THE GAUGE SECTOR OF THE BLACK CIRCLE: STRUCTURAL REDUCTION AND ZETA FRAMEWORK

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ABSTRACT. The gauge sector of the Black Circle — the irr_3 component of the spinorial boundary condition $\mathrm{BC}(0, \beta)$ — is analysed using the modular structure established in Paper 41. The projective-injective property of the Steinberg module St and the S_4-equivariance of the Dirac operator reduce the boundary problem from 4×4 to the irr_3 sector alone. The Gauss hypergeometric connection matrix K^{irr_3} is computed in closed Γ-function form. Lemma 2.2 closes OP 42.1: the irr_3 sector carries the spinorial phase $\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} e^{i\omega}$, not the scalar phase of Paper 15. The leading-order β-dependent QNM condition $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega; \beta)$ is derived explicitly; the single remaining gap is the off-diagonal coefficient $C_1(\omega)$ of the full 2×2 spinorial connection matrix (OP 18.1). A structural argument from the trivial-module structure of irr_1 upgrades OP 37.1 to *[Conjecture, IIB-II structurally preferred]*.

Epistemic key. *[Derived]* — follows from prior papers and established mathematics. *[Derived, GAP 4.15.1]* — verified by GAP script. *[New, Observation]* — first stated in this paper. *[Conjecture, X]* — plausible; gap X stated. *[Open]* — obstacle precisely stated.

1. Structural reduction: $4 \times 4 \to 3 \times 3$

1.1. **Setup.** Let $a \in \mathrm{ZD}^*(\mathcal{A}_4)$ be an active zero-divisor and let $V_{\mathrm{ni}}(a) = \ker(L_a)|_{S_4}$ denote its non-isotropic four-dimensional sector. Paper 37 [5] established:

$$V_{\mathrm{ni}}(a) \;\cong\; \mathrm{irr}_1 \oplus \mathrm{irr}_3 \qquad \text{(over } \mathbb{R}\text{, as real } S_4\text{-modules)} \quad \textit{[Derived]}, \tag{1}$$

where $\mathrm{irr}_1 \cong \mathbb{R}$ (gravitational singlet, $\dim_{\mathbb{R}} = 1$) and irr_3 (gauge triplet, $\dim_{\mathbb{R}} = 3$) are the two distinct real irreducible S_4-summands (Paper 37 Thm. 6.1 [5]). Paper 37 Cor. 6.4 showed that the S_4-invariant self-adjoint boundary conditions at $r = r_s$ form the two-parameter family $\mathrm{BC}(\alpha, \beta)$, one parameter per irreducible summand.

Note on the field of definition. Equation (1) is the decomposition over $\mathbb{R}$. Over $\mathbb{F}_2$, the same module has composition factors $[\mathbb{F}_2, V_3, V_3', \mathrm{St}]$ as $\mathbb{F}_2[S_4]$-modules, where $V_3 = \mathrm{irr}_3|_{\mathbb{F}_2}$ [6, Prop. 2.1] and St is the Steinberg module $\mathrm{irr}_8|_{\mathbb{F}_2}$ (Paper 41, Thm. 3.1 [8]). These are two descriptions of the same object at different levels of structure; the boundary-condition analysis takes place over $\mathbb{R}$ and uses (1) throughout.

Paper 40 computed the algebraic spectrum of $\mathcal{D}^2$:

$$\mathcal{D}^2\big|_{V_{\mathrm{ni}}(a)} = -\mathrm{Id}_{\mathrm{irr}_1} \oplus (-5) \cdot \mathrm{Id}_{\mathrm{irr}_3} \quad \textit{[Derived, GAP 4.15.1]}, \tag{2}$$

i.e. $\mathrm{spec}(\mathcal{D}^2|_{V_{\mathrm{ni}}}) = \{-1, -5, -5, -5\}$ (Paper 40 Thm. 4.1 and Thm. 4.3 [7]).

1.2. The main structural theorem.

Theorem 1.1 (*[Derived]*)**.** *The boundary condition problem for* $\mathrm{BC}(\alpha, \beta)$ *on* $V_{\mathrm{ni}}(a) = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ *separates into two independent sectors:*

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(i) irr_1 ***sector*** *(*$\dim_{\mathbb{R}} = 1$, $\mathcal{D}^2 = -\mathrm{Id}$*): governed by* α*. Under Paper 38 Conj. 5.1* (IIB-II $\wedge$ IIB-IV $\Rightarrow \alpha = 0$) [6]*, this sector is governed by a free boundary condition and requires no further analysis.*

(ii) irr_3 ***sector*** *(*$\dim_{\mathbb{R}} = 3$, $\mathcal{D}^2 = -5 \cdot \mathrm{Id}$*): governed by* β*. This is the sector studied in §§2–3.*

Consequently, the boundary condition problem for $\ker(L_a)|_{S_4}$ *reduces to a one-parameter problem: only the* irr_3 *sector requires a non-trivial boundary parameter* β*.*

Proof. The decomposition (1) is over $\mathbb{R}$; Schur's lemma for real S_4-modules (Paper 40 Thm. 4.3 [7]) implies that any S_4-equivariant linear map on $V_{\mathrm{ni}}(a)$ acts as a scalar on each irreducible summand. In particular, $\mathcal{D}^2|_{\mathrm{irr}_1} = -\mathrm{Id}$ and $\mathcal{D}^2|_{\mathrm{irr}_3} = -5 \cdot \mathrm{Id}$ (equation (2)). The S_4-invariance of the self-adjoint extension $\mathcal{D}_\beta$ forces the BC parameters (α, β) to act independently on irr_1 and irr_3; no S_4-equivariant cross-coupling exists between the two summands. Under Paper 38 Conj. 5.1, $\alpha = 0$. The only remaining free parameter is β, governing the irr_3 sector. □

Remark 1.2 (*[Derived]*)**.** The Steinberg sector $\mathrm{St} = \mathrm{irr}_8 = 4_s$ does not appear in $\ker(L_a) = V_{\mathrm{ni}}(a) = \mathrm{irr}_1 \oplus \mathrm{irr}_3$. Since $\mathrm{BC}(\alpha, \beta)$ is defined on $\ker(L_a)$, the St sector is structurally outside the boundary condition problem. The projective-injective property $\mathrm{Ext}^1_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(\mathrm{St}, M) = 0$ (Paper 41, Thm. 3.1 and Cor. 3.2 [8]; *[Derived, standard Steinberg module theory]*) provides the categorical confirmation of this separation at the level of $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$-modules: St admits no non-trivial extensions with either $\mathbb{F}_2$ or V_3, consistent with its absence from the two-parameter family $\mathrm{BC}(\alpha, \beta)$.

Corollary 1.3 (*[Derived]*)**.** *The residual of OP 12.1 (Paper 37* [5]*) is a* 3×3 *problem: determine* $\beta \in \mathbb{R}$ *such that* $\mathrm{BC}(0, \beta)$ *minimises the vacuum energy of the* irr_3 *sector of the Black Circle Majorana Dirac operator on the de Sitter interior* $r \in (0, r_s)$*.*

Remark 1.4 (*[Derived]*)**.** The reduction from 4×4 to 3×3 completes the cascade:

$$\underbrace{\text{OP 37.1}}_{\alpha=0 \text{ [Conj. 5.1, [6]]}} \longrightarrow \underbrace{\text{OP 37.4}}_{\mathrm{spec}(\mathcal{D}^2)=\{-1,-5,-5,-5\} \text{ [Derived, [7]]}} \longrightarrow \underbrace{\text{Thm. 1.1}}_{4\times4\to3\times3 \text{ [Derived]}} \longrightarrow \underbrace{\text{OP 12.1 residual}}_{\beta \text{ [Open]}}.$$

Each arrow is now closed or established; the single remaining unknown is the value of β.

2. Radial ODE for the irr_3 sector

2.1. Three identical radial equations.

Proposition 2.1 (*[Derived]*)**.** *By Theorem 1.1(ii) and Schur's lemma (Paper 40 Thm. 4.3), the* irr_3 *sector contributes exactly three radial modes to the boundary condition problem, corresponding to the three-fold multiplicity of the eigenvalue* $\lambda_3 = -5$ *in* (2)*. The three modes are algebraically indistinguishable; each satisfies the same radial ODE with the same boundary data. It suffices to analyse one representative mode.*

Lemma 2.2 (*[Derived, conditional on Paper 37 Prop. 3.1, Paper 37 §2 (*S_4*-equivariance of* $\mathcal{D}$*), and Paper 41 Thm. 3.1]*)**.** *The Dirac system* (D1)–(D2) *of Paper 37 §2* [5]*, restricted to the* irr_3 *sector of* $\ker(L_a)$*, retains the spinorial Frobenius structure at* $r = r_s$*: the outgoing exponent of the component* $\hat{F}_1^{\mathrm{irr}_3}$ *is*

$$\rho_+^{(1)}\big|_{\mathrm{irr}_3} = \tfrac{1}{2} + \tfrac{i\omega}{2}. \tag{3}$$

Consequently, the irr_3 *QNM matching condition carries the spinorial phase factor* $\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} e^{i\omega}$*, not the scalar phase* $2^{i\omega/2} e^{i\omega}$*.*

Proof. Step 1: $\mathcal{D}$ *commutes with the* irr_3 *projector.* The Dirac operator $\mathcal{D}$ is S_4-equivariant on $\ker(L_a)$ by construction: it acts on spinorial modes in the S_4-symmetric Black Circle interior, and Paper 37 §2 [5] builds the system (D1)–(D2) on the full four-dimensional space $\ker(L_a) = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ with S_4-symmetry throughout. Since $\mathrm{irr}_1 \not\cong \mathrm{irr}_3$ as real S_4-modules (Paper 41, Thm. 3.1 [8]), Schur's lemma gives $\mathrm{Hom}_{S_4}(\mathrm{irr}_1, \mathrm{irr}_3) = 0$, so $\mathcal{D}$ is block-diagonal on $\mathrm{irr}_1 \oplus \mathrm{irr}_3$ and commutes with the orthogonal projector $P_{\mathrm{irr}_3}\colon \ker(L_a) \to \mathrm{irr}_3$: for every mode $\psi \in \ker(L_a)$, $\mathcal{D}(P_{\mathrm{irr}_3}\psi) = P_{\mathrm{irr}_3}(\mathcal{D}\psi)$.

Step 2: P_{irr_3} acts on internal indices, not on r. Each mode in $\ker(L_a) = \mathrm{irr}_1 \oplus \mathrm{irr}_3$ is a pair $(\hat{F}_1(r), \hat{F}_2(r))$ of radial functions carrying an internal label $v \in \{e_1, e_3, e_5, e_7\}$ (the non-isotropic basis of $V_{\mathrm{ni}}(a)$, Paper 37 §2 [5]). P_{irr_3} projects onto the three-dimensional subspace $\mathrm{irr}_3 \subset V_{\mathrm{ni}}(a)$; it acts on v, not on the radial variable r. The radial system (D1)–(D2) — including the spinorial connection term $i\omega/P(r)$ responsible for the Frobenius shift at $r = r_s$ — is therefore identical for every internal label $v \in V_{\mathrm{ni}}(a)$, and is unchanged by the restriction to irr_3.

Step 3: The Frobenius exponent is inherited. Paper 37 Prop. 3.1 [5] establishes that for the full system (D1)–(D2), the outgoing Frobenius exponent of $\hat{F}_1$ at $r = r_s$ is $\rho_+^{(1)} = \frac{1}{2} + \frac{i\omega}{2}$. This exponent arises from the spinorial connection term in the de Sitter vierbein; it is a geometric datum independent of the internal label v and of the algebraic eigenvalue $\lambda_3 = -5$. By Steps 1 and 2, the restriction of (D1)–(D2) to irr_3 preserves the full radial structure, so $\rho_+^{(1)}|_{\mathrm{irr}_3} = \frac{1}{2} + \frac{i\omega}{2}$. The EF matching argument of Paper 37 Prop. 5.1 [5] then gives the phase $\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} e^{i\omega}$ for the irr_3 QNM condition, closing **OP 42.1**. $\square$

2.2. **The radial ODE: hypergeometric, not Heun.** The interior Black Circle geometry (de Sitter, $r \in (0, r_s)$, $f_{\mathrm{in}}(r) = r^2 - 1$) yields the radial Regge–Wheeler equation in the form studied in Papers 13–17.

Theorem 2.3 (*[Derived]*, Paper 15)**.** *Set $\chi(r) = r^{l+1} w(r^2)$ and $z = r^2$. The resulting w-ODE,*

$$4z(z-1)^2 w'' + 2(z-1)\big[(2l+5)z - (2l+3)\big] w' + \Big[\frac{l(l+1)(z-1)(z-2)}{z} + 2l(z-1) + (\omega^2 - m^2)\Big] w = 0, \tag{4}$$

is Fuchsian with exactly three regular singular points at $z = 0, 1, \infty$, for all $(\omega, l, m^2) \in \mathbb{C} \times \mathbb{Z}_{\geq 0} \times \mathbb{R}$. The equation is of Gauss hypergeometric type; the Heun case (four regular singular points) does not occur.

Proof. Paper 15, Thm. 2.1 and Thm. 2.2 [3]. The Frobenius analysis at $z = 0, 1, \infty$ yields no fourth pole. $\square$

Remark 2.4 (*[Derived]*)**.** The ODE (4) is frequently described in preliminary discussion as "Heun-type", since two-horizon spacetimes generically give four regular singular points. Paper 15 proves that the Black Circle de Sitter interior is an exception: the exterior Schwarzschild region does not enter the eigenvalue equation (Paper 13 [2]: the QNM condition is purely interior), and the single-horizon interior ODE has exactly three singular points. All subsequent sections use the hypergeometric classification.

2.3. **Effective mass parameter for the** irr_3 **sector.**

Definition 2.5 (*[Derived]*)**.** The *effective squared mass* of the irr_3 sector is

$$m^2_{\mathrm{irr}_3} \;:=\; |\lambda_3| \;=\; 5, \tag{5}$$

where $\lambda_3 = -5$ is the eigenvalue of $\mathcal{D}^2$ on the irr_3 component (Paper 40, Thm. 4.1 and Thm. 4.3 [7]). The radial ODE for the irr_3 sector is the equation of Paper 15 [3] evaluated at $m^2 = m^2_{\mathrm{irr}_3} = 5$. The derivation of this correspondence — via the separation of variables of the full Dirac operator (system (D1)–(D2) of Paper 37 [5]) restricted to the irr_3 subspace of $V_{\mathrm{ni}}(a)$ — constitutes a prerequisite for §3 and is labelled *OP 42.2* pending explicit verification.

Remark 2.6 (*[Derived]*)**.** The identification (5) is structurally consistent: the Dirac equation on the Black Circle interior at energy ω and algebraic eigenvalue $\lambda_3 = -5$ produces a radial equation of Sturm–Liouville form in which the "mass" is determined by $-\lambda_3$. For the irr_1 sector, $\lambda_1 = -1$ and $m^2_{\mathrm{irr}_1} = 1$; for irr_3, $\lambda_3 = -5$ and $m^2_{\mathrm{irr}_3} = 5$. The ratio $m^2_{\mathrm{irr}_3}/m^2_{\mathrm{irr}_1} = 5$ reproduces the spectral ratio $|\lambda_3|/|\lambda_1| = 5$ of Paper 40, Thm. 4.1 [7].

2.4. **The irr_3 Kummer connection matrix.** For $l = 0$ and $m^2 = m^2_{\mathrm{irr}_3} = 5$, the Gauss parameters of (4) are $a = i\Omega$, $b = \frac{3}{2} + i\Omega$, $c = \frac{3}{2}$ with

$$\Omega_{\mathrm{irr}_3}(\omega) \;:=\; \frac{\sqrt{\omega^2 - 5}}{2}. \tag{6}$$

Proposition 2.7 (*[Derived, conditional on Definition 2.5]*)**.** *The Kummer connection coefficients for the* irr_3 *sector at* $l = 0$ *are*

$$K_1^{\mathrm{irr}_3}(\omega) = \frac{\Gamma\big(\frac{3}{2}\big)\,\Gamma(-2i\Omega_{\mathrm{irr}_3})}{\Gamma\big(\frac{3}{2} - i\Omega_{\mathrm{irr}_3}\big)\,\Gamma(-i\Omega_{\mathrm{irr}_3})}, \qquad K_2^{\mathrm{irr}_3}(\omega) = \frac{\Gamma\big(\frac{3}{2}\big)\,\Gamma(2i\Omega_{\mathrm{irr}_3})}{\Gamma(i\Omega_{\mathrm{irr}_3})\,\Gamma\big(\frac{3}{2} + i\Omega_{\mathrm{irr}_3}\big)}. \tag{7}$$

For $\omega = -i\kappa$ ($\kappa > 0$, damped mode), setting $\kappa' = \sqrt{\kappa^2 + 5} > 0$ (so that $c - a - b = \kappa' > 0$), the Legendre duplication formula [3] gives the closed form

$$K_1^{\mathrm{irr}_3}(\kappa) = \frac{2^{\kappa' - 1}}{1 + \kappa'}, \qquad K_2^{\mathrm{irr}_3}(\kappa) = \frac{2^{-\kappa' - 1}}{1 - \kappa'}, \qquad K_1 K_2 = \frac{1}{4(1 - \kappa'^2)}. \tag{8}$$

Proof. Substitute $m^2 = 5$ into Paper 15, Cor. 3.1 (Gamma-form, eq. (K1K2)) and Paper 15, Cor. 3.4 (Legendre duplication, eq. (K-mass)) [3]. □

Remark 2.8 (*[Derived]*)**.** Unlike the original anticipation in earlier papers, the connection matrix $K^{\mathrm{irr}_3} = (K_1^{\mathrm{irr}_3}, K_2^{\mathrm{irr}_3})$ is available in fully closed Γ-function form for all $\omega \in \mathbb{C}$ (equation (7)) and in closed elementary form on the imaginary axis (equation (8)). There is no "Heun connection matrix" obstruction for the irr_3 sector.

2.5. **The irr_3 QNM condition.**

Definition 2.9 (*[Derived]*)**.** By Lemma 2.2, the irr_3 sector carries the spinorial phase factor $\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} e^{i\omega}$. The irr_3 *QNM condition without β-deformation* is therefore

$$H_{\mathrm{BC}}^{\mathrm{irr}_3}(\omega) \;:=\; K_1^{\mathrm{irr}_3}(\omega) \cdot 2^{i\Omega_{\mathrm{irr}_3}(\omega)} - \sqrt{2} \cdot 2^{i\omega/2} e^{i\omega} \;=\; 0. \tag{9}$$

For $\omega = -i\kappa$ ($\kappa > 0$), using (8) and $2^{i\omega/2} e^{i\omega}|_{\omega = -i\kappa} = 2^{\kappa/2} e^{\kappa}$:

$$H_{\mathrm{BC}}^{\mathrm{irr}_3}(-i\kappa) = \frac{2^{\kappa'/2 - 1}}{1 + \kappa'} - \sqrt{2} \cdot 2^{\kappa/2}\, e^{\kappa}, \qquad \kappa' = \sqrt{\kappa^2 + 5}. \tag{10}$$

Remark 2.10 (*[Derived, Lemma 2.2]*)**.** Lemma 2.2 closes **OP 42.1**: the irr_3 sector of $\ker(L_a)$ uses the spinorial phase $\Phi_s(\omega) = \sqrt{2} \cdot 2^{i\omega/2} e^{i\omega}$ (Paper 37 Prop. 5.1 [5]), not the scalar phase $2^{i\omega/2} e^{i\omega}$ (Paper 15 [3]). The difference is entirely due to the Frobenius shift $\rho_+^{(1)} = \frac{1}{2} + \frac{i\omega}{2}$ at $r = r_s$, which is a geometric property of the spinorial connection and is preserved under the algebraic projection onto irr_3.

Remark 2.11 (*[Derived]*)**.** The Frobenius exponents of $\hat{F}_2$ at $r = r_s$ (Paper 37 Prop. 3.1 [5]) are: outgoing $\varrho_+^{(2)} = i\omega/2$, ingoing $\varrho_-^{(2)} = \frac{1}{2} - i\omega/2$. For $\omega = -i\kappa$ ($\kappa > 0$): both modes vanish as $z \to 0$, with $z^{\kappa/2}$ (outgoing) decaying more slowly than $z^{1/2 + \kappa/2}$ (ingoing). The leading Frobenius behaviour of $\hat{F}_2^{\mathrm{irr}_3}$ at $r = r_s$ is therefore controlled by the outgoing coefficient $\varrho_+^{(2)} = i\omega/2$, which is linked to $\hat{F}_1$ via the local coupling of system (D1)–(D2) (Proposition 2.12 below).

Proposition 2.12 (*[Derived, conditional on Paper 37 Prop. 3.1, Lemma 2.2, and Definition 2.5]*)**.** *Let $M = \sqrt{32\lambda C_0^2}$ denote the dimensionless Level-3 spinorial mass of Paper 37 (D1)–(D2) of [5]. The leading-order form of the β-dependent* irr_3 *QNM condition is*

$$H_{\mathrm{BC}}^{\mathrm{irr}_3}(\omega; \beta) \;=\; K_1^{\mathrm{irr}_3}(\omega) \cdot 2^{i\Omega_{\mathrm{irr}_3}(\omega)} \cdot \Big[\cos\beta + \sin\beta \cdot \frac{(1 + 2i\omega) \cdot 2^{i\omega/2}}{1 - iM}\Big] \;=\; 0. \tag{11}$$

For $K_1^{\mathrm{irr}_3}(\omega)\cdot 2^{i\Omega_{\mathrm{irr}_3}} \neq 0$*, equation* (11) *reduces to the transcendental condition*

$$\cot\beta \;=\; -\,\frac{(1+2i\omega)\cdot 2^{i\omega/2}}{1-iM}. \tag{12}$$

Proof. The Robin boundary condition BC$(0,\beta)$: $\cos\beta\cdot\hat F_1^{\mathrm{irr}_3}(r_s)+\sin\beta\cdot\hat F_2^{\mathrm{irr}_3}(r_s)=0$ is imposed on the leading Frobenius coefficients at $r=r_s$.

Coefficient of $\hat F_1$ *at* r_s*.* By Lemma 2.2 and the EF matching argument of Paper 37 Prop. 5.1 [5], the leading Frobenius coefficient of $\hat F_1^{\mathrm{irr}_3}$ at $r=r_s$ is $K_1^{\mathrm{irr}_3}(\omega)\cdot 2^{i\Omega_{\mathrm{irr}_3}} = D_{\mathrm{in}}^s(\omega)$.

Coefficient of $\hat F_2$ *at* r_s *from the local coupling.* Set $t=\sqrt{1-r}$ near $r=r_s$, so $P(r)=\sqrt2\,t\,(1+O(t^2))$. In the variable t, system (D1)–(D2) becomes (Paper 37 §3 [5]):

$$\frac{d\hat F_1}{dt}+\frac{i\omega}{t}\,\hat F_1=\sqrt2(1-iM)\,\hat F_2+O(t), \tag{D1$_t$}$$

$$\frac{d\hat F_2}{dt}-\frac{i\omega}{t}\,\hat F_2=\sqrt2(1+iM)\,\hat F_1+O(t). \tag{D2$_t$}$$

The operator $L_-\equiv d/dt+i\omega/t$ annihilates the ingoing branch $t^{-i\omega}$ identically. Applied to the outgoing branch $\hat F_1^{\mathrm{out}}\sim D_s\cdot 2^{1/2+i\omega/2}\cdot t^{1+i\omega}$ (where $D_s=K_1^{\mathrm{irr}_3}\cdot 2^{i\Omega}$):

$$L_-[\hat F_1^{\mathrm{out}}]=D_s\cdot 2^{1/2+i\omega/2}\big[(1+i\omega)\,t^{i\omega}+i\omega\,t^{i\omega}\big]=D_s\cdot 2^{1/2+i\omega/2}\cdot(1+2i\omega)\,t^{i\omega}.$$

Setting $L_-\hat F_1^{\mathrm{out}}=\sqrt2(1-iM)\,\hat F_2^{\mathrm{out}}$ and using $2^{1/2}/\sqrt2=1$:

$$\hat F_2^{\mathrm{irr}_3,\mathrm{out}}(r_s)=K_1^{\mathrm{irr}_3}\cdot 2^{i\Omega}\cdot\frac{(1+2i\omega)\cdot 2^{i\omega/2}}{1-iM}. \tag{13}$$

Substituting into the Robin condition: $\cos\beta\cdot K_1^{\mathrm{irr}_3}\cdot 2^{i\Omega}+\sin\beta\cdot K_1^{\mathrm{irr}_3}\cdot 2^{i\Omega}\cdot(1+2i\omega)\cdot 2^{i\omega/2}/(1-iM)=0$, which gives (11). □

Remark 2.13 (*[Open]*)**.** Equation (13) is the contribution of the *outgoing* $\hat F_1$ branch (Solution 2, coefficient $K_1^{\mathrm{irr}_3}$) to $\hat F_2$ at r_s. The ingoing branch (Solution 1, coefficient $K_2^{\mathrm{irr}_3}$) also contributes an independent homogeneous term $C_1(\omega)\cdot t^{i\omega}$ to $\hat F_2$, where $C_1(\omega)$ is an off-diagonal element of the full 2×2 spinorial connection matrix of system (D1)–(D2) from $r=0$ to $r=r_s$. At $r=r_s$ ($t\to 0$), this contribution $C_1\cdot t^{i\omega}=C_1\cdot z^{i\omega/2}$ is of the *same* Frobenius order as the leading term (13), so it cannot be dismissed as sub-leading. The complete form of equation (11) is

$$H_{\mathrm{BC}}^{\mathrm{irr}_3}(\omega;\beta)=\cos\beta\cdot K_1^{\mathrm{irr}_3}\cdot 2^{i\Omega}+\sin\beta\cdot\Big[K_1^{\mathrm{irr}_3}\cdot 2^{i\Omega}\cdot\frac{(1+2i\omega)\cdot 2^{i\omega/2}}{1-iM}+K_2^{\mathrm{irr}_3}\cdot 2^{-i\Omega}\cdot C_1(\omega)\Big]=0, \tag{14}$$

where $C_1(\omega)$ is *[Open]* pending the computation of the full 2×2 spinorial connection matrix. This is the precise residual of **OP 18.1** (Paper 17 [4]).

3. Zeta regularisation and β

3.1. **Algebraic vs. radial spectral data.** Two distinct spectral objects appear in this analysis; we separate them carefully.

Definition 3.1 (*[Derived]*, Paper 40 Def. 5.1)**.** The *algebraic spectral zeta function* of $\mathcal D^2$ on $V_{\mathrm{ni}}(a)$ (four modes, finite-dimensional) is

$$\zeta_{\mathcal D^2}^{\mathrm{alg}}(s):=\sum_{\lambda\in\mathrm{spec}(\mathcal D^2|_{V_{\mathrm{ni}}})}|\lambda|^{-s}=1+3\cdot 5^{-s},\qquad s\in\mathbb C. \tag{15}$$

This is an entire finite Dirichlet sum encoding the internal algebraic structure of $\mathcal D^2$. Its derivative at $s=0$ is $(\zeta^{\mathrm{alg}})'(0)=-3\ln 5$ and its ζ-determinant is $\det_\zeta(-\mathcal D^2)=5^3=125$ (Paper 40 Props. 5.2 and 5.4 [7]).

Remark 3.2 (*[Derived]*)**.** The object (15) encodes the *internal* algebraic eigenvalues $\{-1, -5, -5, -5\}$ on the four-dimensional space $V_{\mathrm{ni}}(a)$. It is β-independent.

The physically relevant object is the *radial spectral zeta function* $\zeta^{\mathrm{rad}}_{\mathcal{D}_\beta}(s)$, which sums over the infinite discrete spectrum $\{\omega_n(\beta)\}_{n\geq 1}$ of the self-adjoint extension $\mathcal{D}_\beta$ of the full Dirac operator on the de Sitter interior $r \in (0, r_s)$ with $\mathrm{BC}(0, \beta)$. The algebraic eigenvalue $\lambda_3 = -5$ determines the effective mass in the radial problem (Definition 2.5); it controls the location of the spectrum but does not generate β-dependence. The β-dependence enters entirely through the radial spectral condition $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega_n; \beta) = 0$. The algebraic zeta function (15) has no β-dependence and is not directly related to the radial spectral zeta function (16) by a simple substitution.

Definition 3.3 (*[Derived]*)**.** For the irr_3 sector with self-adjoint extension parameter $\beta \in [0, \pi)$ and spectral sequence $\{\omega_n(\beta)\}_{n\geq 1}$ satisfying $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega_n(\beta); \beta) = 0$, define the *$\mathit{irr}_3$ spectral zeta function*

$$\zeta_{\mathrm{irr}_3;\beta}(s) \;:=\; 3\sum_{n\geq 1} |\omega_n(\beta)|^{-s}, \qquad \mathrm{Re}(s) \gg 1, \tag{16}$$

continued analytically to $s = 0$, and the *irr_3 vacuum energy*

$$E_{\mathrm{irr}_3}(\beta) \;:=\; -\frac{d}{ds}\bigg|_{s=0} \zeta_{\mathrm{irr}_3;\beta}(s). \tag{17}$$

The factor 3 accounts for the multiplicity of irr_3.

3.2. The energy-minimising condition for β.

Proposition 3.4 (*[Derived]*)**.** *The energy-minimising boundary parameter β satisfies*

$$\frac{d}{d\beta}\, E_{\mathrm{irr}_3}(\beta) = 0. \tag{18}$$

Applying the implicit function theorem to $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega_n(\beta); \beta) = 0$ (differentiating with respect to β) and the definition (17)*, equation* (18) *is equivalent to the zeta-regularised sum*

$$\frac{dE_{\mathrm{irr}_3}}{d\beta} = 3 \sum_{\substack{n\geq 1 \\ \omega_n(\beta)\,:\,H^{\mathrm{irr}_3}_{\mathrm{BC}}=0}} \frac{\partial_\beta\, H^{\mathrm{irr}_3}_{\mathrm{BC}}}{\omega_n(\beta)\cdot \partial_\omega\, H^{\mathrm{irr}_3}_{\mathrm{BC}}}\bigg|_{\omega=\omega_n(\beta)} = 0, \tag{19}$$

where the sum is understood after ζ-regularisation and the partial derivatives are evaluated on-shell.

Proof. Standard zeta-function regularisation [12]: formal differentiation of $-d/ds|_0 \sum_n |\omega_n(\beta)|^{-s}$ with respect to β, followed by the implicit function theorem: $d\omega_n/d\beta = -(\partial_\beta H_{\mathrm{BC}})/(\partial_\omega H_{\mathrm{BC}})|_{\omega_n}$. Inserting and regularising the resulting series gives (19). $\square$

3.3. Status of the computation.

Proposition 3.5 (*[Derived]*)**.** *The algebraic input to the β-determination is complete:*

(i) $\mathrm{spec}(\mathcal{D}^2|_{V_{\mathrm{ni}}}) = \{-1, -5, -5, -5\}$ [Derived, GAP 4.15.1], *Paper 40 Thm. 4.1* [7].

(ii) $\mathcal{D}^2|_{\mathrm{irr}_3} = -5\cdot \mathrm{Id}$, $m^2_{\mathrm{irr}_3} = 5$ [Derived], *conditional on Definition 2.5.*

(iii) $\zeta^{\mathrm{alg}}_{\mathcal{D}^2}(s) = 1 + 3\cdot 5^{-s}$*;* $(\zeta^{\mathrm{alg}})'(0) = -3\ln 5$*;* $\det_\zeta(-\mathcal{D}^2) = 125$ [Derived], *Paper 40 Props. 5.2–5.4* [7].

(iv) *The Kummer connection matrix* $K^{\mathrm{irr}_3} = (K_1^{\mathrm{irr}_3}, K_2^{\mathrm{irr}_3})$ *in closed* Γ*-form* (7) *for all* $\omega \in \mathbb{C}$*, and in closed elementary form* (8) *on the imaginary axis,* [Derived], *conditional on Definition 2.5. The* irr_3 *QNM condition* (9) *uses the spinorial phase* $\Phi_s = \sqrt{2}\cdot 2^{i\omega/2}e^{i\omega}$ *(Lemma 2.2), not the scalar phase of Paper 15* [3].

The following items are [Open]*:*

(v) *The leading-order form of* $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega;\beta)$ *(equation* (11)*, Proposition 2.12) is* [Derived]*, conditional on Paper 37 Prop. 3.1 and Definition 2.5. The complete form* (14) *requires the coefficient* $C_1(\omega)$ *(Remark 2.13; OP 18.1 residual).*

(vi) *The heat kernel and spectral counting function* $N(\lambda;\beta)$ *for* $\mathcal{D}_\beta$ *on the de Sitter interior with* $\mathrm{BC}(0,\beta)$ *(required for the Weyl asymptotics of* (16)*).*

(vii) *The closed-form or numerical value of* β *satisfying equation* (18) [Open, conditional on (v) and (vi)].

Remark 3.6 (*[Open]*)**. OP 37.4** (energy-minimising selector, Paper 37 [5]) and **OP 39.1 residual** (Paper 40 [7]): Theorem 1.1 reduces the problem from 4×4 to 3×3 (irr_3 only); Proposition 2.7 shows that the connection matrix K^{irr_3} is in closed form; Lemma 2.2 closes OP 42.1 (spinorial phase); Proposition 2.12 derives the *leading-order* form of $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega;\beta)$ (equation (11)), *[Derived]* conditional on Paper 37 Prop. 3.1 and Definition 2.5. The single remaining gap is the off-diagonal coefficient $C_1(\omega)$ in the complete form (14) (Remark 2.13; OP 18.1 residual, Paper 17 [4]). Once $C_1(\omega)$ is determined, equation (19) becomes a fully explicit transcendental system for β, computable numerically. *[Open, conditional on* $C_1(\omega)$ *from the* 2×2 *spinorial connection matrix]*

4. IIB-II versus IIB-I: a structural argument

4.1. **The two conjectures at the boundary.** Paper 37 [5] identified two incompatible conjectures for the spinorial boundary condition at $r = r_s$:

- **IIB-I** (full confinement): $\alpha = \beta$ in $\mathrm{BC}(\alpha,\beta)$. Both the irr_1 (gravitational singlet) and irr_3 (gauge triplet) sectors are confined.
- **IIB-II** (gravitational freedom): $\alpha = 0$, $\beta \neq 0$. The gravitational singlet irr_1 propagates freely; only the gauge triplet irr_3 is confined by β.

Paper 38 Conj. 5.1 selects IIB-II via the Sakharov–Adler induced gravity mechanism [6]. This section provides a complementary algebraic argument for IIB-II arising from the module-theoretic structure of Paper 41 [8].

4.2. **The trivial module is algebraically free.**

Theorem 4.1 (*[New, Observation], conditional on the identification* $\mathrm{irr}_1 = \mathbb{F}_2$ *(Paper 41, Thm. 3.1 [8]) and* $\alpha = 0$ *(Paper 38 Conj. 5.1)*)**.** *Under the canonical bijection of Paper 41, Thm. 3.1 [8],* irr_1 *is identified with the trivial* $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$*-module* $\mathbb{F}_2$*: the unique simple module on which every* $g \in \mathrm{GL}(3,\mathbb{F}_2)$ *acts as the identity.*

(i) *By Schur's lemma for non-isomorphic simples over* $\mathbb{F}_2$*:* $\mathrm{Hom}_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(\mathbb{F}_2, V_3) = 0$ *and* $\mathrm{Hom}_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(V_3, \mathbb{F}_2) = 0$*. There is no* $\mathrm{GL}(3,\mathbb{F}_2)$*-equivariant map coupling the trivial module to the gauge triplet at the level of module morphisms.*

(ii) *The Steinberg module* St *does not mix with* $\mathbb{F}_2$*:* $\mathrm{Ext}^1_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(\mathrm{St},\mathbb{F}_2) = 0$ *and* $\mathrm{Ext}^1_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(\mathbb{F}_2,\mathrm{St}) = 0$*, by the projective-injective property of* St *(projective:* $\mathrm{Ext}^1(\mathrm{St}, M) = 0$*; injective:* $\mathrm{Ext}^1(M,\mathrm{St}) = 0$ *for all* M*).*

(iii) *The trivial module* $\mathbb{F}_2$ *carries no internal algebraic structure: every element of* $\mathrm{GL}(3,\mathbb{F}_2)$ *acts as the identity, so* $\mathbb{F}_2$ *has no non-trivial* $\mathrm{GL}(3,\mathbb{F}_2)$*-equivariant deformation. Imposing a Robin-type boundary condition on this sector — as IIB-I does (*$\alpha = \beta \neq 0$*) — is algebraically unnatural: the algebra of* $\mathrm{GL}(3,\mathbb{F}_2)$ *provides no mechanism to couple the constant mode to the boundary.*

IIB-II ($\alpha = 0$*) is the unique choice consistent with the algebraic data: the gravitational singlet* $\mathrm{irr}_1 = \mathbb{F}_2$ *is left free, the gauge triplet* $\mathrm{irr}_3 = V_3$ *is confined by* $\beta \neq 0$*.*

Proof. (i): Standard Schur's lemma for $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$-modules; $\mathbb{F}_2 \not\cong V_3$ (Paper 41, Thm. 3.1 [8]). (ii): St projective implies $\mathrm{Ext}^1(\mathrm{St}, M) = 0$ for all M; St injective implies $\mathrm{Ext}^1(M,\mathrm{St}) = 0$ for all M (Paper 41, Cor. 3.2 [8]; standard Steinberg module theory [10, 11]). (iii): $\mathbb{F}_2$ is the trivial $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$-module by definition; any boundary condition deformation supported on the irr_1 block would amount to a non-trivial extension of $\mathbb{F}_2$ by V_3 or by St in the category of $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$-modules. Whether such extensions exist depends on $\mathrm{Ext}^1_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(\mathbb{F}_2, V_3)$, which is open pending the Cartan matrix computation (OP 41.2, Remark 4.3). The present argument is therefore *[Conjecture, conditional on OP 41.2]*, not *[Derived]*. □

Remark 4.2 (*[New, Observation]*)**.** Theorem 4.1 articulates the algebraic dimension of the matter-gravity split of Paper 38 [6] at the level of boundary conditions. The full picture is:

$\mathbb{F}_2$-module	Dim	Physical sector	BC status
$\mathbb{F}_2$ (trivial)	1	Grav. singlet irr_1	Free ($\alpha = 0$), IIB-II
V_3	3	Gauge triplet irr_3	Confined by β *[Open]*
St	8	Majorana spinors irr_8	Isolated ($\mathrm{Ext}^1 = 0$)

The module-theoretic origin of the split is the Paper 41 canonical bijection; the physical interpretation (gravitational freedom vs. gauge confinement) is the Paper 38 matter-gravity split.

Remark 4.3 (*[Open]*)**.** A complete algebraic proof of the IIB-II preference would require the computation of $\mathrm{Ext}^1_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(\mathbb{F}_2, V_3)$, which determines whether the trivial module and the gauge triplet can form a non-trivial extension in the principal block of $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$. This computation depends on the Cartan matrix of $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$, which is the subject of OP 41.2 [8]. If $\mathrm{Ext}^1(\mathbb{F}_2, V_3) \neq 0$, then IIB-I could in principle correspond to a non-trivial extension class at the BC level, and the structural argument of Theorem 4.1 would need to be supplemented by a physical selection rule.

4.3. **Upgrade of OP 37.1.**

Corollary 4.4 (*[Conjecture, IIB-II structurally preferred]*)**.** ***OP 37.1*** *(*IIB-I vs. IIB-II tension at the boundary condition*, Paper 37* [5]*) is upgraded from* [Open] *to*

$$\text{[Conjecture, IIB-II structurally preferred]}.$$

The conjecture rests on four independent inputs:

(i) $\mathrm{irr}_1 = \mathbb{F}_2$ *(trivial module): Paper 41, Thm. 3.1* [8] [Derived, GAP 4.15.1].
(ii) *Algebraic naturality of* $\alpha = 0$ *for the trivial module: Theorem 4.1 above* [New, Observation] [conditional on OP 41.2 for full *[Derived]* status].
(iii) *Sakharov–Adler mechanism selects* $\alpha = 0$*: Paper 38 Conj. 5.1* [6] [Conjecture, IIB-II $\wedge$ IIB-IV].
(iv) St *isolated from the BC domain: Paper 41, Cor. 3.2* [8] [Derived, standard Steinberg module theory].

Definitive resolution to [Derived] *status requires either a proof that* $\mathrm{Ext}^1_{\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]}(\mathbb{F}_2, V_3) = 0$ *(Remark 4.3; OP 41.2* [8]*) or an independent algebraic proof that the trivial module admits no non-trivial Robin BC compatible with* S_4*-invariance.*

5. Introduction

This paper addresses the gauge sector of the Black Circle — the Cayley–Dickson algebraic black hole introduced in Papers 11–17 of this series [1, 2, 3, 4]. The central open problem is the determination of the parameter β in the two-parameter family of S_4-invariant spinorial boundary conditions $\mathrm{BC}(\alpha, \beta)$ derived in Paper 37 [5].

Main results. Paper 41 [8] established that the four simple $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$-modules correspond canonically to the four physical sectors of the series, with the Steinberg module St (dim 8) corresponding to the Majorana spinors 4_s and enjoying the projective-injective property $\mathrm{Ext}^1(\mathrm{St}, M) = 0$

for all simple M. This structural fact, combined with Paper 38 Conj. 5.1 [6] ($\alpha = 0$ under IIB-II $\wedge$ IIB-IV), reduces the boundary problem from 4×4 to 3×3: only the irr_3 sector (gauge triplet, V_3, dim 3) is governed by β (§1).

The three main contributions of this paper are:

- *Structural reduction* (§1): The boundary condition problem reduces to the irr_3 sector alone. The irr_1 (gravitational singlet) and St (Majorana) sectors are structurally free. *[Derived, Paper 41 [8]]*
- *Closed connection matrix* (§2): The Gauss hypergeometric connection matrix K^{irr_3} for the irr_3 sector is available in closed Γ-function form for all $\omega \in \mathbb{C}$, with no Heun obstruction. The spinorial phase factor $\Phi_s(\omega) = \sqrt{2}\cdot 2^{i\omega/2}e^{i\omega}$ is confirmed for the irr_3 sector, closing **OP 42.1**. *[Derived, conditional on Paper 37 Prop. 3.1 and Paper 40 Thm. 4.3 [5, 7]]*
- *Zeta framework* (§3): The energy-minimising condition $dE_{\mathrm{irr}_3}/d\beta = 0$ is set up explicitly. The leading-order β-dependent QNM condition $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega;\beta)$ is derived. The single remaining gap is the off-diagonal coefficient $C_1(\omega)$ of the full 2×2 spinorial connection matrix (OP 18.1, Paper 17 [4]). *[Derived/Open]*

Section 4 provides a structural argument that IIB-II is algebraically preferred over IIB-I, upgrading OP 37.1 from *[Open]* to *[Conjecture, IIB-II structurally preferred]*.

Prerequisites. Papers 11, 13, 15, 17, 37, 38, 40, 41.
Notation. $P(r) = \sqrt{1-r^2}$, $z = 1 - r^2/r_s^2$, $t = \sqrt{1-r^2}$ near $r = r_s$. $\Omega_{\mathrm{irr}_3}(\omega) = \sqrt{\omega^2-5}/2$, $\kappa' = \sqrt{\kappa^2+5}$ for $\omega = -i\kappa$.

6. Conclusions and open problems

This paper has achieved three results within the programme of determining the boundary parameter β for the Black Circle gauge sector.

What has been established.
(1) *Structural reduction to* irr_3 (Theorem 1.1). The $\mathrm{BC}(0,\beta)$ problem reduces from 4×4 to the irr_3 sector alone. The gravitational singlet (irr_1) is free under IIB-II, and the Steinberg sector (St, the Majorana spinors 4_s) is outside the BC domain by construction. This reduction uses the structural input of Paper 41 [8] and is *[Derived, conditional on Paper 38 Conj. 5.1 (IIB-II $\wedge$ IIB-IV)]*.
(2) *Closed Γ-function connection matrix* (Proposition 2.7). The Gauss hypergeometric connection matrix K^{irr_3} is available in closed form for all $\omega \in \mathbb{C}$. The previously anticipated "Heun obstruction" does not exist for this sector (Remark 2.8). *[Derived]*
(3) *Closure of OP 42.1* (Lemma 2.2). The irr_3 sector uses the spinorial phase $\Phi_s(\omega) = \sqrt{2}\cdot 2^{i\omega/2}e^{i\omega}$, not the scalar phase of Paper 15 [3]. This is a consequence of the S_4-equivariance of $\mathcal{D}$ (Paper 37 §2 [5]), Schur's lemma, and the geometric origin of the Frobenius shift $\rho^{(1)}_+ = \frac{1}{2} + \frac{i\omega}{2}$ (Paper 37 Prop. 3.1 [5]). *[Derived, conditional on Paper 37 Prop. 3.1 and Paper 40 Thm. 4.3 [7]]*
(4) *Leading-order β-dependent condition* (Proposition 2.12). The explicit form of $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega;\beta)$ at leading Frobenius order is derived from the Robin boundary condition $\cos\beta\cdot \hat{F}^{\mathrm{irr}_3}_1(r_s) + \sin\beta\cdot \hat{F}^{\mathrm{irr}_3}_2(r_s) = 0$. *[Derived, conditional on Paper 37 Prop. 3.1 and Definition 2.5]*
(5) *IIB-II structurally preferred* (Theorem 4.1, Corollary 4.4). The trivial $\mathbb{F}_2[\mathrm{GL}(3,\mathbb{F}_2)]$-module structure of irr_1 provides an algebraic reason to prefer IIB-II over IIB-I. OP 37.1 is upgraded to *[Conjecture, IIB-II structurally preferred]*.

What remains open.
OP 18.1 *[Open, Paper 17 [4]]*. The complete form of $H^{\mathrm{irr}_3}_{\mathrm{BC}}(\omega;\beta)$ (equation (14)) requires the off-diagonal coefficient $C_1(\omega)$ of the full 2×2 spinorial connection matrix for the system (D1)–(D2) of Paper 37. Once $C_1(\omega)$ is determined, equation (18) becomes a fully explicit transcendental equation in β, computable numerically. The coefficient $C_1(\omega)$ requires the global integration of the coupled spinorial system (D1)–(D2) from $r = 0$ to $r = r_s$, which lies beyond the analytic tools developed in this series.

OP 37.4 / OP 39.1 residual. The energy-minimising value of β (equation (19)) and the corresponding spinorial QNM spectrum (equation (9)) remain *[Open, conditional on OP 18.1]*.
OP 41.2 residual *[Open].* The Cartan matrix of $\mathbb{F}_2[G_{84}]$ and the projective-injective property of St as a $\mathbb{F}_2[G_{84}]$-module (rather than a $\mathbb{F}_2[\mathrm{GL}(3, \mathbb{F}_2)]$-module) remain to be computed.

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