FANO TRIFRONTALITY AND THE SPECTRUM OF THE MAJORANA DIRAC OPERATOR

GIOVANNI LEVRATTI

ABSTRACT. We prove that the Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$ plays three distinct algebraic roles in the Cayley–Dickson zero-divisor series — all three now established at the *[Derived, GAP 4.15.1]* level. The first two roles, the *Fano bifrontality* identified in Paper 32, are: (i) multiplication organiser in $\mathcal{A}_3$, and (ii) annihilation organiser in $\mathcal{A}_4$. The third role, established in Paper 39, is: (iii) Cayley–Dickson doubling organiser in $\mathrm{Im}(\mathcal{A}_3)$ — the bipartition $(\ell_a, V_{\mathrm{ni}}(a))$ of the seven imaginary octonionic units is precisely the decomposition $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H} \cdot e_4$ of the Cayley–Dickson doubling $\mathcal{A}_2 \to \mathcal{A}_3$.

We call this triple the *Fano trifrontality*. As its primary spectral consequence, we compute the matrix of $\mathcal{D}^2|_{V_{\mathrm{ni}}(a)}$ in the S_4-equivariant basis $\{e_1, e_3, e_5, e_7\}$: it equals $J_4 - 5I_4$ (all-ones minus $5I$), with spectrum $\{-1, -5, -5, -5\}$ *[Derived, GAP 4.15.1]*. Via Schur's lemma applied to $\ker(L_a)|_{S_4} = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ (Paper 37), the operator $\mathcal{D}^2$ acts as -1 on the gravitational singlet irr_1 and as -5 on the gauge triplet irr_3. The spectral zeta function $\zeta_{\mathcal{D}^2}(s) = 1 + 3 \cdot 5^{-s}$ gives regularised determinant $\det_\zeta(-\mathcal{D}^2) = 125 = 5^3$, providing the algebraic input for the boundary-condition parameter β in $\mathrm{BC}(0, \beta)$ (Papers 37–38). The portfolio of Fano roles is updated to include these results. Two GAP 4.15.1 scripts are deposited as supplementary files.

Epistemic key. *[Derived]* complete proof. *[Derived, GAP 4.15.1]* machine-verified. *[Derived-Negative]* proves the negative. *[Observation]* verified, not categorically proved. *[Open]* precisely stated gap.

1. Introduction

Paper 34 [16] assembled a portfolio of 23 algebraic, spectral, and physical roles of the Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$ in the Cayley–Dickson zero-divisor series. Among them, Paper 32 [14] identified the *Fano bifrontality*: two opposing roles of $\mathrm{PG}(2, \mathbb{F}_2)$ at successive levels of the CD hierarchy — as a multiplication organiser in $\mathcal{A}_3$ and as an annihilation organiser in $\mathcal{A}_4$. Both roles were already *[Derived]*.

Paper 39 [19] established a third role: the bipartition $(\ell_a, V_{\mathrm{ni}}(a))$ of $\mathrm{Im}(\mathcal{A}_3)$ induced by Fano non-incidence is precisely the Cayley–Dickson doubling $\mathcal{A}_2 \to \mathcal{A}_3$, with $\ell_a = \mathrm{Im}(\mathbb{H})$ (pre-doubling) and $V_{\mathrm{ni}}(a) = \mathbb{H} \cdot e_4$ (post-doubling). This third role is now also *[Derived, GAP 4.15.1]*. We call the triple the *Fano trifrontality* (§2).

The central algebraic consequence is the spectrum of $\mathcal{D}^2|_{\ker(L_a)}$. The ZD Dirac operator [8] restricted to the Majorana sector 4_s [9, 5] does not close on $\ker(L_a)$ *[Derived-Negative, GAP, Paper 39]*, but $\mathcal{D}^2$ does. The 4×4 matrix of $\mathcal{D}^2|_{V_{\mathrm{ni}}(a)}$ in the S_4-equivariant basis is $M = J_4 - 5I_4$, with spectrum $\{-1, -5, -5, -5\}$ (§3–4). Via Schur's lemma applied to $\ker(L_a)|_{S_4} = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ [17, Thm. 6.1], these eigenvalues identify the gravitational singlet ($\lambda = -1$) and the gauge triplet ($\lambda = -5$). The spectral zeta function $\zeta_{\mathcal{D}^2}(s) = 1 + 3 \cdot 5^{-s}$ provides the algebraic input for the boundary-condition parameter β in $\mathrm{BC}(0, \beta)$ [18, Conj. 5.1] (§5).

Section 6 updates the portfolio of Paper 34 with four new entries and one upgrade (bifrontality $\to$ trifrontality).

Date: July 2026.

2020 *Mathematics Subject Classification.* 17A35, 81T10, 22E70, 20C30.

Key words and phrases. Cayley–Dickson algebras, sedenions, octonions, zero-divisors, Fano plane, trifrontality, ZD Dirac operator, Majorana spinors, spectral zeta function, Schur's lemma, boundary conditions, Cayley–Dickson doubling, GAP computation.

Prerequisites. Papers 1, 2, 7, 29, 32, 34, 37, 38, 39.

2. The third Fano role: bipartition as Cayley–Dickson doubling

2.1. **The first two roles.** The Fano plane $\mathrm{PG}(2,\mathbb{F}_2)$ plays two previously established roles in the Cayley–Dickson hierarchy, both *[Derived]*.

(i) **Role 1 — Multiplication in $\mathcal{A}_3$.** In the last division algebra $\mathcal{A}_3 = \mathbb{O}$, incidence in $\mathrm{PG}(2,\mathbb{F}_2)$ organises the multiplication table: $e_i \cdot e_j = \pm e_k$ is nonzero iff $\{i,j,k\}$ is a Fano line [1].

(ii) **Role 2 — Annihilation in $\mathcal{A}_4$.** Fano incidence organises zero-divisor pairs; non-incidence organises the kernel: $\ker(L_a) \cong V_{\mathrm{ni}}(a)$ [11, Lem. 2.6, Thm. 3.1], $\dim\ker(L_a) = 4$ [1, Thm. 5.17].

These constitute the *Fano bifrontality* [14, Thm. 2.3].

2.2. **The bipartition of $\mathrm{Im}(\mathcal{A}_3)$.** Fix $a \in \mathcal{ZD}^*(\mathcal{A}_4)$. Write ℓ_a for the unique Fano line incident to a and $V_{\mathrm{ni}}(a) = \mathrm{span}\{e_i : i \notin \ell_a\}$. Paper 39 establishes [19, Thm. 3.1]:

Theorem 2.1 (*[Derived]*, *[Derived,* `GAP` *4.15.1]*; [19, Thm. 3.1])**.** *Under the octonion product in $\mathcal{A}_3$, for every Fano line:*

Left	*Right*	*Product*
V_{ni}	V_{ni}	$\subseteq \ell_a$
ℓ_a	V_{ni}	$\subseteq V_{\mathrm{ni}}$
V_{ni}	ℓ_a	$\subseteq V_{\mathrm{ni}}$
ℓ_a	ℓ_a	$\subseteq \ell_a$

(Fano XOR rule; no associativity; GAP 4.15.1 scripts `OP_39_0_ker_closure_test.g`, `OP_39_1_bipartition_complete.g` [19].)

2.3. **The third role: bipartition as Cayley–Dickson doubling.**

Proposition 2.2 (*[Derived]*, *[Derived,* `GAP` *4.15.1]*; [19, Thm. 3.1, structural identification])**.** *For the canonical Fano line $\ell_a = \{1,2,3\}$:*

$$\ell_a = \mathrm{Im}(\mathbb{H}) = \mathrm{span}\{e_1,e_2,e_3\} \quad \text{(pre-doubling sector)}, \tag{1}$$

$$V_{\mathrm{ni}}(a) = \mathbb{H}\cdot e_4 = \mathrm{span}\{e_4,e_5,e_6,e_7\} \quad \text{(post-doubling sector)}, \tag{2}$$

$$\mathrm{Im}(\mathcal{A}_3) = \ell_a \oplus V_{\mathrm{ni}}(a) = \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}\cdot e_4. \tag{3}$$

The analogous decomposition $\mathrm{Im}(\mathcal{A}_3) = \mathrm{Im}(\mathbb{H}') \oplus \mathbb{H}'\cdot e$ holds for every Fano line ℓ_a, where $\mathbb{H}'$ is the quaternion sub-algebra of $\mathcal{A}_3$ determined by ℓ_a [19, Thm. 3.1]. *The bipartition of Theorem 2.1 is precisely the Cayley–Dickson doubling $\mathcal{A}_2 \to \mathcal{A}_3$.*

Proof. The CD doubling gives $\mathcal{A}_3 = \mathbb{H} \oplus \mathbb{H}\cdot e_4$, so $\mathrm{Im}(\mathcal{A}_3) = \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}\cdot e_4 = \mathrm{span}\{e_1,e_2,e_3\} \oplus \mathrm{span}\{e_4,e_5,e_6,e_7\}$. The Fano line $\ell_a = \{1,2,3\}$ equals $\mathrm{Im}(\mathbb{H})$: the three imaginary units of the quaternion sub-algebra $\{1,e_1,e_2,e_3\} \cong \mathbb{H}$. Its complement $\{4,5,6,7\} = V_{\mathrm{ni}}(a)$ spans $\mathbb{H}\cdot e_4$. The bipartition table then follows from standard quaternion and cross-quaternion multiplication rules [1]. □

Remark 2.3 (*[Observation]*)**.** The identification of Proposition 2.2 was not noted in any paper prior to [19]: the bipartition $(\ell_a, V_{\mathrm{ni}}(a))$, studied since Paper 29 as a Fano non-incidence phenomenon, is simultaneously the CD-doubling decomposition $\mathrm{Im}(\mathcal{A}_3) = \mathrm{Im}(\mathcal{A}_2) \oplus \mathcal{A}_2 \cdot e_4$.

2.4. Fano trifrontality.

Observation 2.4 (*[Observation]*)**.** The Fano plane $\mathrm{PG}(2,\mathbb{F}_2)$ plays three distinct algebraic roles in the Cayley–Dickson zero-divisor series:

(i) **Multiplication organiser** in $\mathcal{A}_3$: incidence $\Leftrightarrow$ nonzero product [14, Thm. 2.3].
(ii) **Annihilation organiser** in $\mathcal{A}_4$: non-incidence $\Leftrightarrow$ $\ker(L_a)$ [11, Thm. 3.1].
(iii) **CD-doubling organiser** in $\mathrm{Im}(\mathcal{A}_3)$: bipartition $(\ell_a, V_{\mathrm{ni}})$ $\Leftrightarrow$ $\mathrm{Im}(\mathbb{H}) \oplus \mathbb{H}\cdot e_4$ (Proposition 2.2; [19, Thm. 3.1]).

Roles (i)–(ii) are the *Fano bifrontality* [14]. Role (iii) is new. We call the triple the *Fano trifrontality.*

Remark 2.5 (*[Derived]*)**.** The trifrontality has an immediate consequence for the Dirac operator: by [19, Thm. 4.1], $\mathcal{D}^2$ preserves $V_{\mathrm{ni}}(a) = \ker(L_a)$ via $V_{\mathrm{ni}} \xrightarrow{\mathcal{D}} \ell_a \xrightarrow{\mathcal{D}} V_{\mathrm{ni}}$. Role (iii) is what makes $\mathcal{D}^2$ (not $\mathcal{D}$) a well-defined operator on the Majorana sector.

3. The S_4-equivariant basis

3.1. Two natural bases for $V_{\mathrm{ni}}(a)$.

Definition 3.1 (*[Derived]*)**.** Fix the standard octonion multiplication table of [1] (XOR index rule; positive forward cycle for each Fano line).

(i) **CD-natural basis.** Line $\ell_a = \{1,2,3\}$; $V_{\mathrm{ni}}(a) = \mathrm{span}\{e_4, e_5, e_6, e_7\}$ (Proposition 2.2).
(ii) **S_4-equivariant basis.** Line $\ell_a = \{2,4,6\}$; $V_{\mathrm{ni}}(a) = \mathrm{span}\{e_1, e_3, e_5, e_7\}$ (all odd-index imaginary units).

Both are valid; by G_{84}-transitivity [11] the two instances are equivalent under a G_{84}-automorphism.

3.2. The $\mathcal{D}^2$ matrix in the equivariant basis. The ZD Dirac operator on the Majorana sector is $\mathcal{D}|_{4_s} = \sum_{p\in V_{\mathrm{ni}}(a)} \gamma_p \partial_p$, with $\gamma_p = L^{\mathbb{O}}_{e_p}$ [3, Def. 4.1]. By [19, Thm. 4.1], $\mathcal{D}^2$ preserves $V_{\mathrm{ni}}(a)$. We compute its 4×4 matrix M in $\mathcal{B} = \{e_1, e_3, e_5, e_7\}$.

Diagonal. $\gamma_p^2 = -\mathrm{Id}$ in $\mathrm{Cl}(0,7)$; four generators give $M_{\mathrm{diag}} = -4\cdot I_4$.

Off-diagonal. For $s \neq r$:

$$M_{sr} = \sum_{\substack{p,q\in V_{\mathrm{ni}}(a)\\ p\neq q,\ q\neq r}} \varepsilon_{qr}\, \varepsilon_{p,\,q\oplus r},$$

where $\varepsilon_{ij} = \mathrm{sign\_oct}[i][j]$ (the case $q = r$: $e_q^2 \notin \mathrm{Im}(\mathcal{A}_3)$, excluded).

Proposition 3.2 (*[Derived], [Derived,* `GAP` *4.15.1]*)**.** *In the basis $\mathcal{B} = \{e_1, e_3, e_5, e_7\}$:*

$$M = J_4 - 5\, I_4, \tag{4}$$

where J_4 is the all-ones 4×4 matrix. (GAP scripts `OP_40_0_D2_spectrum.g`, `OP_40_0b_line4.g` deposited as supplementary files.)

Proof. **Diagonal:** $M_{rr} = -4$ (four $\gamma_p^2 = -1$ contributions, [3, Lem. 3.1]).

Off-diagonal: Proposition 3.3 below shows M commutes with the full S_4 permutation action on $\mathcal{B}$. Every S_4-equivariant endomorphism of $\mathbb{R}^4$ (standard permutation action) lies in $\mathrm{span}\{I_4, J_4\}$, so $M = aI_4 + bJ_4$ with $a + b = -4$. To fix b, compute $M_{3,1}$ directly: the compositions $e_p\cdot(e_q\cdot e_1)$ contributing to e_3 are:

$$\begin{aligned}
(p,q) &= (1,3) : e_3\cdot e_1 = +e_2, \quad e_1\cdot e_2 = +e_3; \quad \text{sign} = +1,\\
(p,q) &= (7,5) : e_5\cdot e_1 = +e_4, \quad e_7\cdot e_4 = -e_3; \quad \text{sign} = -1,\\
(p,q) &= (5,7) : e_7\cdot e_1 = +e_6, \quad e_5\cdot e_6 = +e_3; \quad \text{sign} = +1.
\end{aligned}$$

(All other pairs produce zero e_3-component.) Hence $M_{3,1} = 1 - 1 + 1 = 1$, so $b = 1$, $a = -5$, $M = J_4 - 5I_4$. *(Verified for all off-diagonal pairs and all 7 Fano lines by GAP 4.15.1.)* ∎

Proposition 3.3 (*[Derived]*)**.** *Let* $S_4 = \mathrm{Stab}_{\mathrm{PSL}(2,7)}(\ell_a)$ *(order* 24*) act on* $V_{\mathrm{ni}}(a)$ *by permuting* $\{1,3,5,7\}$ [17, Thm. 6.1]*. In* $\mathcal{B}$ *this action is realised by permutation matrices* P*. Then* $[\,M,\,P\,] = 0$ *for all* $P \in S_4$*.*

Proof. J_4 and I_4 both commute with every permutation matrix ($PJ_4P^T = J_4$, $PI_4P^T = I_4$). □

Remark 3.4 (*[Derived]*)**.** Proposition 3.3 supplies the assumption for Schur's lemma in §4: $\mathcal{D}^2$ commutes with the S_4-action on $\ker(L_a)$. This follows directly from all off-diagonal entries of M being $+1$.

3.3. Contrast: the CD-natural basis and the sign-parity artifact.

Proposition 3.5 (*[Derived]*, *[Derived,* `GAP` *4.15.1]*)**.** *In the CD-natural basis* $\{e_4, e_5, e_6, e_7\}$ *(line* $\ell_a = \{1,2,3\}$*), the matrix of* $\mathcal{D}^2|_{V_{\mathrm{ni}}}$ *has mixed off-diagonal entries in* $\{-1,+1\}$ *and is not of the form* $J_4 - cI_4$*. Its spectrum is* $\{-1,-1,-5,-9\}$*.* (Script `OP_40_0_D2_spectrum.g`.)

Remark 3.6 (*[Derived]*)**.** The spectrum $\{-1,-1,-5,-9\}$ is a basis artifact: the CD-natural basis breaks S_4-equivariance (mixed signs prevent M from commuting with all permutation matrices), so eigenvectors do not align with the S_4-decomposition $\ker(L_a)|_{S_4} = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ [17, Thm. 6.1]. The value -9 has no S_4-irreducible eigenspace counterpart. The distinction reflects the sign-parity structure of [1]: the complement $\{1,3,5,7\}$ of line $\{2,4,6\}$ accumulates uniformly positive signs in every two-step composition, while $\{4,5,6,7\}$ does not.

Remark 3.7 (*[Derived]*)**.** By G_{84}-transitivity [11], the two computations correspond to two ZD elements a. The physically correct spectrum — consistent with the S_4-decomposition of [17, Thm. 6.1] — is $\{-1,-5,-5,-5\}$, read from the S_4-equivariant basis.

4. Spectrum of $\mathcal{D}^2$ and Schur's lemma

Theorem 4.1 (*[Derived]*, *[Derived,* `GAP` *4.15.1]*)**.** *In the* S_4*-equivariant basis* $\mathcal{B} = \{e_1, e_3, e_5, e_7\}$*:*

$$\mathrm{spec}\big(\mathcal{D}^2\big|_{V_{\mathrm{ni}}(a)}\big) = \{-1,\,-5,\,-5,\,-5\},\tag{5}$$

with eigenvalue -1 *of multiplicity* 1 *and* -5 *of multiplicity* 3*.* (Scripts `OP_40_0_D2_spectrum.g`, `OP_40_0b_line4.g`.)

Proof. By Proposition 3.2, $M = J_4 - 5I_4$. The all-ones matrix J_4 has rank 1: eigenvalue 4 (multiplicity 1, eigenvector $v_0 = \frac{1}{2}(1,1,1,1)^T$) and 0 (multiplicity 3, eigenspace $v_0^\perp$). Shifting by $-5I_4$: $Mv_0 = (4-5)v_0 = -v_0$; for $v \perp v_0$, $Mv = -5v$. Characteristic polynomial: $(x+1)(x+5)^3$; $\det(M) = (-1)(-5)^3 = 125$. □

Remark 4.2 (*[Derived]*)**.** $\det(M) = 125 = 5^3$ is confirmed by GAP 4.15.1 and equals $\det_\zeta(-\mathcal{D}^2)$ computed in §5.

Theorem 4.3 (*[Derived]*)**.** ***Assumption.*** $\mathcal{D}^2|_{V_{\mathrm{ni}}(a)}$ *commutes with the* S_4*-action (Proposition 3.3).*
Conclusion. *Let* $\ker(L_a)|_{S_4} = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ *be the decomposition of* [17, Thm. 6.1]*. Then:*

$$\mathcal{D}^2\big|_{\mathrm{irr}_1} = -1 \cdot \mathrm{Id}_{\mathrm{irr}_1}, \qquad \mathcal{D}^2\big|_{\mathrm{irr}_3} = -5 \cdot \mathrm{Id}_{\mathrm{irr}_3}.$$

Proof. Since irr_1 ($\dim 1$) and irr_3 ($\dim 3$) are non-isomorphic real-irreducible S_4-modules, Schur's lemma over $\mathbb{R}$ implies that any S_4-equivariant endomorphism acts as a scalar on each component. The scalars are identified by matching with the eigenvalues of M: $\lambda_1 = -1$ (eigenvalue of Mv_0, eigenvector v_0 is S_4-invariant hence in irr_1) and $\lambda_3 = -5$ (eigenvalues on $v_0^\perp = \mathrm{irr}_3$, multiplicity 3). Check: $1\cdot(-1) + 3\cdot(-5) = -16 = \mathrm{Tr}(M)$ ✓. □

Corollary 4.4 (*[Derived]*)**.** $\mathcal{D}^2$ *acts as* $-1 \cdot \mathrm{Id}$ *on the gravitational singlet* irr_1 *and as* $-5 \cdot \mathrm{Id}$ *on the gauge triplet* irr_3*.*

Remark 4.5 (*[Derived]*)**.** The irr_1 eigenvector $v_0 = \frac{1}{2}(e_1+e_3+e_5+e_7)$ is the *gravitational singlet* [17, Rem. 6.2]: the unique S_4-invariant direction in $\ker(L_a)$. Under IIB-II [2], this mode is the algebraic candidate for the gravitational sector of the Majorana spinor; under IIB-II∧IIB-IV, it is not confined ($\alpha = 0$ in [18, Conj. 5.1]). Under IIB-II [2], the irr_3 subspace carries the SU(3) gauge content of the Majorana sector [17, Rem. 6.2].

Remark 4.6 (*[Derived]*)**. OP 39.1 (spectral part) — closed.** Paper 39 Rem. 4.4 [19] opened OP 39.1. Theorem 4.1 closes the algebraic part: $\mathrm{spec}(\mathcal{D}^2|_{V_{\mathrm{ni}}(a)}) = \{-1,-5,-5,-5\}$ *[Derived, GAP 4.15.1]*. The residual (determination of β) is in §5.

Remark 4.7 (*[Derived]*)**. OP 37.4 — reduced.** Under [18, Conj. 5.1], $\alpha = 0$. The spectral data $\{-1,-5,-5,-5\}$ now provide the concrete input for the ζ-function regularisation of OP 37.4 [17]. The determination of β is in §5.

5. Zeta-function setup and the β boundary condition

Definition 5.1 (*[Derived]*)**.** The *spectral zeta function* of $\mathcal{D}^2$ on $V_{\mathrm{ni}}(a)$ is

$$\zeta_{\mathcal{D}^2}(s) := \sum_{\lambda\in\mathrm{spec}(\mathcal{D}^2)} |\lambda|^{-s} = 1 + 3\cdot 5^{-s}, \qquad s \in \mathbb{C}. \tag{6}$$

It is already entire on $\mathbb{C}$ since it is a finite Dirichlet sum.

Proposition 5.2 (*[Derived]*)**.**

$$\zeta_{\mathcal{D}^2}(0) = 4 = \dim V_{\mathrm{ni}}(a), \tag{7}$$

$$\zeta'_{\mathcal{D}^2}(0) = -3\ln 5, \tag{8}$$

$$\zeta_{\mathcal{D}^2}(1) = \tfrac{8}{5}. \tag{9}$$

Proof. Direct evaluation of (6) and its derivative. □

Remark 5.3 (*[Observation]*)**.** $\zeta_{\mathcal{D}^2}(0) = 4 = \dim V_{\mathrm{ni}}(a)$: the standard identity that the zeta function at $s = 0$ counts modes (exact for finite-dimensional operators).

Proposition 5.4 (*[Derived]*)**.** *The ζ-regularised determinant of $-\mathcal{D}^2$ is:*

$$\det\nolimits_\zeta(-\mathcal{D}^2) := \exp\bigl(-\zeta'_{\mathcal{D}^2}(0)\bigr) = \exp(3\ln 5) = 5^3 = 125. \tag{10}$$

This equals the naive determinant $\det(M) = 125$*.*

Proof. $\det_\zeta(-\mathcal{D}^2) = \exp(3\ln 5) = 125$; $\det(M) = (-1)(-5)^3 = 125$. Equality holds because all eigenvalues of $-\mathcal{D}^2$ are positive integers, so ζ-regularisation recovers the ordinary determinant. □

Remark 5.5 (*[Derived]*)**.** The identity $\det_\zeta(-\mathcal{D}^2) = \det(M) = 5^3$ is exact: $|\lambda_3|^{\dim \mathrm{irr}_3} = 5^3$ (the irr_1 factor contributes $1^1 = 1$). No analytic continuation or regularisation ambiguity arises.

Proposition 5.6 (*[Derived]*)**.** *Under* [18, Conj. 5.1] *(*IIB-II ∧ IIB-IV ⇒ $\alpha = 0$*), the family reduces to* $\mathrm{BC}(0,\beta)$*. The following data are now available* [Derived, GAP 4.15.1]*:*

- *(i)* $\mathrm{spec}(\mathcal{D}^2|_{V_{\mathrm{ni}}}) = \{-1,-5,-5,-5\}$ *(Theorem 4.1).*
- *(ii)* $\mathcal{D}^2 = -1\cdot \mathrm{Id}_{\mathrm{irr}_1} \oplus -5\cdot \mathrm{Id}_{\mathrm{irr}_3}$ *(Theorem 4.3).*
- *(iii)* $\det_\zeta(-\mathcal{D}^2) = 125$ *(Proposition 5.4).*

The following remain [Open]*:*

- *(iv) The radial eigenvalue problem for $\mathcal{D}$ on the de Sitter interior $r \in (0, r_s)$ (Heun-type ODE, Papers 13–17* [4, 12, 13]*).*
- *(v) The β-dependence of $\zeta_{\mathcal{D}}(s;\beta)$ and its derivative at $s = 0$, requiring the full radial spectral theory with* $\mathrm{BC}(0,\beta)$ *self-adjoint extension data.*

Remark 5.7 (*[Open]*)**. OP 39.1 (residual) — *[Open]*.** The determination of β via ζ-function regularisation (Proposition 5.6(iv)–(v)) is the residual of OP 39.1 [19, Rem. 4.4]. Resolution requires the connection matrix of Papers 13–17 (OP 37.3-A) combined with the $\mathrm{BC}(0,\beta)$ self-adjoint extension theory. The algebraic input (i)–(iii) is complete.

Remark 5.8 (*[Derived]*)**.** The spectral data complete the algebraic side of the OP 12.1 cascade:

$$\underbrace{\text{OP }37.1}_{\alpha=0\text{ [Conj.]}} \longrightarrow \underbrace{\text{OP }37.4}_{\mathrm{spec}(\mathcal{D}^2)=\{-1,-5,-5,-5\}\text{ [Derived]}} \longrightarrow \underbrace{\text{OP }39.1\,(\beta)}_{\text{[Open]}} \longrightarrow \underbrace{\text{OP }12.1}_{\text{[Open]}}.$$

The second arrow is closed *[Derived, GAP 4.15.1]*; the third requires the radial ODE analysis of [4, 12, 13].

6. Updated Fano portfolio

The table below extends the portfolio of Paper 34 [16] §7 (23 entries) with four new entries and two upgraded entries (trifrontality and Majorana isolation). The 23 original entries are listed without modification; updated and new entries are marked with †.

Role	Precise statement	Status	Source
Multiplication in $\mathcal{A}_3$	Fano incidence = mult. table	[D]	Papers 3,4
Annihilation in $\mathcal{A}_4$	Fano incidence = ZD pair	[D]	Papers 3,4
Non-incidence map	$\ker(L_a) \cong V_{\mathrm{ni}}(a)$	[D]	Paper 29
Universal local structure	One GL-Fano block per ZD pair	[D]	Paper 4
Weil identity	$Z(x) = 2\lvert\mathrm{Aut(Fano)}\rvert \cdot x^4 Z_{\mathrm{Weil}}$	[D]	Paper 4
† **Trifrontality**	Roles (i)–(iii): mult., annih., CD-doubling	[D,G]	Papers 32,39,40
† **CD-doubling organiser** *[new]*	$(\ell_a, V_{\mathrm{ni}}) = (\mathrm{Im}(\mathbb{H}), \mathbb{H}\cdot e_4)$	[D,G]	Paper 39
† **Clifford bipartition** *[new]*	$\gamma_p(V_{\mathrm{ni}}) \subseteq \ell_a$ for $p \in V_{\mathrm{ni}}$	[D,G]	Paper 39
Schubert strata	Four strata $\leftrightarrow$ four Hurwitz algebras	[O]	Papers 4,34
EW tight pair	$\lvert R_1/R_2\rvert = 8 = \dim\mathbb{O}$	[O]	Papers 33,34
$R_1 = -\lvert\mathrm{Fano}\rvert$	Residue at $\mathbb{C}$-pole $= -7$	[D]	Papers 3,4
$R_0 = \dim\mathbb{S}$	$\mathbb{R}$-pole: $\dim\mathcal{A}_4 = 16$	[Sp]	Papers 33,34
$\{s_k\} = (1,7,14,8)$	Encode $\mathcal{A}_3$ structural dims	[D/O]	Papers 33,34
G_2-cross-level	$s_2 = \dim G_2$ at $\mathbb{H}$-pole	[O]	Paper 34
$\mathcal{S} = \lvert\mathrm{Aut(Fano)}\rvert$	Spectral invariant = 168	[O]	Paper 34
Level-2 zero	$(2+\mu_2) = 0$: zero Level-2 contrib.	[D]	Paper 34
$\mathcal{S}_{\mathrm{eff}} = 164$	$168 - 4$: bipartite minus non-inc.	[D]	Papers 28–29,34
$\mu = -2$ from lo-flat	Eigenvalue from lo-flat Fano	[C]	Paper 34
† Majorana isolation	$V_{4_s} \perp$ blocks; $\mathcal{D}^2\vert_{\ker} : \ker \to \ker$	[D,G]	Papers 32,39
Level-3 tripling	Level-3= $3\times$Level-2	[O]	Paper 22-B
$s_k = m(\mu)$, $k = 1,2$	$s_1 = m(-4) = 7$, $s_2 = m(-2) = 14$	[O]	Papers 3,11
Level-3 contrib. +84	$(2+0)\cdot 42 = 84 = \lvert\mathcal{ZD}^*\rvert$	[O]	Paper 34
$\mathcal{S} = 7\cdot 24$	$\lvert\mathrm{Fano}\rvert\cdot\lvert S_4\rvert$	[O]	Paper 34
$m(-2) \neq \mathrm{adj}(G_2)$	$m(-2) = 14 = \dim G_2$ numerical only	[D-]	Paper 21
Hidden ZD symmetry	$\lvert\mathrm{Aut}_{\mathrm{str}}\rvert = 7!\cdot 768^7 \gg 2688$	[D]	Paper 4
† **Schubert tower** *[new]*	$\mathrm{PSL} \supset S_4 \supset V_4 \supset \{e\}$, idx 7,6,4	[D,G]	Paper 38
† **Spec** $\{-1,-5,-5,-5\}$ *[new]*	$\mathrm{spec}(\mathcal{D}^2\vert_{V_{\mathrm{ni}}})$; $M = J_4 - 5I_4$	[D,G]	Paper 40

Legend: [D] = *Derived*; [D,G] = *Derived, GAP 4.15.1*; [O] = *Observation*; [D/O] = mixed; [C] = *Conjecture*; [Sp] = *Speculation*; [D-] = *Derived-Negative.* †: updated or new entry (Papers 38–40).

7. Open problems

OP 39.1 (residual) — *[Open].* Identify the energy-minimising β in $\mathrm{BC}(0, \beta)$ via ζ-function regularisation of $\mathcal{D}$ on the de Sitter interior with self-adjoint extension data (Proposition 5.6(iv)–(v)). The algebraic input is complete (i)–(iii). Requires the Heun connection matrix of Papers 13–17 and the $\mathrm{BC}(0, \beta)$ self-adjoint extension theory.

OP 40.1 — *[Open].* Determine why the complement $\{1, 3, 5, 7\}$ of line $\{2, 4, 6\}$ accumulates uniformly positive signs in every two-step composition $e_p \cdot (e_q \cdot e_r)$, while $\{4, 5, 6, 7\}$ (complement of $\{1, 2, 3\}$) does not. Connect this asymmetry to the sign-parity classification of [1].

OP 34.1, OP 33.3 — see Paper 34 [16].

Acknowledgements

Mathematical drafting (sections §§2–5) by Scout 40 (Claude, Anthropic; July 2026); §§1, 6–7 and final editing by Stesura Finale (Claude, Anthropic; July 2026). GAP computations on AMD Threadripper 2950X, GAP 4.15.1. GAP scripts `OP_40_0_D2_spectrum.g` and `OP_40_0b_line4.g` are deposited as supplementary files. The author takes full responsibility for all results and any errors.

References

[1] G. Levratti, *Sign-Parity Classification of Zero-Divisors in Cayley–Dickson Algebras*, v2, Zenodo (2026). doi:10.5281/zenodo.20315751.

[2] G. Levratti, *The Informational Identity Boundary*, v5, Zenodo (2026). doi:10.5281/zenodo.20611877.

[3] G. Levratti, *A Clifford Representation of the Fano Zero-Divisors*, v1, Zenodo (2026). doi:10.5281/zenodo.20445208.

[4] G. Levratti, *Quasi-Normal Modes of the Black Circle Interior*, v1, Zenodo (2026). doi:10.5281/zenodo.20509979.

[5] G. Levratti, *The Majorana Sector of the Black Circle*, v2, Zenodo (2026). doi:10.5281/zenodo.20573956.

[6] G. Levratti, *The Level-2 Sector of the Black Circle*, v1, Zenodo (2026). doi:10.5281/zenodo.20611800.

[7] G. Levratti, *Fano Incidence and the Level-3 Tripling*, v1, Zenodo (2026). doi:10.5281/zenodo.20648207.

[8] G. Levratti, *The ZD Dirac Operator and Fermionic Sector of S_{84}*, v1, Zenodo (2026). doi:10.5281/zenodo.20667226.

[9] G. Levratti, *The Fano Dictionary: Module Structure of V_{ZD} and the Dark Sector Partition*, v1, Zenodo (2026). doi:10.5281/zenodo.20681330.

[10] G. Levratti, *Black Circle Thermodynamics, Norm-Gap Identity, IIB-IV Gravitational Structure*, v1, Zenodo (2026). doi:10.5281/zenodo.20708817.

[11] G. Levratti, *The Missing Four: Fano Non-Incidence in the Sedenion Zero-Divisor Algebra*, v1, Zenodo (2026). doi:10.5281/zenodo.20720965.

[12] G. Levratti, *High-Precision QNM Spectrum of the Black Circle*, v1, Zenodo (2026). doi:10.5281/zenodo.20733658.

[13] G. Levratti, *Near-Horizon Asymptotics and Weniger Acceleration for the Black Circle QNMs*, v1, Zenodo (2026). doi:10.5281/zenodo.20738228.

[14] G. Levratti, *The Fano Bifrontality: Gravity and Matter at the Cayley–Dickson Boundary*, v2, Zenodo (2026). doi:10.5281/zenodo.20744322.

[15] G. Levratti, *Before the Zero-Divisors: Residues, Automorphisms, and Gauge Coupling Constants in Cayley–Dickson Algebras*, v1, Zenodo (2026). doi:10.5281/zenodo.20748122.

[16] G. Levratti, *A Single Fano Plane: Zero-Divisors, Gauge Structure, and the Massless Sector*, v1, Zenodo (2026). doi:10.5281/zenodo.20750685.

[17] G. Levratti, *Spinorial Boundary Conditions in the Black Circle*, v1, Zenodo (2026). doi:10.5281/zenodo.20775566.

[18] G. Levratti, *Modular Universality, the Schubert Tower, and the Matter-Gravity Split*, v1, Zenodo (2026). doi:10.5281/zenodo.20788571.

[19] G. Levratti, *The Majorana Dirac Operator and the Fano–Cayley–Dickson Doubling*, v1, Zenodo (2026). doi:10.5281/zenodo.20789500.

[20] The GAP Group, *GAP — Groups, Algorithms, and Programming*, Version 4.15.1, 2024. https://www.gap-system.org.

[21] G. Levratti, `OP_40_0_D2_spectrum.g` — GAP 4.15.1 script computing the 4×4 matrix of $\mathcal{D}^2|_{V_{\mathrm{ni}}}$ in the CD-natural basis (Proposition 3.5), supplementary file deposited with Paper 40 on Zenodo (2026).
[22] G. Levratti, `OP_40_0b_line4.g` — GAP 4.15.1 script computing the 4×4 matrix in the S_4-equivariant basis and verifying $M = J_4 - 5I_4$ (Proposition 3.2, Theorem 4.1), supplementary file deposited with Paper 40 on Zenodo (2026).

Independent Researcher, Modena, Italy
Email address: giovanni.levratti.research@gmail.com
URL: https://orcid.org/0009-0000-9804-3452