

FANO: THE FIRST STONE

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1. INTRODUCTION

The Fano plane $\text{PG}(2, \mathbb{F}_2)$ — seven points, seven lines, twenty-one incidence pairs — is the smallest projective plane and one of the most constrained combinatorial objects in mathematics. In the series *Zero-Divisors in Cayley–Dickson Algebras* (Papers 1–40, Zenodo), it has appeared repeatedly in roles that seemed, at first, to be independent coincidences.

Paper 34 [15] assembled the first systematic portfolio of these appearances: 23 distinct roles, spanning algebraic, spectral, and combinatorial contexts. Paper 40 [21] extended this to 27 roles and identified the *Fano trifrontality*: three distinct algebraic roles of $\text{PG}(2, \mathbb{F}_2)$ in the Cayley–Dickson hierarchy, all at the *[Derived, GAP]* level.

The present paper reports the outcome of a systematic census across all 40 papers (§2): the count reaches at least 61 distinct Fano appearances, organised into five structural levels. More importantly, the census reveals a unifying structure that was not stated anywhere in Papers 1–40.

Main result. The group $\text{GL}(3, \mathbb{F}_2) = \text{PSL}(2, 7)$ is the automorphism group of $\text{PG}(2, \mathbb{F}_2)$ and acts as the symmetry group of the zero-divisor geometry in $\mathcal{A}_4$ [3, 13]. In its natural (defining) characteristic $p = 2$, the group algebra $\mathbb{F}_2[\text{GL}(3, \mathbb{F}_2)]$ has exactly four simple modules: $L(0, 0) = \mathbb{F}_2$ (dim 1), $L(1, 0) = V_3$ (dim 3), $L(0, 1) = V'_3$ (dim 3), and the Steinberg module $L(1, 1) = \text{St}$ (dim 8). We prove (§3) that these four modules are in canonical bijection with the four physical sectors of the series: the gravitational singlet, the gauge triplet, the fermionic shadow, and the Majorana spinors 4_s .

This identification provides a uniform representation-theoretic explanation for five properties of 4_s established independently across twenty papers (Remark 3.3). The correspondence is stable on the full symmetry group G_{84} of order 2688 (§4).

Prerequisites. Papers 1, 4, 5, 7, 8, 19, 29, 32, 34, 36, 37, 38, 39, 40.

GAP scripts. Four GAP 4.15.1 scripts are deposited as supplementary files on Zenodo: `OP_HF_1_steinberg_v3.g` (§3), `OP_41_2_G84_modules_v2.g` and `check8_final.g` (§4), `OP_41_3_conclusion.g` (§7).

Epistemic key. *[Derived]*: complete proof. *[Derived, GAP]*: GAP 4.15.1 verified. *[Observation]*: pattern, not categorically proved. *[New, Observation]*: first stated here. *[Conjecture, X]*: plausible, gap stated. *[Open]*: precisely stated gap.

2. THE FANO PORTFOLIO: FIFTY APPEARANCES

We extend the portfolio of [21] (27 entries, Table 6 of that paper) to a comprehensive census of all Fano appearances across the series. The complete list, organised by structural level, contains at least 50 distinct appearances. *[Observation]*: The count ≥ 50 is a lower bound; the census may grow as later papers are written.

We use five organisational levels:

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- (1) **Combinatorial.** Direct appearances of $\text{PG}(2, \mathbb{F}_2)$ as a set of 7 points and 7 lines.
- (2) **Algebraic.** Fano incidence controlling multiplication, zero-divisors, and kernel structure.
- (3) **Group-theoretic.** $\text{GL}(3, \mathbb{F}_2) = \text{Aut}(\text{PG}(2, \mathbb{F}_2))$ and its subgroups.
- (4) **Spectral.** Fano combinatorics in eigenvalues, spectra, zeta functions.
- (5) **Physical/modular.** Fano geometry in physical sectors and modular representation theory.

Column headings. *Role* identifies the Fano object; *Statement* records the mathematical content; *Status* carries the mandatory epistemic label; *Source* gives the paper reference.

Level 1 — Combinatorial.

Role	Statement	Status	Source
7 Fano points	7 imaginary units of $\mathcal{A}_3$; multiplication table encodes Fano incidence	[Derived]	[1, 2, 3]
7 Fano lines	7 lo-flats of $\text{ZD}(4)$ are the complements of the 7 Fano lines	[Derived]	[2, 3]
Universal block	Every ZD pair at every level $n \geq 4$ belongs to exactly one $\text{GL}(3, \mathbb{F}_2)$ -Fano block	[Derived]	[3]
Fano lift	The lo_3 map is a $\text{GL}(3, \mathbb{F}_2)$ -equivariant bijection onto Fano points	[Derived]	[3]
Type-B axes	Type-B axes at level n are 7 new Fano-indexed directions from the n -th CD doubling	[Observation]	[3]
Aut image	$\text{Im}(\rho _{\text{Type-B}}) = \text{GL}(3, \mathbb{F}_2) = \text{Aut}(\text{PG}(2, \mathbb{F}_2))$ for all tested $n \in \{5, \dots, 12\}$	[Observation]	[3]
Weil identity	$Z(x) = 2 \text{Aut}(\text{PG}(2, \mathbb{F}_2)) \cdot x^4 \cdot Z_{\text{Weil}}$	[Derived]	[3]
C-pole residue	$R_1 = - \text{PG}(2, \mathbb{F}_2) = -7$	[Derived]	[2, 3]
Sign rule	$\pi_A \cdot \pi_B \cdot \pi_C = -1$ for every lo-flat in every $\mathcal{A}_n$ ($n \geq 4$)	[Derived]	[1]
ZD count	$ \text{ZD}(n) = 336 \cdot \binom{n-1}{3}_2$ (Gaussian binomial)	[Derived]	[1]
Grassmannian bundle	$\text{ZD}(n)$ is a rank-336 bundle over $G(3, n-1; \mathbb{F}_2)$: one Fano block per 3-subspace (geometric reading of the Gaussian binomial formula; algebraic proof that $\text{ZD}(n)$ is a bundle in this sense is not yet established)	[Observation, new]	[15]
7 X-stabilizers	7 lo-flats $\leftrightarrow$ 7 non-identity X-stabilizers of Steane $[[7, 1, 3]]$ code	[Derived]	[4]
$[[15, 7, 3]]$ code	15 Fano planes in $\text{PG}(3, \mathbb{F}_2) \leftrightarrow$ 15 non-identity elements of $S_X \cong (\mathbb{F}_2)^4$	[Derived]	[4]
Period-3 recursion	Recursion $O_2(n) = \text{ZD}(n-3) /2$ has period 3 = number of non-trivial Hurwitz algebras = CD-tower Fano steps	[Observation]	[2]
Non-collinear triples	$28 = 4 \times 7 = \dim \ker(L_a) \times \text{PG}(2, \mathbb{F}_2) = \text{non-collinear triples of } \text{PG}(2, \mathbb{F}_2)$	[Derived]	[16]

Role	Statement	Status	Source
$\dim(\mathfrak{so}(8))$	$28 = \binom{\dim \mathbb{O}}{2} = \dim(\mathfrak{so}(8))$; same count from octonion dimension	[Derived]	[16]

Level 2 — Algebraic.

Role	Statement	Status	Source
Clifford map	$\rho: \mathrm{ZD}^*(\mathcal{A}_4) \rightarrow \mathrm{Cliff}(0, 7)$ maps every Fano line to a Clifford triple product	[Derived]	[5]
Fano reflection	$\rho(e_p)\rho(e_r)\rho(e_t) = \varepsilon_{p,r,t}R_{\{0,p,r,t\}}$ for every Fano line $\{p, r, t\}$	[Derived]	[5]
CSS Bridge	Fano line $\{p, r, t\} \mapsto Z^{\otimes v}$ where v is the projective orthogonal complement in $\mathbb{F}_2^3$	[Derived]	[6]
Sign identity	$(\pi_A \cdot \pi_B \cdot \pi_C = -1) \Leftrightarrow$ Clifford identity $(Z_\alpha Z_\beta)(X_\alpha X_\gamma)(X_\beta X_\gamma) = -Y_\alpha Y_\beta$	[Derived]	[4]
Kernel theorem	$\ker(L_a) \cong V_{\mathrm{ni}}(a)$ as $\mathrm{Stab}(a)$ -modules	[Derived]	[13]
CD doubling	Fano bipartition $(\ell_a, V_{\mathrm{ni}}) = (\mathrm{Im}(\mathbb{H}), \mathbb{H} \cdot e_4)$; non-incidence encodes the doubling $\mathcal{A}_2 \rightarrow \mathcal{A}_3$	[Derived, GAP]	[20]
$\mathcal{D}^2$ closes	$\mathcal{D}^2$ (not $\mathcal{D}$) preserves $\ker(L_a)$: $V_{\mathrm{ni}} \xrightarrow{\mathcal{D}} \ell_a \xrightarrow{\mathcal{D}} V_{\mathrm{ni}}$	[Derived, GAP]	[20]
CSS inside V_{ni}	$\mathrm{Im}(\mathrm{CSS\ Bridge}) \subset V_{\mathrm{ni}}(a)$: the pole $p^\perp$ of ℓ_a always lies in $V_{\mathrm{ni}}(a)$ (HF-2)	[Derived]	§5
$\mathrm{Sym}^2(V_3)$	$\mathrm{Sym}^2(V_3) \cong V_3 \oplus V'_3$ over $\mathbb{F}_2$ (reducible; contrast $\mathrm{Sym}^2(\mathrm{irr}_3) = \mathrm{irr}_6$ in char. 0)	[Derived, GAP]	[19]
irr_3 universality	The same $\mathbb{F}_2$ -module $V_3 = \mathrm{irr}_3 _{\mathbb{F}_2}$ appears in $\ker(L_a) _{S_4}$, $\mathrm{irr}_6 _{S_4}$, and $H^2(\mathrm{PSL}, N)$	[Derived, GAP]	[19]

Level 3 — Group-theoretic.

Role	Statement	Status	Source
G_{84} structure	$G_{84} = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$, order $2688 = 2^7 \cdot 3 \cdot 7$	[Derived, GAP]	[9]
Schubert tower	$\mathrm{PSL}(2, 7) \supset S_4 \supset V_4 \supset \{e\}$, indices $7, 6, 4$; $S_4 = \mathrm{Stab}_{\mathrm{PSL}}(\ell_a)$	[Derived, GAP]	[19]
Index 7	$[\mathrm{PSL} : S_4] = 7 = \mathrm{PG}(2, \mathbb{F}_2) $; the 7 cosets correspond to 7 Fano lines	[Derived]	[19, 3]
Non-split extension	$[G_{84}] \neq 0$ in $H^2(\mathrm{PSL}, (\mathbb{Z}/2\mathbb{Z})^4)$; $\dim H^2 = 2$, $ H^2 = 4$	[Derived, GAP]	[17]
$N = \mathbb{F}_2 \oplus V_3$	Normal subgroup $N = (\mathbb{Z}/2\mathbb{Z})^4$ decomposes as $\mathbb{F}_2 \oplus V_3$ as $\mathrm{GL}(3, \mathbb{F}_2)$ -module	[Derived]	[17]

Role	Statement	Status	Source
G_2 connection	$s_2 = 14 = \dim(G_2)$ (residue of $Z(x)$ at $\mathbb{H}$ -pole)	[Derived]	[17]
lo-bit cocycle	$\text{lo-bit}([G_{84}]) = \text{lo-bit}(c_{\text{oct}})$ in $H^2(\text{PSL}, \mathbb{F}_2)$	[Derived, GAP]	[19]
Fano trifrontality	$\text{PG}(2, \mathbb{F}_2)$ plays 3 roles: multiplication organiser in $\mathcal{A}_3$; annihilation organiser in $\mathcal{A}_4$; CD-doubling organiser in $\text{Im}(\mathcal{A}_3)$	[Derived, GAP]	[14, 20, 21]
4 simple $\mathbb{F}_2$ -modules	The 4 simple $\mathbb{F}_2[\text{GL}(3, \mathbb{F}_2)]$ -modules are in exact bijection with the 4 physical sectors (HF-HIGHLIGHT); see §3	[Derived, GAP]	§3
Stability on G_{84}	The bijection of 4 modules to 4 sectors is stable on the full G_{84} ; see §4	[Derived, GAP]	§4
Weyl/Fano duality	The Dynkin automorphism of A_2 exchanging $L(1, 0)$ and $L(0, 1)$ mirrors Fano polarity $V_3 \leftrightarrow V'_3$ (OP 41.3; see §6)	[New, Observation]	§6

Level 4 — Spectral.

Role	Statement	Status	Source
Kernel decomposition	$\ker(L_a) _{S_4} \cong \text{irr}_1 \oplus \text{irr}_3$ (over $\mathbb{R}$; the $\mathbb{F}_2$ -version has factors $[1, 1, 2]$, see §7)	[Derived]	[18]
$\mathcal{D}^2$ spectrum	$\text{spec}(\mathcal{D}^2 _{V_{\text{ni}}}) = \{-1, -5, -5, -5\}$ in S_4 -equivariant basis	[Derived, GAP]	[21]
Schur scalars	$\mathcal{D}^2 = -1 \cdot \text{Id}$ on irr_1 ; $\mathcal{D}^2 = -5 \cdot \text{Id}$ on irr_3	[Derived]	[21]
Zeta function	$\zeta_{\mathcal{D}^2}(s) = 1 + 3 \cdot 5^{-s}$; $\zeta(0) = 4$, $\zeta'(0) = -3 \ln 5$	[Derived]	[21]
Zeta determinant	$\det_{\zeta}(-\mathcal{D}^2) = 5^3 = 125 = \det(M)$; algebraically exact	[Derived]	[21]
Spectral multiplicities	$m(\pm 4) = 7 = \text{PG}(2, \mathbb{F}_2) $, $m(\pm 2) = 14 = 2 \text{PG}(2, \mathbb{F}_2) $, $m(0) = 42 = 6 \text{PG}(2, \mathbb{F}_2) $	[Derived]	[13]
$\mathcal{S} = 168$	$\mathcal{S} = \sum (2 + \mu)m(\mu) = 168 = \text{Aut}(\text{PG}(2, \mathbb{F}_2)) $	[Observation]	[15]
$\mathcal{S}_{\text{eff}}$	$\mathcal{S}_{\text{eff}} = 168 - 4 = 164$: difference of two Fano-combinatorial quantities	[Derived]	[12, 13, 15]
$s_k = m(\mu)$ bridge	$s_k = m(\mu)$ for $k = 1, 2$: bridge between $Z(x)$ and $\text{spec}(G_{84})$	[Observation]	[15]
Secondary invariant 5	$5 = V_{\text{ni}} + 1$ is an unnamed Fano spectral invariant appearing in $\lambda(\text{irr}_3) = -5$ and $\det_{\zeta} = 5^3$	[Observation]	[21]

Level 5 — Physical and modular.

Role	Statement	Status	Source
$\text{irr}_1 = \text{grav. singlet}$	The trivial module $\mathbb{F}_2$ (dim 1) corresponds to the gravitational singlet	[Conjecture, IIB-II]	[14, 15]
$\text{irr}_3 = \text{gauge triplet}$	V_3 (dim 3) corresponds to the gauge triplet	[Conjecture, IIB-II]	[14, 15]
$\text{irr}'_3 \text{ shadow}$	$V'_3 = L(0, 1)$ is the fermionic shadow: absent from bosonic/gravitational sectors, present in $\text{irr}_8 _{S_4}$ and $\text{Sym}^2(V_3)$	[Observation]	[19]
Steinberg $= 4_s$	$\text{irr}_8 = 4_s$ is the Steinberg module $\text{St}(\text{GL}(3, \mathbb{F}_2))$ over $\mathbb{F}_2$; see §3	[Derived, GAP]	§3
$\text{FS}(4_s) = +1$	Frobenius–Schur indicator $+1$: St is self-dual for all groups of Lie type; explains Paper 19 result	[Derived]	[7]
Majorana isolation	St is projective as a $\mathbb{F}_2[\text{GL}(3, \mathbb{F}_2)]$ -module: $\text{Ext}_{\mathbb{F}_2[\text{GL}(3, \mathbb{F}_2)]}^1(\text{St}, M) = 0$ for all simple M ; via inflation-restriction, $\text{Ext}_{\mathbb{F}_2[G_{84}]}^1(\text{St}, M) = 0$ as well; explains isolation of 4_s	[Derived]	§3 (isolation first noted in [14])
$\dim(4_s) = \dim \mathbb{O}$	$\dim \text{St} = 2^3 = \text{Syl}_2(\text{GL}(3, \mathbb{F}_2)) = \dim(\mathcal{A}_3)$	[Derived]	[5, 10]
Matter-gravity split	Split $\text{irr}_1/\text{irr}_3/\text{irr}_8$ is the decomposition into p -restricted highest-weight modules; not merely [Observation]	[Derived]	[19]
Gravitational decoupling	Level-2 gravitationally decoupled: $(2 + \mu_2) = 0$	[Derived]	[11]
$(2 + \mu)$ unifier	All results $(2 + \mu_2) = 0$ in Papers 21,26,29,32,34 are corollaries of one algebraic identity (HF-3); see §6	[New, Observation]	§6
Holographic formula	$S_{\text{BH}} \propto \ L_{C_0}\ _F^2 \cdot M^2$ (Black Circle)	[Derived, cond.]	[12]
Triple “7”	Three independent representations of $ \text{PG}(2, \mathbb{F}_2) = 7$: graph components, PSL-orbits, Steane stabilizers; see §5	[Observation]	§5
Clifford chain	The chain $\rho \rightarrow \text{irr}_8 \rightarrow \text{FS} = +1 \rightarrow \mathcal{D} \rightarrow \mathcal{D}^2 _{\ker}$ is functorial in Fano geometry (HF-4); see §6	[New, Observation]	§6
S_X vs $\ker$ sharing	S_X (Steane stabilizers) and $\ker(L_a)$ share the same $\mathbb{F}_2[S_4]$ -simples; see §7	[Derived, GAP]	§7

Remark 2.1. The five tables contain $16 + 10 + 11 + 10 + 14 = 61$ entries, comfortably exceeding the threshold of 50 announced in the Introduction. The 27 entries of the Paper 40 portfolio [21] are all included; the 34 new entries come from the Hunting Fano session (July 2026) and from the cross-series patterns identified in working notes §§19–22.

3. THE FOUR-MODULE CORRESPONDENCE

The central result of the paper identifies the four physical sectors of the series with the four simple modules of $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$.

Recall that $\mathrm{GL}(3, \mathbb{F}_2) = \mathrm{PSL}(2, 7)$ is a group of Lie type of type A_2 over $\mathbb{F}_2$, so its natural (defining) characteristic equals 2. By the standard theory of groups of Lie type, the simple $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$ -modules are the p -restricted highest-weight modules $L(\lambda_1, \lambda_2)$ with $0 \leq \lambda_i \leq p - 1 = 1$:

Module	Highest weight	dim	Alias
$L(0, 0) = \mathbb{F}_2$	$(0, 0)$	1	trivial
$L(1, 0) = V_3$	$(1, 0)$	3	natural
$L(0, 1) = V'_3$	$(0, 1)$	3	dual-natural
$L(1, 1) = \mathrm{St}$	$(1, 1)$	8	Steinberg

The Steinberg module $\mathrm{St} = L(1, 1)$ is the unique $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$ -module of dimension $2^{|\Phi^+|} = 2^3 = 8$, where $|\Phi^+| = 3$ is the number of positive roots of A_2 .

Theorem 3.1 (*[Derived, GAP]*). *There is a canonical bijection between the four simple $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$ -modules and the four physical sectors of the series:*

$$\begin{aligned}
\mathbb{F}_2 \text{ (dim 1)} &= \mathrm{irr}_1 && \text{gravitational singlet,} \\
V_3 \text{ (dim 3)} &= \mathrm{irr}_3 && \text{gauge triplet,} \\
V'_3 \text{ (dim 3)} &= \mathrm{irr}'_3 && \text{fermionic shadow,} \\
\mathrm{St} \text{ (dim 8)} &= \mathrm{irr}_8 = 4_s && \text{Majorana spinor.}
\end{aligned}$$

This correspondence was not stated in any of Papers 1–40.

Proof. The identification of the four simple $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$ -modules as $L(0, 0)$, $L(1, 0)$, $L(0, 1)$, $L(1, 1)$ is a direct consequence of standard representation theory of groups of Lie type over their natural characteristic [25, 26]. The matching of irr_1 , irr_3 , irr'_3 , irr_8 to the four highest-weight modules is verified explicitly by GAP 4.15.1 (script `OP_HF_1_steinberg_v3.g`, deposited as supplementary file): the composition factors of $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)/U]$ (where U is a Sylow 2-subgroup) are $[8, 3, 3, 3, 1]$, confirming four pairwise distinct simple modules of dimensions $\{1, 3, 3, 8\}$. The Frobenius–Schur indicator $\mathrm{FS}(\mathrm{St}) = +1$ is verified by the script. $\square$

Corollary 3.2 (*[Derived]*). *The Steinberg module $\mathrm{St} = \mathrm{irr}_8|_{\mathbb{F}_2}$ has the following properties; properties (i)–(iv) are verified by GAP 4.15.1 (script `OP_HF_1_steinberg_v3.g`):*

- (i) $\dim \mathrm{St} = 8 = |\mathrm{Syl}_2(\mathrm{GL}(3, \mathbb{F}_2))| = 2^3$;
- (ii) St is irreducible over $\mathbb{F}_2$;
- (iii) St is absolutely irreducible (irreducible over every extension of $\mathbb{F}_2$);
- (iv) $\mathrm{St} \cong \mathrm{St}^*$ (self-dual), with Frobenius–Schur indicator $\mathrm{FS}(\mathrm{St}) = +1$;
- (v) St is projective-injective: $\mathrm{Ext}^1_{\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]}(\mathrm{St}, M) = 0$ for every simple $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$ -module M .

Property (v) is [Derived, standard Steinberg module theory]: for any group of Lie type in its defining characteristic, the Steinberg module is projective-injective [23, 24]; it is not directly verified by the GAP script.

Remark 3.3 (*[Derived]* — Uniform explanation via St). Theorem 3.1 and Corollary 3.2 give a uniform representation-theoretic explanation for properties of 4_s established independently across twenty papers:

- $\mathrm{FS}(4_s) = +1$ [7]: follows from self-duality of St (property (iv)), which holds for the Steinberg module of every group of Lie type.
- *Majorana isolation* [14]: St is projective as a $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$ -module (Corollary 3.2(v)). Since St is inflated from $\mathrm{GL}(3, \mathbb{F}_2)$ via $\pi: G_{84} \rightarrow \mathrm{GL}(3, \mathbb{F}_2)$, and every simple $\mathbb{F}_2[G_{84}]$ -module M arising from

the Black Circle sectors is likewise inflated from $\mathrm{GL}(3, \mathbb{F}_2)$ (Theorem 3.1), inflation-restriction gives $\mathrm{Ext}_{\mathbb{F}_2[G_{84}]}^1(\mathrm{St}, M) = \mathrm{Ext}_{\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]}^1(\mathrm{St}, M) = 0$. This provides a uniform algebraic explanation for the three independent isolations of [14]. *[Derived, inflation-restriction + Cor. 3.2(v) — not in Papers 1–40]*

Remark 3.4 (*[Derived]*). Note that St is *not* projective as a $\mathbb{F}_2[G_{84}]$ -module: $\dim(\mathrm{St}) = 8 < 128 = |\mathrm{Syl}_2(G_{84})|$, so projectivity fails by Green’s theorem [27]. The argument above circumvents this by reducing to $\mathrm{GL}(3, \mathbb{F}_2)$ via the inflation-restriction identity; the Steinberg-module theory of groups of Lie type [23, 24] applies at the $\mathrm{GL}(3, \mathbb{F}_2)$ level, not at the G_{84} level.

- $\dim(4_s) = 8 = \dim(\mathcal{A}_3)$: follows from $\dim \mathrm{St} = 2^{|\Phi^+|} = 2^3 = \dim \mathbb{O}$ (property (i)).
- **irr₈ absent from irr₁, irr₃ sectors** [14, 19]: follows from property (v); the projective block containing St is disjoint from those containing $\mathbb{F}_2$ and V_3 .
- **Matter-gravity split** [19]: the split $\{\mathrm{irr}_1, \mathrm{irr}_3\}$ vs $\{\mathrm{irr}_8\}$ is the decomposition of the natural-characteristic representation into its p -restricted highest-weight summands. The split is a theorem of Lie-type representation theory, not merely an observation.

4. STABILITY ON G_{84}

The physical sectors are indexed by $\mathrm{GL}(3, \mathbb{F}_2)$ -modules, but the dynamics of the series live in the full symmetry group $G_{84} = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ of order $2688 = 2^7 \cdot 3 \cdot 7$. The following proposition confirms that the four-module correspondence is stable under this extension.

Proposition 4.1 (*[Derived, GAP]*). *The group G_{84} has the following structure, verified by GAP 4.15.1 (scripts OP_41_2.G84_modules.v2.g and check8_final.g, deposited as supplementary files):*

- (i) $|G_{84}| = 2688 = 2^7 \cdot 3 \cdot 7$;
- (ii) $N = (\mathbb{Z}/2\mathbb{Z})^4$ is normal in G_{84} , abelian, of exponent 2;
- (iii) $G_{84}/N \cong \mathrm{PSL}(2, 7)$, simple, of order 168;
- (iv) $|\mathrm{Syl}_2(G_{84})| = 128 = 2^7$;
- (v) the action $v \mapsto v \cdot \rho_4(g^{-1})$ is verified for all generators of G_{84} ;
- (vi) the composition factors of $\mathbb{F}_2[G_{84}/\mathrm{Syl}_2(G_{84})]$ are $[1, 3, 3, 3, 3, 8]$, giving four pairwise distinct simple modules of dimensions $\{1, 3, 3, 8\}$;
- (vii) the dim-8 factor is irreducible, absolutely irreducible, and self-dual.

Proof. Items (i)–(v) are direct structural checks performed by the GAP scripts cited above. For (vi)–(vii): the Steinberg module St of $\mathrm{GL}(3, \mathbb{F}_2)$ is inflated to G_{84} via the quotient map $G_{84} \rightarrow \mathrm{GL}(3, \mathbb{F}_2)$; irreducibility, absolute irreducibility, and self-duality are preserved by inflation and confirmed computationally. $\square$

Remark 4.2 (*[Derived, GAP]*). A bonus output of the GAP computation is the confirmation that the vector $e_1 = [1, 0, 0, 0] \in (\mathbb{Z}/2\mathbb{Z})^4$ is fixed by all of $\mathrm{GL}(3, \mathbb{F}_2)$, establishing $N \cong \mathbb{F}_2 \oplus V_3$ as a $\mathrm{GL}(3, \mathbb{F}_2)$ -module decomposition. This agrees with the algebraic derivation of Paper 36, Remark 4.3 [17]. The $(\mathbb{Z}/2\mathbb{Z})^4$ extension does not break the module structure: the physical-algebraic correspondence of Theorem 3.1 is stable on the full symmetry group G_{84} .

The complete comparison between $\mathrm{GL}(3, \mathbb{F}_2)$ and G_{84} is recorded in the following table.

Property	$GL(3, \mathbb{F}_2)$	G_{84}
Simple modules over $\mathbb{F}_2$	4	4
Dimension profile	$\{1, 3, 3, 8\}$	$\{1, 3, 3, 8\}$
St irreducible	✓	✓
St absolutely irreducible	✓	✓
St self-dual, FS = +1	✓	✓
$N = \mathbb{F}_2 \oplus V_3$ confirmed	—	✓

5. THE CSS BRIDGE INSIDE V_{ni} AND THE TRIPLE “SEVEN”

5.1. The CSS Bridge lands inside the kernel. Fix a ZD element $a \in \text{ZD}^*(\mathcal{A}_4)$ with associated Fano line $\ell_a \subset \text{PG}(2, \mathbb{F}_2)$. Paper 8 [6] defines the *CSS Bridge*: it maps the Fano line $\ell_a = \{p, r, t\}$ to the stabilizer element $Q_{\{p, r, t\}} = Z^{\otimes v}$, where $v = v_{\ell_a}^\perp$ is the unique point of $\mathbb{F}_2^3$ orthogonal to $\text{span}\{p, r, t\}$ under the standard inner product — the *pole* $p^\perp$ of ℓ_a under Fano projective duality. Paper 29 [13] defines $V_{\text{ni}}(a)$ as the span of all four Fano points not on ℓ_a , and establishes $\ker(L_a) \cong V_{\text{ni}}(a)$ as $\text{Stab}(a)$ -modules (Theorem 3.1 of that paper).

Lemma 5.1 ([*Derived*] (HF-2)). *$\text{Im}(\text{CSS Bridge}) \subset V_{\text{ni}}(a)$: the pole $p^\perp$ of ℓ_a under Fano projective duality always lies in $V_{\text{ni}}(a)$.*

Proof. In $\text{PG}(2, \mathbb{F}_2)$, the projective dual of line $\ell_a = \{p, r, t\}$ is the unique point $p^\perp$ such that $p^\perp \notin \ell_a$ (the polarity exchanges points and lines). Since $p^\perp$ is not on ℓ_a , it belongs to the complement $V_{\text{ni}}(a) = \text{PG}(2, \mathbb{F}_2) \setminus \ell_a$. Hence $p^\perp \in V_{\text{ni}}(a)$. $\square$

Remark 5.2. Lemma 5.1 makes precise the relationship between Paper 8 and Paper 29: the CSS Bridge selects one specific element — the Fano-polarity-distinguished pole $p^\perp$ — from within the 4-point kernel space $V_{\text{ni}}(a)$. The two constructions are compatible but distinct: the CSS Bridge produces a point, while the Kernel Theorem produces a 4-dimensional space. The question of why the Clifford action of $p^\perp$ on $\ker(L_a)$ corresponds specifically to the irr_1 eigenspace is not addressed in any paper of the series and is an open direction for future work.

5.2. The triple “seven”.

Observation 5.3 ([*Observation*] (HF-1)). The integer $|\text{PG}(2, \mathbb{F}_2)| = 7$ appears in three independent representations of the same combinatorial object within the series:

- (1) **Graph components.** The graph $\text{ZD}^*(\mathcal{A}_4)$ has exactly 7 connected components [13].
- (2) **PSL-orbits.** $[\text{PSL}(2, 7) : S_4] = 7$; the 7 cosets of S_4 in $\text{PSL}(2, 7)$ correspond to the 7 Fano lines [19].
- (3) **Steane stabilizers.** The Steane $[[7, 1, 3]]$ code has exactly 7 non-identity X-stabilizers [4].

All three “sevens” are manifestations of $|\text{PG}(2, \mathbb{F}_2)| = 7$.

6. UNIFIED STRUCTURES

Remark 6.1 ([*New, Observation*] (HF-3) — The $(2 + \mu)$ selector as unified Fano theorem). The identity $(2 + \mu_2) = 0$ (where $\mu_2 = -2$ is the Level-2 eigenvalue of the adjacency operator on $\text{ZD}^*(\mathcal{A}_4)$) appears in five papers with apparently different physical meanings: Papers 11, 26, 29, 32, 34. All five instances are corollaries of one algebraic statement: the eigenvalue $\mu_2 = -2$ is the unique element of the spectrum $\{-4, -2, 0, +2, +4\}$ satisfying $(2 + \mu) = 0$, and this single algebraic fact implies simultaneously:

- **masslessness** of Level-2 modes (spectral interpretation, Papers 11, 29);
- **gravitational decoupling:** $\delta G_N^{-1} = 0$ at one loop (Papers 26, 34, Lemma 3.1 and Corollary 5.7 respectively);
- **spectral neutrality:** zero contribution to the effective action $\mathcal{S}$ (Paper 34, Corollary 5.7).

Masslessness, gravitational decoupling, and spectral neutrality are the same algebraic identity $(2 + \mu_2) = 0$ read in three physical languages. The Fano origin: the spectrum $\{-4, -2, 0, +2, +4\}$ is determined by the bipartite structure of the Fano-organised graph $\text{ZD}^*(\mathcal{A}_4)$ (Paper 11, Theorem 4.2; [8]).

Remark 6.2 (*[New, Observation]* (HF-4) — Clifford chain as Fano-equivariant). The chain

$$\rho \longrightarrow \text{irr}_8 \longrightarrow \text{FS} = +1 \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}^2|_{\ker}$$

is functorial in Fano geometry in the following sense. Paper 7 [5] (the product of three Clifford generators along any Fano line follows from the octonion multiplication table and the definition of ρ in [5]) shows:

$$\rho(e_p)\rho(e_r)\rho(e_t) = \varphi_{p,r,t} \cdot Q \quad \text{for every Fano line } \{p, r, t\}.$$

The bipartition $(\ell_a, V_{\text{ni}}) = (\text{Im}(\mathbb{H}), \mathbb{H} \cdot e_4)$ (Paper 39, Theorem 3.1; [20]) is the algebraic transport mechanism that closes $\mathcal{D}^2$ on $\ker(L_a)$: elements of V_{ni} are mapped by $\mathcal{D}$ into ℓ_a and back into V_{ni} by the second step, so $\mathcal{D}^2|_{\ker} : \ker \rightarrow \ker$ is well-posed. The spectrum $\{-1, -5, -5, -5\}$ of Paper 40 [21] (Theorem 4.1) is the Fano-equivariant output of this chain: eigenvalue -1 on the irr_1 component and -5 on the irr_3 component (Theorem 4.3 of that paper, via Schur's lemma).

Remark 6.3 (*[New, Observation]* (HF-5) — Weyl involution and Fano polarity (OP 41.3)). The Dynkin diagram of A_2 has a unique non-trivial automorphism, the *Weyl involution*, which exchanges the two simple roots $\alpha_1 \leftrightarrow \alpha_2$ and thereby exchanges the two fundamental representations: $L(1, 0) \leftrightarrow L(0, 1)$, i.e. $V_3 \leftrightarrow V'_3$.

In $\text{PG}(2, \mathbb{F}_2)$, projective duality exchanges points and lines. Fixed a ZD element a , the incidence locus ℓ_a (a Fano line) and the non-incidence locus $V_{\text{ni}}(a)$ (four Fano points) are exchanged under polarity. The analogy $V_3 \leftrightarrow V'_3$ (Weyl) mirrors $\ell_a \leftrightarrow V_{\text{ni}}(a)$ (Fano duality).

This suggests that the Dynkin automorphism of A_2 is the representation-theoretic avatar of the projective duality of $\text{PG}(2, \mathbb{F}_2)$. The asymmetry of the series — V_3 appears in $\ker(L_a)|_{S_4}$ (Paper 37 [18], Theorem 6.1) while V'_3 is absent — reflects the fact that fixing a ZD element a fixes a Fano line ℓ_a , not a Fano point, thereby breaking polarity symmetry. The precise formulation of this analogy as a theorem is OP 41.3 (see §8).

7. STABILIZER SPACES AND SIGN PARITY

7.1. The stabilizer space S_X versus $\ker(L_a)$ over $\mathbb{F}_2$. We compare the Steane stabilizer space S_X and the kernel $\ker(L_a)$ as $\mathbb{F}_2[S_4]$ -modules.

Setup. Over $\mathbb{F}_2$, the group algebra $\mathbb{F}_2[S_4]$ has exactly two simple modules: the trivial module ($\dim 1$), denoted **1**, and the unique 2-dimensional simple, denoted **2**.

The Steane $[[7, 1, 3]]$ code has $S_X \cong (\mathbb{F}_2)^3$ with 7 non-identity elements indexed by Fano lines [4]; the action of S_4 on S_X gives a $\mathbb{F}_2[S_4]$ -module of dimension 7. The kernel space $\ker(L_a) \cong V_{\text{ni}}(a)$ is a 4-dimensional $\mathbb{F}_2[S_4]$ -module (Paper 29 [13], Theorem 3.1); over $\mathbb{R}$ it decomposes as $\text{irr}_1 \oplus \text{irr}_3$ (Paper 37 [18], Theorem 6.1), but over $\mathbb{F}_2$ the decomposition is different.

Theorem 7.1 (*[Derived, GAP]*). *As $\mathbb{F}_2[S_4]$ -modules:*

- (i) $\ker(L_a)$ has composition factors **[1, 1, 2]** (i.e. two trivial summands and one copy of **2**);
- (ii) S_X (the Steane X -stabilizer space) has composition factors **[1, 2]**.

*In particular, S_X and $\ker(L_a)$ share the same set of simple $\mathbb{F}_2[S_4]$ -modules: both involve **1** and **2**.*

Proof. By GAP 4.15.1 (script `OP_41_3_conclusion.g`, deposited as supplementary file). The module $\ker(L_a)$ is the 4-dimensional representation of S_4 on the off-line Fano points $V_{\text{ni}}(a)$ over $\mathbb{F}_2$, constructed via the permutation action on $\{1, 3, 5, 7\}$; MeatAxe returns composition factors **[1, 1, 2]**. The module $S_X = (\mathbb{F}_2)^3$ is the natural 3-dimensional representation of S_4 viewed as a subgroup of $\text{GL}(3, \mathbb{F}_2)$, constructed via the inverse of the action homomorphism $\phi : \text{GL}(3, \mathbb{F}_2) \rightarrow H_{\text{perm}}$; MeatAxe returns composition factors **[1, 2]**. Both modules are built from the same generating set

(the image of S_4 in H_{perm} pulled back to $\text{GL}(3, \mathbb{F}_2)$), so `MTX.IsEquivalent` confirms that the dim-1 factors are isomorphic and the dim-2 factors are isomorphic. $\square$

Remark 7.2. The statement is correctly formulated over $\mathbb{F}_2$. Over $\mathbb{R}$, Paper 37 [18] (Theorem 6.1) establishes $\ker(L_a)|_{S_4} \cong \text{irr}_1 \oplus \text{irr}_3$ (the standard 3-dimensional real S_4 -irreducible, not the sign-twisted irr'_3). The passage from $\mathbb{R}$ to $\mathbb{F}_2$ changes the module category: the simple $\mathbb{F}_2[S_4]$ -module **2** (dim 2) is the mod-2 reduction of the real irr_3 in the sense that they share the same simple $\mathbb{F}_2$ -constituents, but irr_3 over $\mathbb{R}$ is not the same object as **2** over $\mathbb{F}_2$. Theorem 7.1 is a statement about $\mathbb{F}_2$ -modules only.

Remark 7.3. The shared simple modules **{1, 2}** of S_X and $\ker(L_a)$ provide an algebraic bridge between the quantum error correction structure of Papers 5–8 (encoded in S_X) and the zero-divisor kernel structure of Papers 29–40 (encoded in $\ker(L_a)$). Both spaces are, over $\mathbb{F}_2$, built from the same two S_4 -irreducibles. This connection is the $\mathbb{F}_2$ manifestation of the pattern described in Remark 5.2.

7.2. Sign-parity asymmetry (OP 40.1).

Proposition 7.4 (*[Derived, computational] — OP 40.1 resolved*). *Fix the octonion sign table of [1] (XOR index rule; positive forward cycle $a \rightarrow b \rightarrow c \rightarrow a$ for each Fano line). The off-diagonal matrix entry of $\mathcal{D}^2$ in a basis $\{e_s\}_{s \in V_{\text{ni}}(a)}$ is:*

$$M_{sr} = \sum_{\substack{q \in V_{\text{ni}}(a) \\ q \neq r}} \varepsilon_{qr} \varepsilon_{p, q \oplus r}, \quad p = s \oplus q \oplus r.$$

For $\ell_a = \{2, 4, 6\}$, $V_{\text{ni}}(a) = \{1, 3, 5, 7\}$: $M_{sr} = +1$ for every $s \neq r$. For every other Fano line, $M_{sr} \in \{-1, +1, +3\}$. Hence $\ell_a = \{2, 4, 6\}$ is the unique Fano line for which $M = J_4 - 5I_4$ (uniform off-diagonal +1) in the Paper 1 convention.

Proof. Fix $s \neq r$ in $V_{\text{ni}}(a)$. Decompose the sum over $q \in V_{\text{ni}}(a) \setminus \{r\}$ into three terms: $q = s$ and $\{q_1, q_2\} = V_{\text{ni}}(a) \setminus \{r, s\}$.

Step 1 (universal): the term $q = s$ contributes +1. When $q = s$: $p = s \oplus s \oplus r = r$, so the factor is $\varepsilon_{sr} \varepsilon_{r, s \oplus r}$. The triple $(s, r, s \oplus r)$ is a Fano line with positive cycle $a \rightarrow b \rightarrow c \rightarrow a$. In any such cycle, consecutive pairs carry the same orientation: if $(s, r) = (a, b)$ is forward then $(r, s \oplus r) = (b, c)$ is forward ($+1 \cdot +1 = +1$); if $(s, r) = (b, a)$ is backward then $(r, s \oplus r) = (a, c)$ is backward ($(-1)(-1) = +1$). In every case $\varepsilon_{sr} \varepsilon_{r, s \oplus r} = +1$. This holds for all seven Fano lines (verified from Paper 1 [1]).

Step 2 (specific to $\ell_a = \{2, 4, 6\}$): the two remaining terms cancel. With $p_i = s \oplus q_i \oplus r$, the sum $\varepsilon_{q_1 r} \varepsilon_{p_1, q_1 \oplus r} + \varepsilon_{q_2 r} \varepsilon_{p_2, q_2 \oplus r}$ equals 0 for every one of the 12 off-diagonal pairs (s, r) when $\ell_a = \{2, 4, 6\}$. This is verified by exhaustive case-check from the Paper 1 sign table [1] (all $7 \times 12 = 84$ instances computed).

Conclusion. $M_{sr} = (+1) + 0 = +1$ for every $s \neq r$ when $\ell_a = \{2, 4, 6\}$, giving $M = J_4 - 5I_4$. $\square$

Remark 7.5. The affine-plane interpretation of [21] (Remark in the proof of Proposition 3.5) remains valid: $\{1, 3, 5, 7\} = \{x \in \mathbb{F}_2^3 : x_3 = 1\}$ ensures that $q \oplus r \in \{2, 4, 6\}$ for every $q, r \in V_{\text{ni}}(a)$, so every two-step composition passes through ℓ_a . Proposition 7.4 shows that this geometric routing alone does not determine the sign pattern; the uniform cancellation in Step 2 is an additional sign-parity property of the specific cycle $2 \rightarrow 4 \rightarrow 6 \rightarrow 2$ within the Paper 1 convention, which singles out $\{2, 4, 6\}$ as the unique S_4 -equivariant line. The claim that for the other six Fano lines the cancellation of Step 2 is partial (i.e. holds for fewer than all 12 off-diagonal pairs) is supported by the same exhaustive computation but is not proved analytically here; it is recorded as *[Observation]*.

8. OPEN PROBLEMS

OP 41.0 [*Open, Speculation*]. Is there a functor $F: \text{Geom}(\text{PG}(2, \mathbb{F}_2)) \rightarrow \text{Mod}(\mathbb{F}_2[G_{84}])$ such that all 61 Fano roles of §2 are natural morphisms in $\text{Im } F$? The existence of such a functor would unify the five structural levels of the portfolio into a single categorical framework. The module category of $\mathbb{F}_2[G_{84}]$ has 4 simple objects (Theorem 3.1, Proposition 4.1); its Cartan matrix and projective covers remain to be computed.

OP 41.2 [*Partially closed, GAP 4.15.1*]. The four simple $\mathbb{F}_2[G_{84}]$ -modules are $\{\text{irr}_1, \text{irr}_3, \text{irr}'_3, \text{St}\}$ of dimensions $\{1, 3, 3, 8\}$ [*Derived, GAP 4.15.1*]. The Steinberg module St is *not* projective as a $\mathbb{F}_2[G_{84}]$ -module: $\dim(\text{St}) = 8 < 128 = |\text{Syl}_2(G_{84})|$, so Green's theorem gives the obstruction [*Derived-Negative, GAP 4.15.1*]. Furthermore, $\text{Ext}_{\mathbb{F}_2[G_{84}]}^1(\mathbb{F}_2, V_3) \cong \mathbb{F}_2 \neq 0$ [*Derived, Hochschild-Serre + GAP 4.15.1*]. The full Cartan matrix of $\mathbb{F}_2[G_{84}]$ and the decomposition matrix remain [*Open*]: the BrauerTable computation for groups of order 2688 is currently out of reach for GAP 4.15.1.

OP 41.3 [*Derived*]. The Weyl involution $\theta: g \mapsto (g^T)^{-1}$ of A_2 (exchanging $L(1, 0) \leftrightarrow L(0, 1)$, i.e. $V_3 \leftrightarrow V'_3$) and the projective duality of $\text{PG}(2, \mathbb{F}_2)$ (exchanging points and lines) are both realised by the same outer automorphism of $\text{GL}(3, \mathbb{F}_2)$, acting as the twist functor $M \mapsto M^\theta$ on the category of $\mathbb{F}_2[\text{GL}(3, \mathbb{F}_2)]$ -modules. This identifies $V_3 \leftrightarrow V'_3$ canonically with points $\leftrightarrow$ lines in $\text{PG}(2, \mathbb{F}_2)$. [*Derived, algebraic argument; closed post-Paper 41, July 2026*]

OP 40.1 (status update). Proposition 7.4 establishes the sign-parity asymmetry of $\{e_1, e_3, e_5, e_7\}$ as [*Derived, computational*] from the Paper 1 [1] sign table. The companion claim that the remaining six Fano lines each have partial cancellation is [*Observation*] (same computation; not proved analytically).

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