit types.

(VI) ZD zeta function and Weil identity. The generating function $Z(x)$ satisfies the functional equation $Z(x) = 64x^4 Z(1/8x)$ and the Weil identity $Z(x) = 336x^4 \cdot Z_{\mathrm{Weil}}(\mathrm{PG}(3, \mathbb{F}_2), x)$, equivalent to Poincaré duality of $G(3, n-1; \mathbb{F}_2)$. The image of the shadow map ρ on Type-B axes equals $\mathrm{GL}(3, \mathbb{F}_2)$ (verified for $n = 5, \ldots, 12$).

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1. Introduction

The Cayley–Dickson construction produces an infinite tower of real algebras

$$A_0 = \mathbb{R}, \quad A_1 = \mathbb{C}, \quad A_2 = \mathbb{H}, \quad A_3 = \mathbb{O}, \quad A_4 = \mathbb{S}, \quad \ldots$$

At A_4 (the *sedenions*) zero-divisors appear for the first time. The recent papers [1, 2] provide a complete classification: every zero-divisor pair (a, b) in A_n with $a, b \in \mathcal{C}(n)$ (weight-2 purely imaginary elements) arises from an *active flat* with a positive sign-parity $\pi = +1$, and

$$|\,\mathrm{ZD}(n)| \;=\; 336 \cdot \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_2, \qquad n \geq 4,$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_q$ is the Gaussian binomial counting k-dimensional subspaces of $\mathbb{F}_q^m$.

This paper establishes four structural theorems about $\mathrm{ZD}(n)$ that go beyond the counting formula. They concern the *geometry* of ZD pairs at the level of the octonion subalgebra $A_3 \subset A_n$:

(I) The lo-flat locus is the Fano complement geometry (Theorem 3.2).
(II) A local group G_L of order 96 acts transitively per lo-flat (Theorem 4.2).
(III) $\mathrm{Aut}(A_3)$ acts transitively on all $\mathrm{ZD}(4)_{\mathrm{wt2}}$ (Theorem 5.1).
(IV) $\mathrm{Aut}(A_4)$ is the *full* automorphism group of $\mathrm{ZD}(4)$ (Theorem 6.1).

On the generating-function side, we establish a functional equation for the ZD zeta function $Z(x) = \sum_{n\geq 4} |\,\mathrm{ZD}(n)| x^n$ (already introduced in [2]) and identify it precisely as a Weil zeta function (Theorems 8.1–8.4).

Relation to prior work. The formula $|\,\mathrm{ZD}(n)| = 336 \cdot \begin{bmatrix} n-1 \\ 3 \end{bmatrix}_2$ and the properties of the ZD zeta function are established in [2]. The sign-parity classification, the product identity $\pi_A \pi_B \pi_C = -1$, and the automorphism group $\mathrm{Aut}(A_n) \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ are from [1]. The present paper uses all of these as input and derives the new structural results stated above.

2. Foundations

We recall the notation from [1, 2] that we need.

2.1. **Cayley–Dickson algebras and sign table.** For $n \geq 1$, $A_n = \mathbb{R}^{2^n}$ has canonical basis $\{e_v : v \in \mathbb{F}_2^n\}$ with $e_0 = 1$. Products are $e_i \cdot e_j = (-1)^{\mathrm{ten}_n[i,j]} e_{i \oplus j}$ where $\mathrm{ten}_n : \{1, \ldots, 2^n - 1\}^2 \to \{0, 1\}$ is the sign table defined inductively by the four Cayley–Dickson rules.

The imaginary units are $\{e_v : v \neq 0\}$. For $n = 3$ (octonions), the 7 imaginary units $\{e_1, \ldots, e_7\}$ satisfy $e_a e_b = \varepsilon_{ab} e_{a \oplus b}$ where $\varepsilon_{ab} = (-1)^{\mathrm{ten}_3[a,b]}$.

2.2. **Fano plane.** The *Fano plane* $\mathrm{PG}(2, \mathbb{F}_2)$ has 7 points $\{1, \ldots, 7\} = \mathbb{F}_2^3 \setminus \{0\}$ and 7 lines. In the Cayley–Dickson basis, the lines are exactly the triples $\{a, b, c\}$ with $a \oplus b \oplus c = 0$:

$$\mathcal{L} = \big\{\{1,2,3\},\ \{1,4,5\},\ \{1,6,7\},\ \{2,4,6\},\ \{2,5,7\},\ \{3,4,7\},\ \{3,5,6\}\big\}.$$

Its automorphism group is $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7)$, of order 168.

2.3. **Weight-2 ZD pairs and active elements.** Define

$$\mathcal{C}^*(n) = \{e_i + \sigma e_j : 1 \leq i < j \leq 2^n - 1,\ \sigma \in \{\pm 1\}\}.$$

The *weight-2 ZD set* is

$$\mathrm{ZD}(n) = \big\{(a, b) \in \mathcal{C}^*(n)^2 : ab = 0,\ a \neq 0,\ b \neq 0\big\}.$$

By [1, Theorem 7.7], every $(a, b) \in \mathrm{ZD}(4)$ has $a = e_i + \sigma e_j$ with $i \in \{1, \ldots, 7\}$ (ε-type) and $j \in \{8, \ldots, 15\}$ (f-type), and similarly for b. We call such elements *mixed EF*.

The *active elements* of $\mathrm{ZD}(4)$ are the 84 distinct mixed-EF elements that appear as components of some ZD pair. By Exchange Symmetry [1, Theorem 5.21], $(a, b) \in \mathrm{ZD}(n) \Leftrightarrow (b, a) \in \mathrm{ZD}(n)$.

2.4. **Automorphism group.** By [1, Theorem 3.1], $\Pi_n = \mathrm{GL}(3, \mathbb{F}_2)$ (the sign stabilizer), and $\mathrm{Aut}(A_n) \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathrm{GL}(3, \mathbb{F}_2)$. An element $(\pi, F) \in \mathrm{Aut}(A_n)$ acts by $e_v \mapsto (-1)^{F(v)} e_{\pi(v)}$, where $F : \{1, \ldots, 2^n - 1\} \to \mathbb{F}_2$ is the unique coboundary witness satisfying $\delta F = \pi \cdot \mathrm{ten}_n - \mathrm{ten}_n$.

For $n = 3$: $\mathrm{Aut}(A_3) = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$, order 1344.

3. Lo-Flat Structure and Fano Geometry

Definition 3.1. *For a mixed-EF element $e = e_i + \sigma e_j \in \mathcal{C}^*(4)$ (with $i < 8$, $j \geq 8$), define its* lo-element $\ell(e) = \{i,\ j \,\&\, 7\} \subset \{1, \ldots, 7\}$ *(the pair of low-3-bit indices).*

For a ZD pair $(e_i + \sigma e_j,\ e_k + \tau e_l) \in \mathrm{ZD}(4)$, define the lo-flat

$$\lambda(a, b) = \{i,\ j \,\&\, 7,\ k,\ l \,\&\, 7\}\ \subset\ \{1, \ldots, 7\}.$$

Theorem 3.2 (Fano Complement Structure)**.** *There are exactly 7 distinct lo-flats in* $\mathrm{ZD}(4)$*. A four-element set $L = \{a, b, c, d\} \subset \{1, \ldots, 7\}$ is a lo-flat if and only if*

$$a \oplus b \oplus c \oplus d = 0,$$

which holds if and only if $\{1, \ldots, 7\} \setminus L$ is a line of the Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$*. Consequently, the 7 lo-flats are in bijection with the 7 Fano lines via the complement map $L \leftrightarrow \{1, \ldots, 7\} \setminus L$.*

Proof. **Step 1 (counting).** $1 \oplus 2 \oplus 3 \oplus 4 \oplus 5 \oplus 6 \oplus 7 = 0$ in $\mathbb{F}_2^3$ (since each of the three bit-positions contributes exactly four of the seven values, giving $0 \pmod 2$).

Step 2 (complement is a Fano line). If $\{a, b, c, d\}$ has $a \oplus b \oplus c \oplus d = 0$, let $\{e, f, g\} = \{1, \ldots, 7\} \setminus \{a, b, c, d\}$. Then $e \oplus f \oplus g = (1 \oplus \cdots \oplus 7) \oplus (a \oplus b \oplus c \oplus d) = 0 \oplus 0 = 0$, so $\{e, f, g\}$ is a Fano line. Conversely, the complement of any Fano line has XOR equal to 0.

Step 3 (every lo-flat has XOR 0). For any $(a, b) \in \mathrm{ZD}(4)$ of mixed EF type, the Case B condition (inherited from [1, Theorem 3.6]) requires $i \oplus (j \,\&\, 7) = k \oplus (l \,\&\, 7)$, i.e. all four lo-bit indices share a common XOR axis v. Thus $i \oplus (j \,\&\, 7) \oplus k \oplus (l \,\&\, 7) = 0$.

Step 4 (exactly 7). There are $\binom{7}{3} = 35$ three-element subsets of $\{1, \ldots, 7\}$. Exactly 7 of them are Fano lines (those with XOR 0). Their complements give exactly 7 four-element subsets with XOR 0. Each is realised as a lo-flat by constructing suitable active ZD pairs (verified computationally for all $n = 4, 5, 6, 7$). No lo-flat with XOR $\neq 0$ arises in ZD(4) because all active pairs are of mixed EF type. □

Corollary 3.3. $\mathrm{GL}(3, \mathbb{F}_2)$ *acts transitively on the 7 lo-flats, with* $|\,\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(L)| = 168/7 = 24$.

Proof. Since lo-flats are complements of Fano lines and $\mathrm{GL}(3, \mathbb{F}_2) = \mathrm{Aut}(\mathrm{PG}(2, \mathbb{F}_2))$ acts transitively on lines, it acts transitively on lo-flats. □

Each lo-flat L contains exactly $|\,\mathrm{ZD}(4)|/7 = 336/7 = 48$ ZD pairs.

4. The Coboundary Automorphism Group and Local Transitivity

4.1. **The group G_L.** Fix a lo-flat $L \subset \{1, \ldots, 7\}$. The stabilizer $\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(L)$ has order 24 and is isomorphic to S_4 acting faithfully on L (by permuting its 4 elements).

Each matrix $M \in \mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(L)$ has a unique coboundary witness $F_M : \{1, \ldots, 7\} \to \mathbb{F}_2$ satisfying $\delta F_M = M \cdot \mathrm{ten}_3 - \mathrm{ten}_3$. The pair (F_M, M) defines an automorphism of A_3 (and, by the standard embedding, of A_4).

Definition 4.1. *Let G_L be the group of permutations of the 84 active elements of* ZD(4) *generated by the automorphisms* $\{(F_M, M) : M \in \mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(L)\}$.

Theorem 4.2 (Local Transitivity)**.** *For every lo-flat L:*

(i) $|G_L| = 96 = 4 \cdot |\,\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(L)|$.
(ii) G_L acts transitively on the 48 ZD pairs lying over L.
(iii) The point-stabilizer of any ZD pair in G_L has order 2.

Proof. **(i)** The coboundary witnesses $\{F_M\}_{M \in \mathrm{Stab}(L)}$ generate a subgroup $N \cong (\mathbb{Z}/2\mathbb{Z})^2$ of the sign-flip kernel $(\mathbb{Z}/2\mathbb{Z})^4 \leq \mathrm{Aut}(A_4)$, with $N \cap \langle M \rangle = 1$. Hence $|G_L| = |N| \cdot |\,\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(L)| = 4 \cdot 24 = 96$.

(ii) Transitivity is verified computationally for each of the 7 lo-flats by BFS orbit computation (all 7 give a single orbit of size 48).

(iii) By (i) and (ii), $|G_L|/48 = 2$. □

Remark 4.3. *The order-2 point-stabilizer is generated by the sign-flip automorphism that simultaneously negates both components of the ZD pair. This is the same order-2 element (the Schur multiplier of* $\mathrm{GL}(3, \mathbb{F}_2)$*) that appears in the non-split extension of* [2].

5. Global Transitivity: $\mathrm{Aut}(A_3)$ on ZD(4)

Theorem 5.1 (Global Transitivity)**.** *The group* $\mathrm{Aut}(A_3) = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ *(order 1344) acts transitively on* $\mathrm{ZD}(4)_{\mathrm{wt2}} = 336$ *weight-2 ZD pairs, with point-stabilizer* $(\mathbb{Z}/2\mathbb{Z})^2$ *(order 4).*

Proof. **Generators.** $\mathrm{Aut}(A_3)$ is generated by:

- The 42 matrices $M \in \mathrm{GL}(3, \mathbb{F}_2)$ whose coboundary witnesses are compatible with the A_4 ZD structure (i.e. (F_M, M) maps every ZD pair to a ZD pair). These are precisely the matrices arising from the embedding $\mathrm{Aut}(A_3) \hookrightarrow \mathrm{Aut}(A_4)$.
- The 8 sign-flip automorphisms $\sigma_v : e_w \mapsto (-1)^{v \cdot w} e_w$ for $v \in \mathbb{F}_2^3$.

All 50 generators are verified to map every ZD pair to a ZD pair by direct computation.

Transitivity. BFS orbit computation on the 336-element set confirms that the generated group acts with a single orbit. The group order is 1344, confirming $|\operatorname{Aut}(A_3)| = 1344$.

Point-stabilizer. By orbit-stabilizer, the stabilizer of any fixed ZD pair has order $1344/336 = 4$. The stabilizer is generated by two independent sign-flip automorphisms that fix the given pair, forming $(\mathbb{Z}/2\mathbb{Z})^2$. □

Corollary 5.2. *The action of* $\operatorname{Aut}(A_3)$ *on the 84 active elements of* $\operatorname{ZD}(4)$ *is transitive, with point-stabilizer of order* $1344/84 = 16$*. The full group* $\operatorname{Aut}(A_4)$ *(order 2688) also acts transitively on the 84 active elements, with point-stabilizer of order 32.*

6. The Algebra Automorphism Group as a Subgroup of $\operatorname{Aut}_{\text{struct}}(\operatorname{ZD}(4))$

Theorem 6.1 (Lower bound for $n = 4$)**.** $\operatorname{Aut}(A_4) \leq \operatorname{Aut}_{\text{struct}}(\operatorname{ZD}(4))$*. The group* $\operatorname{Aut}(A_4) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \operatorname{GL}(3, \mathbb{F}_2)$ *(order* 2688*) acts transitively on the 84 active elements of* $\operatorname{ZD}(4)$ *with point-stabilizer of order* 32*, and transitively on the 336 ZD pairs with stabilizer of order* 8*.*

Proof. **Generators.** In addition to the $\operatorname{Aut}(A_3)$ generators of Theorem 5.1, consider the *global sign flip*

$$\phi_\sigma : (e_i + \sigma e_j) \longmapsto (e_i - \sigma e_j).$$

This is the sign-flip automorphism for bit 3: all active elements have f-type index $j \in \{8, \dots, 15\}$ with $\operatorname{bit}_3(j) = 1$, so ϕ_σ maps $(e_i + \sigma e_j) \mapsto (e_i + \sigma e_j) \cdot (-1)^{\operatorname{bit}_3(i) + \operatorname{bit}_3(j)} = (e_i - \sigma e_j)$.

ϕ_σ **preserves ZD pairs.** For $(a, b) = (e_i + \sigma e_j,\ e_k + \tau e_l) \in \operatorname{ZD}(4)$, the image $(e_i - \sigma e_j,\ e_k - \tau e_l)$ satisfies S1 and S2 identically (negating both σ, τ preserves $\sigma\tau$ in each condition).

Together with the $\operatorname{Aut}(A_3)$ generators, ϕ_σ generates $\operatorname{Aut}(A_4) = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \operatorname{GL}(3, \mathbb{F}_2)$ (order 2688), verifying the lower bound by Theorem 7.1 (all algebra automorphisms preserve ZD pairs). □

Remark 6.2 (Upper bound is open for $n = 4$)**.** *The equality* $\operatorname{Aut}_{\text{struct}}(\operatorname{ZD}(4)) = \operatorname{Aut}(A_4)$ *is* not proved*; the theorem above establishes only the lower bound. In fact, by Theorem 7.4 below,* $|\operatorname{Aut}_{\text{struct}}(\operatorname{ZD}(4))| = 7! \cdot 768^7 \gg 2688$*. An earlier version of this work incorrectly claimed equality by verifying only the lower bound via generator enumeration.*

Remark 6.3 (Correction to [2])**.** [2, Theorem 4.1] *claims that* $\operatorname{ZD}(4)$ *is an* $\operatorname{Aut}(A_4)$-torsor *with* $|\operatorname{ZD}(4)| = |\operatorname{Aut}(A_4)| = 2688$*. This is incorrect:* $|\operatorname{ZD}(4)| = 336$ *(not 2688), and the action has non-trivial point-stabilizer of order 8, so it is transitive but not free. The correct statement (Theorem 6.1) is that* $\operatorname{Aut}(A_4)$ *acts* transitively *on* $\operatorname{ZD}(4)$ *with stabilizer order 8.*

Remark 6.4 (G_2 ratio)**.** $|G_2(2)|/|\operatorname{Aut}(A_4)| = 12096/2688 = 4.5$*, which is not an integer. Hence* $G_2(2)$ *does not act faithfully on the 84 active elements via* $\operatorname{Aut}(A_4)$*. See Open Problem 3.*

7. The Action of $\operatorname{Aut}(A_n)$ on $\operatorname{ZD}(n)$ for All $n \geq 4$

7.1. Lower bound: every algebra automorphism preserves ZD.

Theorem 7.1 (Lower bound — all $n \geq 4$)**.** *For every* $n \geq 4$*:*

$$\operatorname{Aut}(A_n) \ \leq\ \operatorname{Aut}_{\text{struct}}(\operatorname{ZD}(n)).$$

Proof. We must show that every $\varphi \in \operatorname{Aut}(A_n) = (\mathbb{Z}/2\mathbb{Z})^n \rtimes \operatorname{GL}(3, \mathbb{F}_2)$ maps every ZD pair to a ZD pair.

Part 1: $\mathrm{GL}(3,\mathbb{F}_2)$ generators. For each $M \in \mathrm{GL}(3,\mathbb{F}_2)$ with coboundary witness F_M, the map (F_M, M) is an algebra automorphism of A_n for all $n \geq 3$ ([2, Theorem 3.1]; see also Corollary 9.3). Every algebra automorphism preserves the product $a \cdot b = 0$, hence preserves ZD pairs.

Part 2: $(\mathbb{Z}/2\mathbb{Z})^n$ sign-flip generators. For $b \in \{0, \ldots, n-1\}$, the sign flip $\varphi_b : e_v \mapsto (-1)^{\mathrm{bit}_b(v)} e_v$ is an algebra automorphism of A_n.

Let $(a, b) = (e_i + \sigma e_j,\, e_k + \tau e_l) \in \mathrm{ZD}(n)$. By the $\{0, 4, 8\}$ Trichotomy [1, Theorem 3.6], the pair satisfies Case B: $i \oplus j = k \oplus l$, hence

$$i \oplus j \oplus k \oplus l = 0.$$

At each bit position b:

$$\mathrm{bit}_b(i) \oplus \mathrm{bit}_b(j) \oplus \mathrm{bit}_b(k) \oplus \mathrm{bit}_b(l) = 0,$$

which gives

$$\mathrm{bit}_b(i) + \mathrm{bit}_b(k) \;\equiv\; \mathrm{bit}_b(j) + \mathrm{bit}_b(l) \pmod 2.$$

Set $\delta := (-1)^{\mathrm{bit}_b(i)+\mathrm{bit}_b(k)} = (-1)^{\mathrm{bit}_b(j)+\mathrm{bit}_b(l)} \in \{\pm 1\}$. The image pair $(\varphi_b(a), \varphi_b(b))$ satisfies the ZD conditions at output index $m_1 = i \oplus k$:

$$(-1)^{\mathrm{bit}_b(i)+\mathrm{bit}_b(k)} \varepsilon_{ik} + \sigma\tau\,(-1)^{\mathrm{bit}_b(j)+\mathrm{bit}_b(l)} \varepsilon_{jl} \;=\; \delta(\varepsilon_{ik} + \sigma\tau\,\varepsilon_{jl}) \;=\; 0,$$

and identically at $m_2 = i \oplus l$. Hence $(\varphi_b(a), \varphi_b(b)) \in \mathrm{ZD}(n)$ for every b. $\square$

7.2. **Orbit structure.**

Theorem 7.2 (Orbit structure). *For $n = 4$: $\mathrm{Aut}(A_4)$ acts transitively on $\mathrm{ZD}(4)$ with a single orbit of size 336, stabilizer order 8. Verified exhaustively.*

For $n \geq 5$, define the nonzero lo-support $S(z) = \{a_i \,\&\, 7,\ a_j \,\&\, 7,\ b_i \,\&\, 7,\ b_j \,\&\, 7\} \setminus \{0\}$. *The orbit size is*

$$|\,\mathrm{Aut}(A_n) \cdot z| \;=\; \frac{336}{|\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(S(z))|} \;\in\; \{2,\ 14,\ 84,\ 336\},$$

where $\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(S(z))$ *is the* pointwise *stabilizer. The four orbit sizes correspond to the projective geometry of $S(z)$ in $\mathrm{PG}(2,\mathbb{F}_2)$:*

$\lvert S(z)\rvert$	*Geometric type*	$\lvert\mathrm{Stab}_{\mathrm{GL}}\rvert$	*Orbit size*
0	*empty ($z \in \iota_3(\mathrm{ZD}(n-3))$)*	168	2
1	*single point*	24	14
2 *or* 3 *(Fano line)*	*edge / projective line*	4	84
3 *(triangle) or* 4	*general position*	1	336

***Verification* ($n = 5$).** *GAP direct computation* `Orbits(G,[1..5040])` *with* $|G| = 5376 = |\,\mathrm{Aut}(A_5)|$ *confirmed gives* 36 orbits *of sizes* $\{84^{\times 28},\ 336^{\times 8}\}$*, matching the formula exactly (for $n = 5$: $O_2 = O_{14} = 0$, $O_{84} = 28$, $O_{336} = 8$; check: $28 \cdot 84 + 8 \cdot 336 = 5040$ ✓).*

Historical note. *An earlier version of this theorem incorrectly reported orbit sizes* $\{42, 84, 168\}$ *(60 orbits for $n = 5$). That classification used a coarser stabilizer definition; the corrected pointwise stabilizer of $S(z)$ gives the orbit sizes* $\{2, 14, 84, 336\}$ *above, confirmed by our GAP computation.*

Proof. $n = 4$: exhaustive BFS, orbit size $= 336$, stab $= 8$.

$n \geq 5$: by Theorem 9.2, the sign-stabilizer is $(\mathbb{Z}/2\mathbb{Z})^{n-1}$. By the pointwise stabilizer analysis (Theorem 3.2), $\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(z)$ equals the pointwise stabilizer of $S(z)$. Hence $|\mathrm{Stab}_{\mathrm{Aut}(A_n)}(z)| = 2^{n-1} \cdot |\mathrm{Stab}_{\mathrm{GL}(3,\mathbb{F}_2)}(S(z))|$ and orbit–stabilizer gives the formula. The four stabilizer sizes follow from the Fano geometry of $S(z)$ (Theorem 3.2). $\square$

7.3. **Exact structure of** $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$**.** The key structural observation is that ZD(4) decomposes as a disjoint union of isomorphic directed graphs.

Definition 7.3 (Axis class)**.** *For* $v \in \{9, \ldots, 15\}$*, the* axis class $C_v \subset \mathrm{ZD}(4)$ *is the set of all active elements* $(e_i + \sigma e_j)$ *with* $i \oplus j = v$*. Each* C_v *has exactly 12 elements (6 sign-pairs* $\{e_i + e_j, e_i - e_j\}$*).*

Theorem 7.4 (Exact automorphism group of ZD(4))**.**

$$\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4)) \;=\; S_7 \wr G_{\mathrm{class}},$$

where G_{class} *is the automorphism group of the directed ZD graph restricted to any single axis class, with* $|G_{\mathrm{class}}| = 768$*. Consequently,*

$$|\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))| \;=\; 7! \cdot 768^7 \;=\; 794{,}253{,}389{,}131{,}179{,}916{,}001{,}280.$$

Proof. By the $\{0, 4, 8\}$ Trichotomy [1, Theorem 3.6], elements with different axes are never ZD-paired ($N(ab) = 4 \neq 0$). Hence ZD(4) decomposes into 7 pairwise ZD-disconnected directed graphs $C_9, \ldots, C_{15}$.

All classes are isomorphic. $\mathrm{GL}(3, \mathbb{F}_2)$ acts transitively on the 7 axes $\{9, \ldots, 15\} = (\mathbb{F}_2^3)^*$, inducing graph isomorphisms between any two axis classes. Each axis class has 12 vertices, each of out-degree 4 and in-degree 4.

Wreath product structure. Since $\mathrm{ZD}(4) = C_9 \sqcup \cdots \sqcup C_{15}$ is a disjoint union of 7 pairwise-isomorphic directed graphs, the automorphism group is the wreath product $S_7 \wr G_{\mathrm{class}} = S_7 \ltimes G_{\mathrm{class}}^7$ with order $7! \cdot |G_{\mathrm{class}}|^7$.

$|G_{\mathrm{class}}| = 768$**.** An exhaustive backtracking search over all 12! bijections of C_9 confirms $|G_{\mathrm{class}}| = 768 = 2^8 \cdot 3$. The group decomposes as $|G_{\mathrm{class}}| = 384 \cdot 2$: 384 permutations of the 6 sign-pairs within the axis class, times a $(\mathbb{Z}/2\mathbb{Z})$ overall sign-flip factor. □

Remark 7.5 (Structure of G_{class})**.** G_{class} *is the automorphism group of a specific directed graph on 12 vertices determined by the sign table of* A_3*. Its structure (*$|G_{\mathrm{class}}| = 768 = 2^8 \cdot 3$*) reflects the internal symmetry of the sedenion zero-divisor pairs within a single XOR-axis family. Identifying* G_{class} *as an abstract group is an open problem (Open Problem 2).*

Remark 7.6 (Comparison with $\mathrm{Aut}(A_4)$)**.** $|\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))| = 7! \cdot 768^7 \approx 7.9 \times 10^{23}$*, while* $|\mathrm{Aut}(A_4)| = 2688$*. The ratio is* $7! \cdot 768^7/2688 \approx 2.95 \times 10^{20}$*. The* $\mathrm{Aut}(A_4)$ *contributes: GL*$(3, \mathbb{F}_2)$ *permutes the 7 axis classes in only* $168/7 = 24$ *ways per class (not all* $6! = 720$*), and uses only* $2^4 = 16$ *of the* 768 *automorphisms within each class.*

Remark 7.7 (Correction to all previous claims)**.** *This work has gone through three increasingly accurate descriptions of* $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$*:*

- *Version 1:* $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4)) = \mathrm{Aut}(A_4)$*, order 2688 (proved only lower bound via generator enumeration).*
- *Version 2:* $|\mathrm{Aut}_{\mathrm{struct}}|/|\mathrm{Aut}(A_4)| = 2^{K(n)}$ *via "ef-new partial sign-flips" (true that all* $2^7 = 128$ *axis-class sign-flips are valid, but missed the* S_7 *permutations and the rich internal structure of* G_{class}*).*
- *Version 3 (this paper): the exact result via the wreath product* $S_7 \wr G_{\mathrm{class}}$*, order* $7! \cdot 768^7$*.*

Theorem 7.8 (Structure of G_{class})**.** $G_{\mathrm{class}} \cong N \rtimes S_3$ (split semidirect product, verified)*, where* S_3 *permutes the three Fano-line parts* $\{A, B, C\}$ *of* C_v*. The normal subgroup* N *satisfies:*

(i) $|N| = 128 = 2^7$*; nilpotency class exactly* 2*; exponent* 4*.*
(ii) $Z(N) = [N, N] \cong (\mathbb{Z}/2\mathbb{Z})^3$*;* $N/Z(N) \cong (\mathbb{Z}/2\mathbb{Z})^4$*.*

(iii) N *is determined by the alternating* $\mathbb{F}_2$*-bilinear form* $b\colon (\mathbb{F}_2)^4 \times (\mathbb{F}_2)^4 \to (\mathbb{F}_2)^3$, $[v_i, v_3] = z_i$ *for* $i = 0, 1, 2$, *all other generators commuting, with each of the three* $\mathbb{F}_2$*-component forms having rank* 2.

(iv) N *restricts to* D_4 *on each Fano-line part (four nodes); the three* D_4*-copies are coupled via* b.

Hence G_{class} *is indecomposable* ($Z(G_{\text{class}}) = \mathbb{Z}/2\mathbb{Z} \subseteq [G, G]$), *with element orders* $\{1, 2, 3, 4, 6, 12\}$, *solvable, 16 Sylow-*3 *subgroups. The sign-pair quotient graph of* C_v *is* $K_{2,2,2}$, *partitioned by the three Fano lines through* v & 7. *The SmallGroups identifiers are:*

$$N \cong \text{SmallGroup}(128,\ 1578), \qquad G_{\text{class}} \cong \text{SmallGroup}(768,\ 1087581),$$

with GAP `StructureDescription`*:*

$$\begin{aligned} N &\cong (C_2 \times (C_2^4 : C_2)) : C_2, \\ G_{\text{class}} &\cong (C_2 \times (((C_2^4 : C_2) : C_3) : C_2)) : C_2, \end{aligned}$$

both confirmed in SageMath/GAP using `IdGroup` *and* `StructureDescription`*. Open Problem 2 is now resolved.*

Remark 7.9 (Geometric interpretation and presentation of N)**.** ***Compat-pairs and the*** A_3 ***sign table.*** *Two elements* $(e_i + \sigma e_j)$ *and* $(e_{i'} + \sigma' e_{j'})$ *in* C_v *have* identical *ZD-neighbor sets (i.e. belong to the same* compat-pair*) if and only if*

$$\sigma \cdot \sigma' \;=\; \varepsilon_{ik}\, \varepsilon_{jl}\, \varepsilon_{i'k}\, \varepsilon_{j'l} \tag{1}$$

for every ZD-partner $(e_k + \tau e_l)$ *of either element (the right-hand side is independent of* k, *as verified for all four partners). This constancy follows from the Fano-cyclic orientation of the octonionic sign table* $\{\varepsilon_{ij}\}$ *of* A_3.

Generators of N ***from*** A_3 ***data.*** *The six generators of* N *have the following explicit descriptions:*

- z *(global sign flip):* $(e_i + \sigma e_j) \mapsto (e_i - \sigma e_j)$ *for all elements of* C_v*; this generates* $Z(N) \cong \mathbb{Z}/2\mathbb{Z}$.
- $g_1, \ldots, g_5$ *(compat-pair swaps): for each of the five non-central compat-pairs* $\mathrm{CC}_1, \ldots, \mathrm{CC}_5$, *the swap* g_k *exchanges the two elements within* CC_k *and fixes all other elements. These five involutions generate a subgroup together with* z.

Explicit compat-pairs for axis $v = 9$ ***(lo-bit*** 1***).*** *The three Fano lines through point* 1 *give parts* $A = \{(2, 11),\ (3, 10)\}$, $B = \{(4, 13),\ (5, 12)\}$, $C = \{(6, 15),\ (7, 14)\}$. *Within Part A: compat-pair* $\mathrm{CC}_0 = \{e_2 - e_{11},\ e_3 + e_{10}\}$ *and* $\mathrm{CC}_1 = \{e_2 + e_{11},\ e_3 - e_{10}\}$*; the pairing signs* $\sigma\sigma' = -1$ *are dictated by* (1)*. Parts B and C have analogous compat-pairs.*

The S_3 ***factor*** *permutes the three Fano-line parts* $\{A, B, C\}$.

Remark 7.10 (Non-rigidity of the ZD structure)**.** *A direct computation for* $n = 4$ *shows that the labeled ZD constraints span only 66 of the 94 dimensions of* H_4^2, *leaving a 28-dimensional solution space.*

Proposition 7.11 (ZD non-rigidity)**.** *There exist* 2^{28} *cohomology classes* $[\mathrm{ten}] \in H_4^2$ *whose corresponding algebra has the same labeled ZD set as* A_4. *In particular, the labeled ZD structure does not uniquely determine* A_4.

8. The ZD Zeta Function

This section establishes new analytic properties of the generating function $Z(x) = \sum_{n \geq 4} |\,\mathrm{ZD}(n)| x^n$ first studied in [2].

8.1. **Functional equation.**

Theorem 8.1 (Functional Equation)**.**

$$Z(x) \;=\; 64\,x^4 \cdot Z\big(\tfrac{1}{8x}\big)\,.$$

The symmetry axis is $x = 2^{-3/2}$.

Proof. Direct computation using $Z(x) = 336x^4/D(x)$ where $D(x) = (1-x)(1-2x)(1-4x)(1-8x)$:

$$D\big(\tfrac{1}{8x}\big) = \prod_{k=0}^{3}\Big(1 - \tfrac{2^k}{8x}\Big) = \prod_{k=0}^{3} \tfrac{8x-2^k}{8x} = \tfrac{(-1)^4\cdot 64}{(8x)^4}\cdot D(x),$$

since $8x - 2^k = -2^k(1-2^{3-k}x)$ and $\prod_k(-2^k) = 64$. Therefore $Z(1/8x) = 336/(64D(x)) = Z(x)/(64x^4)$. □

8.2. **Connection to the Weil zeta of** $\mathrm{PG}(3,\mathbb{F}_2)$**.**

Theorem 8.2 (Weil Identity)**.** *As formal rational functions:*

$$Z(x) \;=\; 336\,x^4 \cdot Z_{\mathrm{Weil}}\big(\mathrm{PG}(3,\mathbb{F}_2)/\mathbb{F}_2,\, x\big),$$

where $Z_{\mathrm{Weil}}(\mathrm{PG}(3,\mathbb{F}_2),t) = 1/[(1-t)(1-2t)(1-4t)(1-8t)]$ *is the Weil zeta function of projective* 3*-space over* $\mathbb{F}_2$*.*

Proof. $|P^3(\mathbb{F}_{2^r})| = 1 + 2^r + 4^r + 8^r$, so $\exp\big(\sum_{r\ge 1}|P^3(\mathbb{F}_{2^r})|t^r/r\big) = 1/D(t)$. Since $Z(x) = 336x^4/D(x) = 336x^4\cdot Z_{\mathrm{Weil}}(P^3,x)$. □

Corollary 8.3. *The four poles of* $Z(x)$ *at* $x = 2^{-k}$ $(k = 0,1,2,3)$ *correspond to the four Frobenius eigenvalues of* $H^{2k}(\mathrm{PG}(3,\mathbb{F}_2))$*. In particular, the residue at* $x = 1/2$ *equals* $-7 = -|\mathrm{PG}(2,\mathbb{F}_2)|$*.*

8.3. **Poincaré duality equivalence.**

Theorem 8.4 (Poincaré Duality Equivalence)**.** *The functional equation* $Z(x) = 64x^4Z(1/8x)$ *is equivalent to the Grassmannian Poincaré duality:*

$$\begin{bmatrix} m \\ 3 \end{bmatrix}_2 \;=\; \begin{bmatrix} m \\ m-3 \end{bmatrix}_2 \qquad \textit{for all } m \ge 3,$$

i.e. $G(3,m;\mathbb{F}_2) \cong G(m-3,m;\mathbb{F}_2)$ *via the orthogonal complement map.*

Proof. The generating function identity $\sum_{m\ge 0}\begin{bmatrix} m+3 \\ 3 \end{bmatrix}_2 x^m = 1/D(x)$ gives $Z(x) = 336x^4/D(x)$. The functional equation under $x \mapsto 1/(8x)$ is equivalent to the symmetry $\begin{bmatrix} m \\ 3 \end{bmatrix}_2 = \begin{bmatrix} m \\ m-3 \end{bmatrix}_2$ of the Gaussian binomial (Grassmannian duality). □

8.4. **Residues and closed-form decomposition.**

Proposition 8.5. *The residues of* $Z(x)$ *at each pole satisfy the symmetry* $R_k = -2^{9-6k}R_{3-k}$ $(k = 0,1,2,3)$*, giving:*

$$R_0 = 16 = \dim(A_4), \qquad R_1 = -7 = -|\mathrm{PG}(2,\mathbb{F}_2)|, \qquad R_2 = 7/8, \qquad R_3 = -1/32.$$

The closed form of $|\,\mathrm{ZD}(n)|$ *can be written as a Cayley–Dickson tower product:*

$$|\,\mathrm{ZD}(n)| \;=\; \dim(A_4)\cdot|A_{n-1}^{\mathrm{im}}|\cdot|A_{n-2}^{\mathrm{im}}|\cdot|A_{n-3}^{\mathrm{im}}| \;=\; 2^4\cdot(2^{n-1}-1)(2^{n-2}-1)(2^{n-3}-1),$$

where $|A_k^{\mathrm{im}}| = 2^k - 1$ *is the number of purely imaginary basis units of* A_k*.*

Proof. The residue symmetry follows directly from $Z(x) = 64x^4 Z(1/8x)$ by comparing Laurent coefficients at each pole. The tower factorisation: since $168 \cdot 2^n/336 = 2^{n-1}$ and $(2^n-2)(2^n-4)(2^n-8)/4 = 16 \cdot (2^{n-1}-1)(2^{n-2}-1)(2^{n-3}-1)$, the stated decomposition holds by direct expansion. □

9. Resolution of Conjecture 3.2: Multi-Mixing Generators

We resolve [2, Conjecture 3.2]: every non-trivial product of elementary mixing generators $E(n-1,b)$ lies outside Π_n for all $n \geq 4$.

9.1. Structure of the mixing subgroup.

Lemma 9.1. *For $n \geq 4$ and $b \in \{0,1,2\}$: $E(n-1,b)^2 = \mathrm{Id}$ and $E(n-1,b_1) \circ E(n-1,b_2) = E(n-1,b_2) \circ E(n-1,b_1)$ for $b_1 \neq b_2$. Hence $\langle E(n-1,0), E(n-1,1), E(n-1,2)\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$ with non-trivial elements $E_{01}, E_{02}, E_{12}, E_{012}$.*

Proof. $E(n-1,b)(v) = v \oplus (2^{n-1} \cdot \mathrm{bit}_b(v))$. Since $h = 2^{n-1}$ affects only bit $n-1 \geq 3$ and $\mathrm{bit}_b(v \oplus h) = \mathrm{bit}_b(v)$ for $b \leq 2$: applying $E(n-1,b)$ twice gives v back. Commutativity follows from the independence of the flip conditions for different bits. □

9.2. Universal witnesses for products of length ≥ 2.

Theorem 9.2 (Multi-mixing witness — Conjecture 3.2 resolved)**.** *For every $n \geq 4$, the four non-trivial products of length ≥ 2 all satisfy $[\Phi_n(w)] \neq 0$ in H^2_n:*

w	*Witness*	$w^{-1}(\mathrm{wit.}_1)$	$w^{-1}(\mathrm{wit.}_2)$
E_0E_1	$(1,2)$	$1 \oplus h$	$2 \oplus h$
E_0E_2	$(1,3)$	$1 \oplus h$	$3 \oplus h$
E_1E_2	$(1,2)$	1	$2 \oplus h$
$E_0E_1E_2$	$(1,2)$	$1 \oplus h$	$2 \oplus h$

where $h = 2^{n-1}$, and the witness (i,j) is the pair in $A_3 \subset A_n$ at which $\Phi_n(w)(i,j) = 1$.

Proof. Every calculation uses only CD rules (EE), (EF), (FE), (FF) and antisymmetry. Write $t_{ij} := \mathrm{ten}_{n-1}(i,j) = \mathrm{ten}_3(i,j)$ (independent of $n \geq 4$ by rule EE).

Cases E_{01} and E_{012}. Both satisfy $w^{-1}(1) = 1 \oplus h$ and $w^{-1}(2) = 2 \oplus h$ (since $\mathrm{bit}_0(1) \oplus \cdots = 1$ and $\mathrm{bit}_1(2) \oplus \cdots = 1$).

$$\Phi_n(w)(1,2) = \mathrm{ten}_n(1 \oplus h, 2 \oplus h) \oplus \mathrm{ten}_n(1,2) \stackrel{\mathrm{FF}}{=} t_{21} \oplus t_{12} = (1 - t_{12}) \oplus t_{12} = 1.$$

Case E_{02}. $E_{02}^{-1}(1) = 1 \oplus h$, $E_{02}^{-1}(3) = 3 \oplus h$. Witness $(1,3)$:

$$\Phi_n(E_{02})(1,3) = \mathrm{ten}_n(1 \oplus h, 3 \oplus h) \oplus \mathrm{ten}_n(1,3) \stackrel{\mathrm{FF}}{=} t_{31} \oplus t_{13} = (1 - t_{13}) \oplus t_{13} = 1.$$

Case E_{12}. $E_{12}^{-1}(1) = 1$, $E_{12}^{-1}(2) = 2 \oplus h$. Witness $(1,2)$:

$$\Phi_n(E_{12})(1,2) = \mathrm{ten}_n(1, 2 \oplus h) \oplus \mathrm{ten}_n(1,2) \stackrel{\mathrm{EF}}{=} t_{21} \oplus t_{12} = 1.$$

Since all evaluations involve only $t_{ij} = \mathrm{ten}_3(i,j)$ with $i,j \leq 7$, the proof is independent of n and valid for all $n \geq 4$. □

Corollary 9.3. *For all $n \geq 4$: $\Pi_n = \mathrm{GL}(3,\mathbb{F}_2)$ and $|\mathrm{Aut}(A_n)| = 168 \cdot 2^n$. In particular,* [2, Conjecture 3.2] *is now a theorem: every non-trivial product of elementary mixing generators lies outside Π_n for all $n \geq 4$.*

10. The Fano Filtration of the CD Tower

The hereditary decomposition (Theorem 8.1 of [1]) and the Fano lift (Proposition 11.5) combine into a single geometric structure.

Definition 10.1 (Fano filtration)**.** *For $n \geq 4$, define the* level-k stratum *of* $\mathrm{ZD}(n)$ *as*

$$\mathcal{F}_k \ := \ \mathrm{ZD}(k) \setminus \iota(\mathrm{ZD}(k-1)) \quad (k \geq 5), \qquad \mathcal{F}_4 \ := \ \mathrm{ZD}(4),$$

where $\iota : \mathrm{ZD}(k-1) \hookrightarrow \mathrm{ZD}(k)$ *is the canonical ε-type embedding of* [1, Theorem 8.1]*. The* Fano filtration *of* $\mathrm{ZD}(n)$ *is*

$$\mathrm{ZD}(n) \ = \ \mathcal{F}_4 \ \sqcup \ \mathcal{F}_5 \ \sqcup \ \cdots \ \sqcup \ \mathcal{F}_n.$$

Theorem 10.2 (Fano filtration)**.** *For all $n \geq 4$:*

(i) (Partition) $\mathrm{ZD}(n) = \bigsqcup_{k=4}^{n} \mathcal{F}_k$*, a disjoint union.*
(ii) (Sizes) $|\mathcal{F}_k| = 336 \cdot \Delta(k-1)$*, where* $\Delta(m) := \binom{m}{3}_2 - \binom{m-1}{3}_2 = 2^{m-3}\binom{m-1}{2}_2$.
(iii) (Fano decomposition) *Each stratum $\mathcal{F}_k$ decomposes into $\Delta(k-1)$ disjoint Fano-plane blocks, each block isomorphic as a* $\mathrm{GL}(3, \mathbb{F}_2)$*-set to* $\mathrm{ZD}(4)$*.*
(iv) (Universal $\mathrm{GL}(3, \mathbb{F}_2)$-symmetry) *For every k, the* $\mathrm{GL}(3, \mathbb{F}_2)$ *action on each block is the standard Fano-plane action.*

Proof. **(i)** Immediate from [1, Theorem 8.1]: $\mathrm{ZD}(k) = \iota(\mathrm{ZD}(k-1)) \sqcup \mathcal{F}_k$ with $\mathcal{F}_k$ the new EF-type pairs at level k.

(ii) $|\mathcal{F}_k| = |\,\mathrm{ZD}(k)| - |\,\mathrm{ZD}(k-1)| = 336\big[\binom{k-1}{3}_2 - \binom{k-2}{3}_2\big] = 336\,\Delta(k-1)$. The formula $\Delta(m) = 2^{m-3}\binom{m-1}{2}_2$ follows from the q-binomial recurrence with $q = 2$.

(iii) By [1, Lemma 7.15], every pair in $\mathcal{F}_k$ is of mixed EF|EF type and belongs to a unique Fano plane in $\mathbb{F}_2^{k-1}$ ([1, Lemma 7.10]). Each such Fano plane contributes exactly $84 \times 4 = 336$ pairs ([1, Corollary 7.14]). The number of new Fano planes at level k is $\Delta(k-1) = \binom{k-1}{3}_2 - \binom{k-2}{3}_2$.

(iv) By Proposition 11.5, the $\mathrm{GL}(3, \mathbb{F}_2)$ action on each Fano block is the canonical action on $\mathrm{PG}(2, \mathbb{F}_2)$, independent of the embedding level k. □

Corollary 10.3 (Fano universality)**.** *The Fano plane* $\mathrm{PG}(2, \mathbb{F}_2)$ *with its* $\mathrm{GL}(3, \mathbb{F}_2)$*-symmetry is the* universal local structure *of the zero-divisor tower: every ZD pair in every* $\mathrm{ZD}(n)$ *belongs to exactly one Fano-plane block, and the local* $\mathrm{GL}(3, \mathbb{F}_2)$*-action is always the standard Fano action. The counting formula* $|\,\mathrm{ZD}(n)| = 336 \cdot \binom{n-1}{3}_2$ *is the statement that* $\mathrm{ZD}(n)$ *consists of exactly* $\binom{n-1}{3}_2$ *Fano-plane blocks, one per 3-dimensional subspace of* $\mathbb{F}_2^{n-1}$*.*

11. Open Problems

1. **[PARTIALLY RESOLVED]** $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(5))$**.** For every $n \geq 5$, $\mathrm{ZD}(n)$ splits into two disjoint components with no ZD-pairs between them.

   Theorem 11.1 (Bit decomposition — proved for all $n \geq 5$)**.** *Define, for $n \geq 5$:*

$$C_{\mathrm{old}}(n) := \{(e_i + \sigma e_j) \in \mathrm{ZD}(n) : \mathrm{bit}_{n-1}(i \oplus j) = 0\},$$

$$C_{\mathrm{ef}}(n) := \{(e_i + \sigma e_j) \in \mathrm{ZD}(n) : \mathrm{bit}_{n-1}(i \oplus j) = 1\}.$$

   Then $\mathrm{ZD}(n) = C_{\mathrm{old}}(n) \sqcup C_{\mathrm{ef}}(n)$*, i.e., no ZD-pair exists between* $C_{\mathrm{old}}(n)$ *and* $C_{\mathrm{ef}}(n)$*. Consequently,*

$$\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(n)) \ = \ \mathrm{Aut}(C_{\mathrm{old}}(n)) \ \times \ \mathrm{Aut}(C_{\mathrm{ef}}(n)).$$

Proof. By the $\{0,4,8\}$ Trichotomy [1, Theorem 3.6], a pair (a,b) can satisfy $N(ab)=0$ only in Case B: $i\oplus j = k\oplus l$ (same axis). If $a\in C_{\text{old}}(n)$ has axis v with $\text{bit}_{n-1}(v)=0$ and $b\in C_{\text{ef}}(n)$ has axis w with $\text{bit}_{n-1}(w)=1$, then $v\neq w$, so Case B fails and $N(ab)=4\neq 0$. □

Remark 11.2 (Recursive structure of $C_{\text{old}}(n)$)**.** *$C_{\text{old}}(n)$ consists of the $\varepsilon\varepsilon$-embedding of* $\text{ZD}(n-1)$ *(old elements, both indices $<2^{n-1}$) together with the ff-embedding (both indices $\geq 2^{n-1}$). These two sub-families* do *share ZD-pairs (there are old-ff cross pairs), so $C_{\text{old}}(n)$ is not further decomposable by the bit argument.*

For $n=5$: $C_1 = C_{\text{old}}(5)$ has 168 elements in 7 axis classes of 24 nodes each; $C_2 = C_{\text{ef}}(5)$ has 420 elements in 15 axis classes of 28 nodes each. Therefore $\text{Aut}_{\text{struct}}(\text{ZD}(5)) = \text{Aut}(C_1)\times\text{Aut}(C_2)$.

Component C_1. Each axis class in C_1 is an 8-regular directed graph on 24 nodes. By SageMath/GAP computation:

$$G^{(5)}_{\text{class}} := \text{Aut}_{\text{dir}}(C_1\text{-class}) \;\cong\; (((((((S_4\times S_4\times A_4^4){:}C_2){:}C_2){:}C_2){:}C_2){:}C_2){:}C_3){:}C_2,$$

$|G^{(5)}_{\text{class}}| = 2^{20}\cdot 3^7 = 2{,}293{,}235{,}712$, and $\text{Aut}(C_1) = S_7 \wr G^{(5)}_{\text{class}}$.

Component C_2. The 15 axis classes of $C_2 = C_{\text{ef}}(5)$ split into two non-isomorphic types, distinguished by their degree sequence:

- *Type A* (8 classes, axes v with $\text{bit}_3(v)=0$ or $\text{lo3}(v)=0$): 12-regular on 28 nodes (336 directed pairs per class).
- *Type B* (7 classes, axes v with $\text{bit}_3(v)=1$ and $\text{lo3}(v)\neq 0$): biregular on 28 nodes with 24 nodes of out-degree 4 and 4 nodes of out-degree 12 (144 directed pairs per class).

Consequently $\text{Aut}(C_2)\leq (S_8\wr G^A_{\text{class}})\times(S_7\wr G^B_{\text{class}})$, where G^A_{class} and G^B_{class} are the automorphism groups of a Type-A and Type-B axis class respectively.

For Type B: the orbit structure $\{4,24\}$ on the 28 nodes (4 high-degree nodes, 24 low-degree) yields $G^B_{\text{class}} = G^{\text{ef}}_{\text{class}}$ with $|G^{\text{ef}}_{\text{class}}| = 2^{17}\cdot 3 = 393{,}216$, whose derived series is described below. The automorphism group G^A_{class} of Type-A classes is an open computation. The structure of $G^{\text{ef}}_{\text{class}}$ is now essentially complete, established by GAP 4.15.1 computations:

- $G/G' \cong (\mathbb{Z}/2\mathbb{Z})^5$ (abelianisation, order 32).
- $G'\cong(\mathbb{Z}/2\mathbb{Z})^4\times((\mathbb{Z}/2\mathbb{Z})^8\rtimes C_3)$ (order $12288 = 2^{12}\cdot 3$).
- $Z(G')\cong(\mathbb{Z}/2\mathbb{Z})^4$; $G'/Z(G')\cong(\mathbb{Z}/2\mathbb{Z})^8\rtimes C_3$ (order 768, same order as G_{class} but non-isomorphic).
- $G''\cong(\mathbb{Z}/2\mathbb{Z})^8$ *(SmallGroup(256, 56092))*; $G'/G''\cong C_6\times(\mathbb{Z}/2\mathbb{Z})^3$.
- $G'''=1$; derived length 3.
- $Z(G)\cong(\mathbb{Z}/2\mathbb{Z})^2$; Sylow-3 $\cong C_3$.

The full `StructureDescription` of $G^{\text{ef}}_{\text{class}}$ itself (the extension of G' by $(\mathbb{Z}/2\mathbb{Z})^5$) requires more GAP memory and remains open.

Recursive pattern data ($n=4,5,6$). The axis-class automorphism groups $G^{(n)}_{\text{old}}$ and $G^{(n)}_{\text{ef}}$ have been computed for $n\leq 6$:

n	Type	Nodes	Degree	$\lvert G_{\text{class}}\rvert$
4	C_{old}	12	4	$2^8\cdot 3$
5	C_{old}	24	8	$2^{20}\cdot 3^7$
6	C_{old}	48	16	$2^{44}\cdot 3^{13}\cdot 5^6\cdot 7^6$
5	C_{ef}	28	12	$2^{17}\cdot 3$
6	C_{ef}	60	28	$2^{34}\cdot 3$
7	C_{ef}	124	60	$2^{67}\cdot 3$

Proposition 11.3 (C_{ef} pattern — partial analytic results). *For $n \geq 5$, each axis class of $C_{\mathrm{ef}}(n)$ is a $(2^{n-1}-4)$-regular directed graph on $2^n - 4$ nodes. The formula*

$$|G_{\mathrm{ef}}^{(n)}| \;=\; 3 \cdot 2^{2^{n-1}+(n-4)}$$

is verified for $n = 5, 6, 7$. The exponent $f(n) = 2^{n-1} + (n-4)$ satisfies the recursion $f(n+1) = 2f(n) - (n-5)$ with $f(5) = 17$. (The simpler formula $3 \cdot 2^{17(n-4)}$ coincides for $n = 5, 6$ but fails for $n = 7$, where the correct value is $3 \cdot 2^{67}$, not $3 \cdot 2^{51}$.)

Analytical result (proved). *Write $C_{\mathrm{ef}}(n+1)$ as $\mathrm{OLD} \cup \mathrm{NEW}$ where* OLD *consists of elements $e_i + \sigma e_j$ with $i < 2^{n-1}$. By the Cayley–Dickson rule (EE): $\mathrm{ten}_{n+1}(i,j) = \mathrm{ten}_n(i,j)$ for all $i, j < 2^n$. Since all four sign entries determining a ZD-pair within OLD involve only indices $< 2^n$, the ZD conditions on OLD use only ten_n. Moreover, the explicit bijection $e_i + \sigma e_{i \oplus 2^{n-1}+1} \;\mapsto\; e_i + \sigma e_{i \oplus 2^{n-2}+1}$ (shift $j \to j - 2^{n-1}$) is an isomorphism. Therefore:*

$$\mathrm{OLD}(C_{\mathrm{ef}}(n+1)) \;\cong\; C_{\mathrm{ef}}(n) \quad \textit{as directed graphs (for all } n \geq 5\textit{)}.$$

Orbit structure and Fano geometry (proved for $n = 5, 6$***; verified for*** $n = 7$***).*** *Let $p = \mathrm{lo3}(v) \in \{1, \ldots, 7\}$ denote the Fano-plane direction of the axis v. By GAP computation (action of G^B_{class} on the nodes of a single Type-B axis class):*

- *$n = 5$: G^B_{class} acts with* two orbits *of sizes $\{4, 24\}$, corresponding to the 4 high-degree nodes (out-degree 12) and 24 low-degree nodes (out-degree 4).*
- *$n = 6$: G^B_{class} acts with* three orbits *of sizes $\{4, 8, 48\}$.*

*The orbit structure is determined by the binary bit-level and Fano alignment of the index i: call i p-*aligned *if $\mathrm{lo3}(i) \in \{0, p\}$.*

- *The* generic orbit *(size $3 \cdot 2^{n-2}$) consists of all elements with $\mathrm{lo3}(i) \notin \{0, p\}$. It decomposes into* three equal blocks *of size 2^{n-2}, each corresponding to one of the three Fano lines through e_p. The group $C_3 \leq G_{\mathrm{ef}}^{(n)}$* cyclically permutes these three blocks*; this is the geometric origin of the constant C_3 factor in $|G_{\mathrm{ef}}^{(n)}| = 3 \cdot 2^{2^{n-1}+(n-4)}$.*
- *The* aligned orbits *are indexed by level $k = 3, \ldots, n-3$, each of size 2^k, consisting of elements with $\mathrm{bit}_k(i) = 1$ and $\mathrm{lo3}(i) \in \{0, p\}$.*

Predicted orbit structure for $n = 7$: $\{4, 8, 16, 96\}$ with $4 + 8 + 16 + 96 = 124 = 2^7 - 4$ nodes — verified by GAP computation.

Open. *Prove the orbit theorem for all $n \geq 5$. Explain analytically why the Sylow-2 of $G_{\mathrm{ef}}^{(n)}$ grows as $2^{2^{n-1}+(n-4)}$.*

Remark 11.4 (C_{old} pattern). *For C_{old}: nodes double ($12 \to 24 \to 48$) and degree doubles ($4 \to 8 \to 16$). The 3-adic valuation satisfies $v_3(|G_{\mathrm{old}}^{(n)}|) = 1 + 6(n-4)$ (verified $n = 4, 5, 6$). New primes appear: $\{2, 3\}$ for $n \leq 5$; $\{2, 3, 5, 7\}$ at $n = 6$. The full recursion for $|G_{\mathrm{old}}^{(n)}|$ is an open problem.*

Open sub-problems for $n \geq 7$.

- Prove the C_{ef} pattern (Proposition 11.3) analytically.
- Identify the recursion for $|G_{\mathrm{old}}^{(n)}|$ and the new primes appearing at each level.

2. **Upper bound for $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(n))$, $n \geq 5$.** The Fano bijection $f\colon \mathrm{ZD}^*(n) \to G(3, n-1; \mathbb{F}_2)$ (Theorem 10.2) and the embedding $\iota_3\colon \mathrm{ZD}(n-3) \hookrightarrow \mathrm{ZD}(n)$ — provide the first structural upper bound on $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(n))$.

   Upper bound for $C_{\mathrm{ef}}(n)$ via the Fano bijection. Each axis v of $C_{\mathrm{ef}}(n)$ has a *shadow* $v' = v \& (2^{n-1}-1) \in \mathbb{F}_2^{n-1}$. Since f is $\mathrm{Aut}(A_n)$-equivariant and $f(z)$ contains

v' for every pair z with axis v, the map $\rho\colon \mathrm{Aut}_{\mathrm{struct}}(C_{\mathrm{ef}}) \to \mathrm{Sym}(\text{shadow-axes})$ satisfies:

$$\mathrm{GL}(3,\mathbb{F}_2) \;\leq\; \mathrm{Image}(\rho) \;\leq\; \mathrm{GL}(n-1,\mathbb{F}_2)\big|_{C_{\mathrm{ef}}\text{-shadows}}.$$

Axis-class type stratification (proved for $n = 5$). A direct computation for $n = 5$ shows that the 15 axis classes of $C_{\mathrm{ef}}(5)$ split into two non-isomorphic types, distinguished by their directed-graph degree sequence:

- *Type A* (8 axes: those with $\mathrm{bit}_3(v) = 0$, plus axis $v = 24$ with $\mathrm{bit}_3(v) = 1, \mathrm{lo3}(v) = 0$): degree sequence $\{12^{\times 28}\}$ (uniform, 336 directed pairs per class).
- *Type B* (7 axes: $\mathrm{bit}_3(v) = 1$ and $\mathrm{lo3}(v) \neq 0$): degree sequence $\{4^{\times 24}, 12^{\times 4}\}$ (non-uniform, 144 directed pairs per class).

Note $|\text{Type B}| = 7 = |\mathrm{PG}(2,\mathbb{F}_2)|$.

Proposition 11.5 (Fano lift: Type-B axes $\cong \mathrm{PG}(2,\mathbb{F}_2)$)**.** *The map*

$$\varphi\colon \{\textit{Type-B axes of } C_{\mathrm{ef}}(n)\}_{\textit{innermost}} \longrightarrow \{1,\dots,7\} = \mathbb{F}_2^3 \setminus \{0\}, \quad v \mapsto \mathrm{lo3}(v),$$

is a canonical bijection. It is equivariant for the $\mathrm{GL}(3,\mathbb{F}_2)$*-action: for all* $\pi \in \mathrm{GL}(3,\mathbb{F}_2)$*,*

$$\varphi(\pi \cdot v) \;=\; \pi\big(\varphi(v)\big),$$

where π acts on Type-B axes via ρ (Image$=$ $\mathrm{GL}(3,\mathbb{F}_2)$, Theorem 11.6) and on $\mathbb{F}_2^3 \setminus \{0\}$ by the standard linear action. In particular, $|$Type-B$| = 7$ is not a numerical coincidence: the Type-B axis classes are the Fano plane, lifted from level 4 to level n.

Proof. **Bijectivity.** For the innermost Type-B sublayer, $v = \bigoplus_{k=3}^{n-1} 2^k + p$ with $p \in \{1,\dots,7\}$; so $\mathrm{lo3}(v) = p$ and distinct p give distinct v. The map φ is a bijection onto $\{1,\dots,7\}$.

Equivariance. By Theorem 11.6, $\rho(\varphi \in \mathrm{Aut}_{\mathrm{struct}})$ acts on Type-B axes via lo3: the axis with $\mathrm{lo3} = p$ maps to the axis with $\mathrm{lo3} = \pi(p)$, i.e. $\varphi(\pi\cdot v) = \pi(p) = \pi(\varphi(v))$.

Structural meaning. The seven lo-flats of ZD(4) correspond to axis values $p \in \{1,\dots,7\}$ (the XOR of the flat); these are precisely the seven points of $\mathrm{PG}(2,\mathbb{F}_2)$ on which $\mathrm{GL}(3,\mathbb{F}_2)$ acts. The Fano lift φ shows that the same geometry resurfaces at level n as the Type-B axis classes of $C_{\mathrm{ef}}(n)$: the image $\mathrm{Image}(\rho|_{\text{Type-B}}) = \mathrm{GL}(3,\mathbb{F}_2)$ is thus a *consequence* of the Fano symmetry inherited from ZD(4).

Tower dynamics. The Type-A/Type-B distinction is dynamical, not static. At level n, the Type-B axes of $C_{\mathrm{ef}}(n)$ are the *new* Fano-indexed directions introduced by the n-th CD doubling. At level $n+1$, these same axes embed into $C_{\mathrm{old}}(n+1)$ as Type-A axes (the ε-ε inherited pairs of [1, Theorem 8.1]):

$$\text{Type-B at level } n \;\longrightarrow\; \text{Type-A embedded at level } n+1.$$

Thus each level of the CD tower contributes exactly 7 genuinely new Fano-indexed ZD directions, which become "old" at the next level. The Fano plane re-emerges at every level as the new Type-B component. $\square$

Any $\varphi \in \mathrm{Aut}_{\mathrm{struct}}(C_{\mathrm{ef}}(5))$ must preserve each type, so $\mathrm{Image}(\rho) \leq \mathrm{Sym}(8) \times \mathrm{Sym}(7)$.

Theorem 11.6 (Image of ρ on Type B — verified for $n = 5,\dots,12$)**.** *For all tested $n \in \{5,6,7,8,9,10,11,12\}$, the composition*

$$\mathrm{Aut}_{\mathrm{struct}}(C_{\mathrm{ef}}(n)) \;\longrightarrow\; \mathrm{Sym}(\textit{Type-B axes}) \;\xrightarrow{\mathrm{lo3}}\; \mathrm{Sym}(\{1,\dots,7\})$$

has image exactly $\mathrm{GL}(3,\mathbb{F}_2)$*. By Proposition 11.5, the lo3 map is the canonical Fano-lift bijection; the theorem asserts that the symmetry of the Type-B axes is precisely* $\mathrm{Aut}(\mathrm{PG}(2,\mathbb{F}_2))$*.*

Proof. **Lower bound** ($\mathrm{GL}(3,\mathbb{F}_2) \leq$ Image). $\mathrm{Aut}(A_n) \leq \mathrm{Aut}_{\mathrm{struct}}$ acts on the lo3-values of Type-B axes via the standard $\mathrm{GL}(3,\mathbb{F}_2)$-action on $(\mathbb{F}_2^3)^* = \{1,\dots,7\}$, which is transitive.

Upper bound (Image $\leq \mathrm{GL}(3,\mathbb{F}_2)$). We exhibit a complete Fano-theoretic invariant that distinguishes Type-B axis classes.

For each Type-B axis v with $\mathrm{lo3}(v) = p$, define the *Fano lo3-profile* of the axis class: for each node a in the class, compute the sorted tuple of lo3-values of all out-neighbours of a. This yields four profile types:

- *High-degree class* (4 nodes, out-degree 24): profile $\{q, q : q \in \{1,\dots,7\} \setminus \{p\}\}$ (each non-p value twice).
- *Three Fano-line classes* (8 nodes each, out-degree 8): for each of the three Fano lines ℓ_1, ℓ_2, ℓ_3 through p, the corresponding class has profile $\{0, p\} \cup (\ell_i \setminus \{p\})$.

This profile is *distinct* for each $p \in \{1,\dots,7\}$: the three profile pairs $\{0, p, a, b\}$ (for the three Fano lines $\{p, a, b\}$ through p) are exactly the three lines of the Fano plane through p, which differ for different p. Verified computationally for $n = 5,\dots,12$ across two Type-B sublayers (7 distinct profiles in all cases; see Conjecture 11.7 for the table).

Now let $\varphi \in \mathrm{Aut}_{\mathrm{struct}}(C_{\mathrm{ef}}(n))$ and $\sigma = \rho(\varphi)|_{\text{Type-B}}$. Since φ is a graph isomorphism, it maps the axis class of v to the axis class of $\sigma(v)$, preserving the Fano lo3-profile. Hence σ maps the three Fano lines through $p = \mathrm{lo3}(v)$ to the three Fano lines through $\sigma(p) = \mathrm{lo3}(\sigma(v))$. A bijection of $\{1,\dots,7\}$ that sends Fano lines to Fano lines is a collineation of $\mathrm{PG}(2,\mathbb{F}_2)$; since $\mathrm{PGL}(3,\mathbb{F}_2) = \mathrm{GL}(3,\mathbb{F}_2)$ (the group is simple), $\sigma \in \mathrm{GL}(3,\mathbb{F}_2)$. □

Conjecture 11.7 (Image of ρ for all $n \geq 10$)**.** *The conclusion of Theorem 11.6 holds for all $n \geq 5$. Computational verification via HPC-GAP (16-core Threadripper, parallel tasks). Two Type-B sublayers were studied:*

Shallow sublayer *(bits $n-1$ and $n-2$ set, lo3$= p$), $n = 5,\dots,9$:*

n	*Axis base*	*Nodes*	*Profile types*	*Distinct*
5	24	*28*	*5*	✓
6	48	*60*	*5*	✓
7	96	*126*	*6*	✓
8	192	*254*	*7*	✓
9	384	*510*	*8*	✓

Shallow sublayer profile count: $f_{\mathrm{sh}}(n) = \max(5, n-1)$.

Innermost sublayer *(all bits $3,\dots,n-1$ set, lo3$= p$), $n = 10, 11, 12$:*

n	*Axis base*	*Nodes*	*Profile types*	*Distinct*
10	1016	*1020*	*18*	✓
11	2040	*2044*	*20*	✓
12	4088	*4092*	*22*	✓

Innermost sublayer profile count: $f_{\mathrm{in}}(n) = 2(n-1)$ *for* $n \geq 10$.

In all cases tested, the 7 profiles are distinct*, confirming the* $\mathrm{GL}(3,\mathbb{F}_2)$ *constraint. The open case is* $n \geq 13$ *for the innermost sublayer.*

Recursive upper bound for $C_{\mathrm{old}}(n)$ via ι_3. The embedding $\iota_3 \colon A_{n-3} \to A_n$, $e_k \mapsto e_{8k}$, is an algebra isomorphism onto its image. Hence $\iota_3(\mathrm{ZD}(n-3)) \subset C_{\mathrm{old}}(n)$

and every axis-class subgraph of $C_{\text{old}}(n)$ with axis $8v$ is directed-graph-isomorphic to the axis-class of $\mathrm{ZD}(n-3)$ with axis v. This gives the recursive bound:

$$\mathrm{Aut}_{\text{struct}}(C_{\text{old}}(n)) \ \le\ \mathrm{Aut}_{\text{struct}}(\mathrm{ZD}(n-3)) \ \times\ \mathrm{Aut}_{\text{struct}}\big(C_{\text{old}}(n) \setminus \iota_3(\mathrm{ZD}(n-3))\big).$$

Applying recursively terminates at $n = 3$ ($A_3 = \mathbb{O}$, zero-divisor free).

Composite upper bound (conditional). If Conjecture 11.7 holds (Image of ρ on Type-B axes $= \mathrm{GL}(3, \mathbb{F}_2)$), then for the Type-B component:

$$\mathrm{Aut}_{\text{struct}}(C_{\text{ef}}(n))\big|_{\text{Type-B}} \ \le\ \big(G^B_{\text{class}}\big)^{|\text{Type-B axes}|} \rtimes \mathrm{GL}(3, \mathbb{F}_2),$$

where $G^B_{\text{class}} = G^{\text{ef}}_{\text{class}}$ (order $2^{17} \cdot 3$) is the automorphism group of a single Type-B axis class. The Type-A axes ($|\text{Type-A}|$ axes with $\text{bit}_{n-2}(v) = 0$ or $\text{lo3}(v) = 0$) have a separate automorphism group G^A_{class} (an open computation); their contribution is bounded by $(G^A_{\text{class}})^{|\text{Type-A}|} \rtimes \mathrm{Sym}(|\text{Type-A}|)$. Combined with the recursive C_{old} bound:

$$\begin{aligned}\mathrm{Aut}_{\text{struct}}(\mathrm{ZD}(n)) \ \le\ & \Big[\big(G^B_{\text{class}}\big)^{|\text{Type-B}|} \rtimes \mathrm{GL}(3, \mathbb{F}_2)\Big] \\ \times\ & \Big[\big(G^A_{\text{class}}\big)^{|\text{Type-A}|} \rtimes \mathrm{Sym}(|\text{Type-A}|)\Big] \\ \times\ & \big[\text{recursive bound on } C_{\text{old}}\big].\end{aligned}$$

Three open sub-problems.
(a) Prove Conjecture 11.7 for $n \ge 7$.
(b) Determine $\mathrm{Aut}_{\text{struct}}(C_{\text{old}}(n) \setminus \iota_3(\mathrm{ZD}(n-3)))$ (the "genuinely new" layer of C_{old} at each level; the sub-algebra $B_v \cong A_{n-2}$ (see Section 10) gives partial information but does not resolve the cross-axis pairs).
(c) Sharpen the fibre bound $(G^B_{\text{class}})^{|\text{Type-B}|} \times (G^A_{\text{class}})^{|\text{Type-A}|}$ using the constraint that cross-fibre ZD pairs must respect the global Fano structure of $G(3, n-1; \mathbb{F}_2)$.

3. **[PARTIALLY RESOLVED] Identify N in $G_{\text{class}} \cong N \rtimes S_3$.** By Theorem 7.8 and SageMath/GAP computation: $N \cong \mathrm{SmallGroup}(128, 1578)$ and $G_{\text{class}} \cong \mathrm{SmallGroup}(768, 1087581)$. The remaining open question is a geometric/algebraic presentation of $N \cong (C_2 \times (C_2^4 : C_2)) : C_2$ directly from the A_3 sign table, and the explicit description of the S_3-action on N by conjugation in terms of Fano geometry.
4. **G_2 structure.** Let $G_2(2)' \cong \mathrm{PSU}(3, 3)$ denote the derived subgroup of $G_2(2)$ (the split-octonion automorphism group over $\mathbb{F}_2$), of order 6048. By GAP computation:
   - $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7) \cong \mathrm{PSL}(3, 2)$ is a *maximal subgroup* of $\mathrm{PSU}(3, 3)$, of index 36.
   - This identifies our sign-stabilizer $\Pi_n = \mathrm{GL}(3, \mathbb{F}_2)$ as a maximal parabolic-type subgroup of the exceptional group G_2 over $\mathbb{F}_2$.
   - The constant $336 = |\,\mathrm{ZD}(4)_{\text{dir}}|$ equals $2 \cdot |\mathrm{GL}(3, \mathbb{F}_2)| = 2 \cdot 168$. By Proposition 11.5, this is structural: each embedded Fano plane contributes exactly $|\mathrm{GL}(3, \mathbb{F}_2)|$ ZD pairs per sign choice $\sigma \in \{+1, -1\}$, giving $2|\mathrm{GL}(3, \mathbb{F}_2)| = 336$ per Fano plane. The constant 336 is thus $|\text{Fano symmetry group}| \times |\text{sign choices}|$. It is also the order of the maximal subgroup $\mathrm{SL}(3, 2){:}2 \le G_2(2)$.

The 36 cosets of $\mathrm{GL}(3, \mathbb{F}_2)$ in $\mathrm{PSU}(3, 3)$ represent 36 distinct "octonion-compatible" Fano-plane structures. Their geometric interpretation in relation to ZD(4) is an open problem. The full $G_2(2)$ (order 12096) acts on 36 objects via its maximal subgroup $\mathrm{SL}(3, 2){:}2$ of order 336; identifying these 36 objects with a ZD-geometric structure would prove a transitive $G_2(2)$ action.

5. **Motivic interpretation.** Make precise $M_{\mathrm{ZD}} = L[\mathrm{SL}(2,7)] \otimes h(\mathrm{PG}(3,\mathbb{F}_2))$ and relate it to the variety of active ZD pairs inside $G(3, n-1; \mathbb{F}_2)$.
6. **[RESOLVED] GAP–analytic orbit verification.** The apparent discrepancy between the GAP result (36 orbits, sizes $\{84, 336\}$ for $n = 5$) and an earlier analytic prediction (60 orbits, sizes $\{42, 84, 168\}$) is resolved by the corrected pointwise stabilizer definition: the earlier prediction used a coarser set-stabilizer of the lo-flat rather than the pointwise stabilizer of $S(z)$. The corrected formula (Theorem 7.2 above) gives $\{2, 14, 84, 336\}$ as orbit sizes, and for $n = 5$ predicts exactly 36 orbits $\{84^{\times 28}, 336^{\times 8}\}$, in perfect agreement with the GAP computation.
7. **Analytic proof of Theorem 11.6 — current state.** The following results have been established for the innermost Type-B sublayer ($v_n = \sum_{k=3}^{n-1} 2^k + p$):

   Result 1 (proved): Sign-parity recursion. For pairs (i, j) and (k, l) in the innermost Type-B axis class at level n, with $j = i \oplus v_n$, $l = k \oplus v_n$, $j' = j - 2^{n-1}$, $l' = l - 2^{n-1}$:

   $$\varepsilon_n(i,k) = \varepsilon_{n-1}(i,k), \qquad \varepsilon_n(j,l) = -\varepsilon_{n-1}(j',l'),$$
   $$\varepsilon_n(i,l) = -\varepsilon_{n-1}(i,l'), \qquad \varepsilon_n(j,k) = -\varepsilon_{n-1}(j',k).$$

   These follow from the CD rules (EE), (FF), (EF) and verified for $n = 7$ over all 3720 non-degenerate index pairs (zero violations). Substituting into the ZD conditions $c_1 = c_2 = 0$ gives the sign-parity relation:

   $$\pi_n(i,j;k,l) \;=\; -\pi_{n-1}(i,j';k,l'), \tag{2}$$

   where $\pi_m(a,b;c,d) := \varepsilon_m(a,c)\varepsilon_m(b,c)\varepsilon_m(a,d)\varepsilon_m(b,d)$ is the sign parity of [1, Definition 7.2]. In particular, (i,j) and (k,l) are ZD at level n iff $\pi_{n-1}(i,j';k,l') = -1$.

   Result 2 (computational): Base case in A_4. For the projected flat in A_4 = sedenions (level 4) with witness element i and axis $v_4 = 8 + p$:

   $$\pi_4(i, i\oplus v_4;\; k,\; k\oplus v_4) \;=\; \begin{cases} -1 & \text{if } \mathrm{lo3}(k) \in \{0,p\}, \\ +1 & \text{if } \mathrm{lo3}(k) \notin \{0,p\}. \end{cases}$$

   Verified exhaustively over all valid pairs in A_4. Applying (2) recursively $(n-4)$ times gives $\pi_n = (-1)^{n-4}\pi_4$; combined with the base case, this yields ZD-compatible at level n iff $(-1)^{n-4}\pi_4 = +1$, i.e., $\mathrm{lo3}(k) \in \{0,p\}$ for the correct parity of n.

   Open: Parity Lemma. Identify the precise witness class (the set of elements i in the innermost axis class for which the recursion terminates cleanly at level 4) and prove analytically that this class has lo3-support $\subseteq \{0,p\}$ for all $n \geq 5$. Computational verification confirms the Theorem for $n = 5, \ldots, 12$; the analytic proof is the remaining open step.
8. **Algebraic and cohomological meaning of the poles of $Z(x)$.** $Z(x)$ is a rational function, so its meromorphic continuation to all of $\mathbb{C}$ is immediate. The open problem is the *algebraic meaning* of the four poles $x = 2^{-k}$ ($k = 0,1,2,3$), each corresponding to a Hurwitz algebra A_k. Clarify the cohomological interpretation via $H^{2k}(\mathrm{PG}(3,\mathbb{F}_2))$ and the L-function $L(M_{\mathrm{ZD}}, s) = Z(2^{-s})$.

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