A CLIFFORD REPRESENTATION OF THE FANO ZERO-DIVISORS

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Abstract. We construct an explicit map $\rho\colon \mathrm{ZD}^*(4) \to \mathrm{Cliff}(0,7)$ from the 84 active zero-divisors of the sedenion algebra A_4 into the Clifford algebra $\mathrm{Cliff}(0,7) \cong M_8(\mathbb{R})$; see [7] for the general theory of Clifford algebras. The map is defined by $\rho(e_p+\sigma e_{8+k}) := L^{\mathcal{O}}_{e_p}$, where $L^{\mathcal{O}}_{e_p}$ denotes left-multiplication by e_p in the octonions $\mathcal{O} = A_3$. We prove three exact properties: (i) $\mathrm{GL}(3,\mathbb{F}_2)$-equivariance; (ii) anti-commutation $\{\rho(A),\rho(B)\} = 0$ for every ZD pair, via the *distinct ε-index theorem* (proved algebraically); and (iii) the Fano reflection identity $\rho(e_p)\rho(e_r)\rho(e_t) = \varepsilon_{p,r,t}\, R_{\{0,p,r,t\}}$ for every Fano line $\{p,r,t\}$. A remark records the precise relation between these results and a naïve "Cliff(8), Pauli-weight-4" formulation originally proposed. All results are verified computationally for $n = 4, 5, 6, 7$.

Contents

1. Introduction

The Sign-Parity Classification [1] established that every ZD pair $(A,B) \in \mathrm{ZD}(4)$ has the form $A = e_p + \sigma e_{8+k}$, $B = e_r + \tau e_{8+l}$ with indices satisfying the Fano-plane incidence structure. The present paper asks whether this combinatorial structure can be encoded as an algebraic representation in a Clifford algebra.

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The answer is yes, and the construction is unexpectedly clean: it uses nothing beyond the octonion left-multiplication operators $L^{\mathcal{O}}_{e_k}$, which are the standard generators of $\mathrm{Cliff}(0,7)$.

Notation. We write A_n for the Cayley–Dickson algebra of dimension 2^n over $\mathbb{R}$, with $A_3 = \mathcal{O}$ (octonions) and $A_4 = \mathbb{S}$ (sedenions). Basis elements are $e_0 = 1, e_1, \dots, e_{2^n-1}$, with

$$e_i \cdot e_j = \varepsilon_{ij}\, e_{i\oplus j}, \quad \varepsilon_{ij} \in \{+1, -1\},$$

where $\oplus$ denotes bitwise XOR; see [6] for the general theory of non-associative algebras. The Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$ is identified with $\mathbb{F}_2^3 \setminus \{0\} = \{1, \dots, 7\}$; its seven lines are $\{p, r, t\}$ with $p \oplus r \oplus t = 0$.

The *ε-index* of an active element $A = e_p + \sigma e_{8+k}$ is $p \in \{1, \dots, 7\}$, i.e. the index of its octonion component.

2. Structural Preliminaries

2.1. Active zero-divisors.

Definition 2.1. An element $A \in A_4$ is *active* if $A = e_p + \sigma e_{8+k}$ for some $p \in \{1, \dots, 7\}$, $k \in \{1, \dots, 7\}$, $\sigma \in \{+1, -1\}$, and there exists $B \neq 0$ with $AB = 0$. Write $\mathrm{ZD}^*(4)$ for the set of active elements.

From [1]: $|\mathrm{ZD}^*(4)| = 84$ and $|\mathrm{ZD}(4)| = 336$ ordered ZD pairs.

2.2. The distinct ε-index theorem.

The following theorem is the structural core of the construction.

Theorem 2.2 (Distinct ε-indices). *Let $(A, B) \in \mathrm{ZD}(n)$ with $A = e_p + \sigma_A e_{N+k}$ and $B = e_r + \sigma_B e_{N+l}$, where $N = 2^{n-1}$. Then $p \neq r$. Moreover, every ordered pair (p, r) with $p \neq r$ and $p, r \in \{1, \dots, N-1\}$ arises as the ε-indices of some ZD pair.*

Proof. We prove both assertions algebraically.

Distinct ε-indices. Suppose for contradiction that $p = r$.

Step 1 (XOR forces $k = l$). By the $\{0, 4, 8\}$ Trichotomy ([1], Theorem 3.6, Case B), $A \cdot B = 0$ requires

$$p \oplus (N+k) \oplus r \oplus (N+l) = 0.$$

Since $(N+k) \oplus (N+l) = k \oplus l$ (the leading bits 2^{n-1} cancel), this gives $p \oplus r = k \oplus l$. With $p = r$: $k \oplus l = 0$, hence $k = l$.

Step 2 (Direct product). With $p = r$ and $k = l$, using $e_p^2 = e_{N+k}^2 = -e_0$ and anti-commutativity $\varepsilon_{N+k,p} = -\varepsilon_{p,N+k}$:

$$\begin{aligned} A \cdot B &= e_p^2 + \sigma_B\, \varepsilon_{p,N+k}\, e_{p\oplus(N+k)} + \sigma_A\, \varepsilon_{N+k,p}\, e_{p\oplus(N+k)} + \sigma_A\sigma_B\, e_{N+k}^2 \\ &= -(1 + \sigma_A\sigma_B)\, e_0 + (\sigma_B - \sigma_A)\, \varepsilon_{p,N+k}\, e_{p\oplus(N+k)}. \end{aligned}$$

Step 3 (Contradiction). For $A \cdot B = 0$ both coefficients must vanish:

- e_0 coefficient: $1 + \sigma_A\sigma_B = 0 \Rightarrow \sigma_A\sigma_B = -1 \Rightarrow \sigma_A \neq \sigma_B$.
- $e_{p\oplus(N+k)}$ coefficient: $(\sigma_B - \sigma_A)\, \varepsilon_{p,N+k} = 0$. Since $\varepsilon_{p,N+k} \neq 0$: $\sigma_B = \sigma_A$.

These two conditions contradict each other, so $A \cdot B \neq 0$ whenever $p = r$.

All 42 pairs appear. Given any ordered pair (p, r) with $p \neq r$ and $p, r \in \{1, \dots, 7\}$, set $t := p \oplus r \in \{1, \dots, 7\}$ (nonzero since $p \neq r$). By the $\{0, 4, 8\}$ Trichotomy, any ZD pair with ε-indices (p, r) satisfies $k \oplus l = t$; pick any $k \in \{1, \dots, 7\}$ with $k \neq t$ and set $l := k \oplus t \in \{1, \dots, 7\}$. By the Sign-Parity Criterion ([1], Theorem 7.16), among the four $(\sigma_A, \sigma_B) \in \{+1, -1\}^2$ choices for the flat $\{p,\, N+k,\, r,\, N+l\}$, at least one has sign-parity $+1$, yielding a ZD pair $(e_p + \sigma_A e_{N+k},\, e_r + \sigma_B e_{N+l})$ with ε-indices exactly (p, r). ☐

Remark 2.3. The map $A \mapsto \varepsilon(A) = p$ separates every ZD pair: ZD adjacency implies $\varepsilon(A) \neq \varepsilon(B)$.

3. The Clifford Generators

The octonion left-multiplication operators are defined by

$$L^{\mathcal{O}}_{e_k} \colon \mathcal{O} \to \mathcal{O}, \quad x \mapsto e_k \cdot x.$$

As 8×8 real matrices they satisfy $\{L^{\mathcal{O}}_{e_i}, L^{\mathcal{O}}_{e_j}\} = -2\delta_{ij} I_8$ for all $i, j \in \{1, \dots, 7\}$, making them the seven generators of $\mathrm{Cliff}(0, 7) \cong M_8(\mathbb{R})$; see [7] for the general theory of Clifford algebras.

Lemma 3.1. *For every $i, j \in \{1, \dots, 7\}$ with $i \neq j$: $L^{\mathcal{O}}_{e_i} L^{\mathcal{O}}_{e_j} + L^{\mathcal{O}}_{e_j} L^{\mathcal{O}}_{e_i} = 0$.*

Proof. Immediate from $\{L_{e_i}, L_{e_j}\} = -2\delta_{ij} I_8$. ☐

Remark 3.2 (Non-associativity warning). Because the octonions are *not* associative, $L^{\mathcal{O}}_{e_i} \circ L^{\mathcal{O}}_{e_j} \neq L^{\mathcal{O}}_{e_i \cdot e_j}$ in general as matrix composition. The product $L_i L_j$ is a Clifford bivector of grade 2, not a grade-1 generator.

4. The Representation

Definition 4.1. Define $\rho \colon \mathrm{ZD}^*(4) \to \mathrm{Cliff}(0, 7) = M_8(\mathbb{R})$ by

$$\rho\big(e_p + \sigma e_{8+k}\big) := L^{\mathcal{O}}_{e_p}.$$

Remark 4.2. The map depends only on the ε-index p: it is constant on each of the 7 fibers $\{A \in \mathrm{ZD}^*(4) : \varepsilon(A) = p\}$.

5. Main Theorems

5.1. **Anti-commutation.**

Theorem 5.1 (Anti-commutation)**.** *For every ZD pair $(A, B) \in \mathrm{ZD}(4)$:*

$$\{\rho(A),\, \rho(B)\} = 0.$$

Proof. Let $A = e_p + \sigma e_{8+k}$ and $B = e_r + \tau e_{8+l}$. By Theorem 2.2, $p \neq r$. Therefore

$$\{\rho(A), \rho(B)\} = \{L^{\mathcal{O}}_{e_p}, L^{\mathcal{O}}_{e_r}\} = -2\delta_{pr}\, I_8 = 0.$$

☐

5.2. $\mathrm{GL}(3, \mathbb{F}_2)$**-equivariance.**

Theorem 5.2 ($\mathrm{GL}(3, \mathbb{F}_2)$-equivariance)**.** *For every $g \in \mathrm{GL}(3, \mathbb{F}_2)$ acting on $\{1, \dots, 7\} = \mathbb{F}_2^3 \setminus \{0\}$:*

$$\rho(g \cdot A) = g \cdot \rho(A),$$

where the right-hand action permutes the generators $\{L_{e_1}, \dots, L_{e_7}\}$ by index.

Proof. The ε-index satisfies $\varepsilon(g \cdot A) = g(\varepsilon(A))$ since g acts on the octonion component. Since $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7)$ is a subgroup of $G_2 = \mathrm{Aut}(\mathcal{O})$ (see [5], §4.1), every $g \in \mathrm{GL}(3, \mathbb{F}_2)$ acts on $\{e_1, \dots, e_7\}$ as an octonion automorphism. The induced map $\varphi(g) \colon L_{e_i} \mapsto L_{e_{g(i)}}$ satisfies $\{L_{e_{g(i)}}, L_{e_{g(j)}}\} = -2\delta_{g(i), g(j)} I_8 = -2\delta_{ij} I_8$, hence is an automorphism of $\mathrm{Cliff}(0, 7)$. The automorphism group $\mathrm{GL}(3, \mathbb{F}_2) \cong \mathrm{PSL}(2, 7)$ and its action on the Fano geometry are studied in detail in [2]. ☐

5.3. Fano reflection identity.

Definition 5.3. For a Fano line $\{p, r, t\}$ (with $p \oplus r \oplus t = 0$), the *Fano 4-flat* is $F = \{0, p, r, t\} \subset \{0, \dots, 7\}$. The *Fano reflection* is

$$R_F := 2P_F - I_8 = \operatorname{diag}\big(\chi_F(j)\big)_{j=0}^{7}, \quad \chi_F(j) = \begin{cases} +1 & j \in F, \\ -1 & j \notin F. \end{cases}$$

Note $R_F^2 = I_8$ (involution).

We need two preparatory lemmas.

Lemma 5.4 (ε-identity)**.** *For every Fano line $\{p, r, t\}$: $\varepsilon_{p,r,t} := \operatorname{sign}(e_p \cdot (e_r \cdot e_t)) = -\varepsilon_{rt}$.*

Proof. Since $r \oplus t = p$: $e_r \cdot e_t = \varepsilon_{rt}\, e_p$. Hence $e_p \cdot (e_r \cdot e_t) = \varepsilon_{rt}(e_p \cdot e_p) = \varepsilon_{rt} \cdot (-1) = -\varepsilon_{rt}$. □

Lemma 5.5 (Line sign consistency)**.** *For every Fano line $\{p, r, t\}$, let $\alpha := \varepsilon_{rt}$. Then $\varepsilon_{pr} = \varepsilon_{tp} = \alpha$.*

Proof. Wrap(p, r) with $p \oplus r = t$: $\varepsilon_{pr} \cdot \varepsilon_{t,r} = -1$. Since $\varepsilon_{t,r} = -\varepsilon_{rt} = -\alpha$: $\varepsilon_{pr}(-\alpha) = -1$, so $\varepsilon_{pr} = \alpha$. Wrap$(r, p)$ with $r \oplus p = t$: $\varepsilon_{rp} \cdot \varepsilon_{t,p} = -1$. Since $\varepsilon_{rp} = -\varepsilon_{pr} = -\alpha$: $(-\alpha)\varepsilon_{tp} = -1$, so $\varepsilon_{tp} = \alpha$. □

Theorem 5.6 (Fano Reflection Identity)**.** *For every Fano line $\{p, r, t\}$:*

$$L^{\mathcal{O}}_{e_p} L^{\mathcal{O}}_{e_r} L^{\mathcal{O}}_{e_t} \;=\; \varepsilon_{p,r,t}\, R_F \;=\; -\varepsilon_{rt}\, R_F,$$

where $F = \{0, p, r, t\}$.

Proof. Set $\alpha := \varepsilon_{rt}$, so $\varepsilon_{p,r,t} = -\alpha$ (Lemma 5.4).

Step 1 (Diagonal form). Since $p \oplus r \oplus t = 0$, the operator $G := L_{e_p} L_{e_r} L_{e_t}$ acts on each basis vector e_j as $e_p \cdot (e_r \cdot (e_t \cdot e_j)) = \lambda_j\, e_j$, where

$$\lambda_j \;=\; \varepsilon_{tj}\, \varepsilon_{r,\, t\oplus j}\, \varepsilon_{p,\, r\oplus t\oplus j}.$$

Hence G is diagonal.

Step 2 ($j \in F$, four sub-cases). Write $\alpha = \varepsilon_{pr} = \varepsilon_{tp} = \varepsilon_{rt}$ (Lemma 5.5). We show $\lambda_j = -\alpha$ for each $j \in \{0, p, r, t\}$.

- $j = 0$: $\lambda_0 = \varepsilon_{t,0} \cdot \varepsilon_{r,t} \cdot \varepsilon_{p,p} = 1 \cdot \alpha \cdot (-1) = -\alpha.\checkmark$
- $j = p$: $t \oplus p = r$ and $r \oplus t \oplus p = 0$ (Fano), so $\lambda_p = \varepsilon_{tp} \cdot \varepsilon_{r,r} \cdot \varepsilon_{p,0} = \alpha \cdot (-1) \cdot 1 = -\alpha.\checkmark$
- $j = r$: $t \oplus r = p$ and $r \oplus t \oplus r = t$ (Fano), so $\lambda_r = \varepsilon_{tr} \cdot \varepsilon_{r,p} \cdot \varepsilon_{p,t}$. Now $\varepsilon_{tr} = -\alpha$, $\varepsilon_{rp} = -\alpha$ (anti-commutativity), and Wrap(p, t) with $p \oplus t = r$ gives $\varepsilon_{pt} \cdot \varepsilon_{rt} = -1$, so $\varepsilon_{pt} = -\alpha$. Hence $\lambda_r = (-\alpha)(-\alpha)(-\alpha) = -\alpha^3 = -\alpha.\checkmark$
- $j = t$: $t \oplus t = 0$ and $r \oplus t \oplus t = r$, so $\lambda_t = \varepsilon_{tt} \cdot \varepsilon_{r,0} \cdot \varepsilon_{p,r} = (-1) \cdot 1 \cdot \alpha = -\alpha.\checkmark$

Thus $\lambda_j = -\alpha = \varepsilon_{p,r,t}$ for all $j \in F$.

Step 3 (Lo-flat: $j \notin F$). For $j \in \{1, \dots, 7\} \setminus \{p, r, t\}$, the indices j, p, r, t are all distinct and all lie in $\{1, \dots, 7\}$, so L_j anti-commutes with each of L_p, L_r, L_t:

$$L_j\, G = L_j L_p L_r L_t = (-L_p L_j) L_r L_t = L_p L_r L_j L_t = -L_p L_r L_t L_j = -G\, L_j.$$

Hence $\{L_j, G\} = 0$. Since $G = \operatorname{diag}(\lambda_0, \dots, \lambda_7)$, this gives $\lambda_{j\oplus l} = -\lambda_l$ for every l. Setting $l = 0$: $\lambda_j = -\lambda_0 = -(-\alpha) = \alpha = -\varepsilon_{p,r,t}.\checkmark$

Conclusion. G has eigenvalue $-\alpha = \varepsilon_{p,r,t}$ on $\operatorname{span}(F)$ and eigenvalue $+\alpha = -\varepsilon_{p,r,t}$ on the lo-flat, so $G = \varepsilon_{p,r,t}\, R_F$. □

Corollary 5.7. *$\Gamma_{prt} := L_{e_p} L_{e_r} L_{e_t}$ is an involution: $\Gamma_{prt}^2 = I_8$.*

Proof. $\Gamma_{prt}^2 = \varepsilon_{p,r,t}^2\, R_F^2 = I_8$. □

6. Comparison with the Original Desiderata

Remark 6.1 (Comparison with the Cliff(8) formulation). The construction was originally framed as a map $\rho\colon \mathrm{ZD}^*(4) \to \mathrm{Cliff}(8) = M_{16}(\mathbb{R})$ satisfying four properties: (a) $\mathrm{GL}(3, \mathbb{F}_2)$-equivariance, (b) "Pauli weight = 4", (c) $\{\rho(A), \rho(B)\} = 0$ for ZD pairs, and (d) $\rho(e_A)\rho(e_B)\rho(e_C) = -I_{16}$ for covering triples. The construction above differs in the following exact ways.

Target algebra. Our ρ maps into $\mathrm{Cliff}(0, 7) = M_8(\mathbb{R})$ (7 generators). The proposed $\mathrm{Cliff}(8) = M_{16}(\mathbb{R})$ (8 generators) admits a natural embedding $M_8 \hookrightarrow M_{16}$ via $M \mapsto M \otimes I_2$, but this adds no structure relevant to the zero-divisor problem. $\mathrm{Cliff}(0, 7)$ is the correct and minimal target.

Pauli weight. In $\mathrm{Cliff}(0, 7)$ acting on $(\mathbb{R}^2)^{\otimes 3}$ (3 qubits), the generators L_{e_k} have Pauli-decomposition weight at most 3. The condition "weight = 4" was predicated on a 4-qubit target and is not a natural invariant of the zero-divisor structure.

Triple product identity. The property $\rho(e_A)\rho(e_B)\rho(e_C) = -I_{16}$ was overly strong. The correct identity (Theorem 5.6) gives $\Gamma_{prt} = \varepsilon_{p,r,t}\, R_F$, a *Fano reflection*, not a scalar multiple of the identity. Since $\pi_A\pi_B\pi_C = -1$ for every 2-flat ([1], Theorem 7.18), this encodes exactly the sign constraint of the Sign-Parity framework.

The exact properties of ρ are summarised in Table 1.

	Property	Original formulation	Exact result
(1)	Equivariance	$\mathrm{GL}(3, \mathbb{F}_2)$-equivariant	**proved** (same)
(2)	Triple product	$= -I_{16}$	$= \varepsilon_{prt}\, R_F$ (Fano reflection)
(3)	Anti-commutation	$\{\rho(A), \rho(B)\} = 0$ for ZD pairs	**proved** (same)
(4)	Pauli weight	$= 4$ in Cliff(8)	**dropped** (wrong target)

Table 1. Original desiderata vs. exact results.

6.1. **Connection to Paper 5 stabilizer codes.**

Remark 6.2 (Pauli decomposition and Fano duality). **Pauli decomposition.** Each $L^{\mathcal{O}}_{e_k} \in M_8(\mathbb{R})$, viewed as an operator on 3 qubits ($\dim = 2^3 = 8$), decomposes as

$$L^{\mathcal{O}}_{e_k} = \frac{-i}{2}\sum_{j=1}^{4} s_j^{(k)}\, P_j^{(k)}, \quad s_j^{(k)} \in \{+1, -1\},$$

with all four coefficients of equal magnitude $\frac{1}{2}$. This is forced by the Frobenius identity $\|L^{\mathcal{O}}_{e_k}\|_F^2 = 8$ (since $L^{\mathcal{O}}_{e_k}$ is an isometry of $\mathbb{R}^8$, hence orthogonal: $(L^{\mathcal{O}}_{e_k})^\top L^{\mathcal{O}}_{e_k} = I_8$, giving $\mathrm{Tr}(I_8) = 8$), which requires $\sum_P |c_{k,P}|^2 = 1 = 4 \cdot \frac{1}{4}$.

The explicit decompositions are

$$\begin{aligned}
L^{\mathcal{O}}_{e_1} &= \tfrac{-i}{2}(\mathtt{IIY}+\mathtt{IZY}+\mathtt{ZIY}-\mathtt{ZZY}),\\
L^{\mathcal{O}}_{e_2} &= \tfrac{-i}{2}(\mathtt{IYI}+\mathtt{IYZ}-\mathtt{ZYI}+\mathtt{ZYZ}),\\
L^{\mathcal{O}}_{e_3} &= \tfrac{-i}{2}(\mathtt{IXY}+\mathtt{IYX}-\mathtt{ZXY}+\mathtt{ZYX}),\\
L^{\mathcal{O}}_{e_4} &= \tfrac{-i}{2}(-\mathtt{YII}+\mathtt{YIZ}+\mathtt{YZI}+\mathtt{YZZ}),\\
L^{\mathcal{O}}_{e_5} &= \tfrac{-i}{2}(-\mathtt{XIY}+\mathtt{XZY}+\mathtt{YIX}+\mathtt{YZX}),\\
L^{\mathcal{O}}_{e_6} &= \tfrac{-i}{2}(\mathtt{XYI}-\mathtt{XYZ}+\mathtt{YXI}+\mathtt{YXZ}),\\
L^{\mathcal{O}}_{e_7} &= \tfrac{-i}{2}(\mathtt{XXY}-\mathtt{XYX}+\mathtt{YXX}-\mathtt{YYY}).
\end{aligned}$$

The X-part encodes the Fano points. For every $k \in \{1,\dots,7\}$, all four Pauli strings in the decomposition of $L^{\mathcal{O}}_{e_k}$ share the same X-part (positions where X or Y appears), and this X-part equals the binary representation of k in $\mathbb{F}_2^3$:

$$\text{x-part}\big(L^{\mathcal{O}}_{e_k}\big) \;=\; \text{bin}(k) \;\in\; \mathbb{F}_2^3 \setminus \{0\}.$$

This gives a canonical bijection $e_k \leftrightarrow \text{bin}(k)$ between the 7 octonion generators and the 7 points of $\mathrm{PG}(2,\mathbb{F}_2)$.

The 8 anti-commuting single-Pauli selections. An exhaustive search over all $4^7 = 16\,384$ ways of selecting one Pauli string per generator from its four components shows that exactly 8 selections satisfy $\{\tilde\rho(e_i),\tilde\rho(e_j)\} = 0$ for all 21 pairs $i \neq j$. These 8 selections share the same fixed X-parts and differ only in which Z-correction is chosen from the 4-element coset $\{I,Z\}^{\otimes 3}$.

Triple products and Fano duality. For every one of the 8 valid selections, and for every Fano line $\{p,r,t\}$, the triple product equals

$$\tilde\rho(e_p)\,\tilde\rho(e_r)\,\tilde\rho(e_t) \;=\; \phi_{p,r,t}\cdot Q_{\{p,r,t\}}, \qquad \phi_{p,r,t} \in \{+i,-i\},$$

where $Q_{\{p,r,t\}} \in \{I,Z\}^{\otimes 3}$ is a specific non-identity Z-type Pauli string, the same for all 8 selections. The assignment $\{p,r,t\} \mapsto Q_{\{p,r,t\}}$ is a bijection from the 7 Fano lines to the 7 nonzero elements of $\{I,Z\}^{\otimes 3} \cong \mathbb{F}_2^3 \setminus \{0\}$, and coincides with the *projective duality* of $\mathrm{PG}(2,\mathbb{F}_2)$: the dual (in the combinatorial sense) of each Fano line.

The twisted group algebra is the octonions. Define the cocycle $c_{\text{oct}}(\text{bin}(i),\text{bin}(j)) := \varepsilon_{ij}$ (the octonion sign table of [1]). The twisted group algebra $\mathbb{C}[(\mathbb{Z}/2)^3]^{c_{\text{oct}}}$ with multiplication $u_\alpha \cdot u_\beta = c_{\text{oct}}(\alpha,\beta)\,u_{\alpha\oplus\beta}$ is non-associative and its left-regular representation sends $u_{\text{bin}(k)} \mapsto L^{\mathcal{O}}_{e_k}$. This algebra is therefore isomorphic to the octonions $\mathcal{O}$ themselves: the twisted group algebra construction recovers the original representation, not a new one.

The cocycle condition fails on non-Fano triples. The map c_{oct} is *not* a group 2-cocycle: the cocycle identity $c(\alpha,\beta)\,c(\alpha\oplus\beta,\gamma) = c(\alpha,\beta\oplus\gamma)\,c(\beta,\gamma)$ fails for exactly the 28 non-collinear triples in $\mathrm{PG}(2,\mathbb{F}_2)$. The 7 Fano lines satisfy it — these are precisely the triples for which both parenthesisations of the triple product agree: $(u_p\cdot u_r)\cdot u_t = u_p\cdot(u_r\cdot u_t) = \varepsilon_{p,r,t}\cdot u_0$. The failure on non-Fano triples is the octonion associator.

The fundamental obstruction. Every Pauli string squares to $+I$, while $L^2_{e_k} = -I$ in $\mathrm{Cliff}(0,7)$. This group-theoretic mismatch is the precise obstruction: no assignment of single Pauli strings to the generators of $\mathcal{O}$ can reproduce the octonion multiplication, even projectively.

Open problem (Paper 8). The Fano duality $\{p,r,t\} \mapsto Q_{\{p,r,t\}}$ identified above is the duality between X-type and Z-type stabilizers in the CSS codes of [4]. Establishing this correspondence as a theorem — and determining whether the non-Fano failure of the cocycle condition has a code-theoretic interpretation — is the natural problem for a subsequent paper.

7. Discussion

7.1. **Geometric meaning of the Fano reflection.** The operator $R_F = 2P_F - I_8$ acting on $\mathcal{O} = \mathbb{R}^8$ is the reflection fixing the Fano 4-flat $F = \{0, p, r, t\}$ and negating its orthogonal complement (the lo-flat). The sub-algebra $\mathbb{H}_{prt} := \operatorname{span}\{e_0, e_p, e_r, e_t\} \cong \mathbb{H}$ (quaternions) is a Cayley–Dickson subalgebra of $\mathcal{O}$, and R_F is its grading involution: the $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathcal{O}$ with even part $\mathbb{H}_{prt}$ and odd part its orthocomplement. Thus $\Gamma_{prt} = L_{e_p} L_{e_r} L_{e_t}$ is the *chirality element* of the quaternionic subalgebra associated to the Fano line $\{p, r, t\}$.

7.2. **Connection to the sign-parity framework.** The Fano reflection identity provides a Clifford-algebraic realisation of the sign-parity discriminant. The octonion triple sign $\varepsilon_{p,r,t}$ coincides with $\pi(F, \text{split})$ for the corresponding 2-flat in A_4 ([1], Definition 7.2), bridging the sign-parity classification of Paper 1 with the present Clifford representation. The ZD zeta function of [3], whose poles encode the four division-algebra thresholds, acquires a Clifford-algebraic interpretation through this bridge: the residue at each pole corresponds to the count of Fano triples acting via Γ_{prt}.

7.3. **Verified extension to $n \le 7$.** The construction $\rho(A) = L^{\mathcal{O}}_{\varepsilon(A)}$ and all three main theorems extend to all $n \ge 4$, with ρ defined on the EF-type active elements of $\mathrm{ZD}^*(n)$.

Theorem 7.1 (Extension to $n \le 7$). *For every $n \in \{4, 5, 6, 7\}$:*

(1) All EF-type ZD pairs of A_n have distinct ε-indices (Theorem 2.2).
(2) $\{\rho(A), \rho(B)\} = 0$ for all EF-type ZD pairs (Theorem 5.1).
(3) $L_{e_p} L_{e_r} L_{e_t} = \varepsilon_{p,r,t} R_F$ for all 7 Fano lines (Theorem 5.6).

Proof. Items (2) and (3) concern only the A_3 structure and are independent of n. Item (1) is verified computationally for $n \le 7$ (Appendix A). The algebraic proof for all n follows by induction on the hereditary decomposition ([1], Theorem 8.1): every EF-type ZD pair in A_{n+1} arises from an EF-type ZD pair in A_n under the Cayley–Dickson embedding, which preserves the ε-index structure. □

Appendix A. Numerical Verification

All theorems were verified computationally for $n = 4, 5, 6, 7$ using Python/NumPy on an AMD Ryzen Threadripper 2950X (16 physical cores). GL(3,$\mathbb{F}_2$)-equivariance (Theorem 5.2) was additionally verified in GAP 4.15.

ZD counts by block type. Active elements of A_n are classified as EF (one component in the E-block $\{1, \ldots, 2^{n-1} - 1\}$, one in the F-block $\{2^{n-1}, \ldots, 2^n - 1\}$), EE (both in E), or FF (both in F). EE and FF pairs are zero-divisors inherited from the embedded copy of A_{n-1} ([1], Theorem 8.1); the map ρ is defined on EF-type elements.

Theorem 2.2 (distinct ε-indices): 0 failures among EF-type ZD pairs at every n.

Theorem 5.6 (Fano Reflection Identity): $\|L_{e_p} L_{e_r} L_{e_t} - \varepsilon_{p,r,t} R_F\|_{\max} = 0$ for all 7 Fano lines, verified independently of n.

Theorem 5.2 ($\mathrm{GL}(3, \mathbb{F}_2)$-equivariance): verified in GAP 4.15 by constructing the permutation group

$$G = \langle \sigma_1, \sigma_2 \rangle \subset \mathrm{Sym}(\{1, \ldots, 7\})$$

with $\sigma_1 = (2\ 6)(3\ 7)$ and $\sigma_2 = (1\ 4\ 2)(3\ 5\ 6)$. GAP confirms $|G| = 168$, $G \cong \mathrm{PSL}(2, 7)$, G is 2-transitive on $\{1, \ldots, 7\}$, and all 7 Fano lines are preserved by all 168 elements of G.

n	dim	EF	EE	FF	Cross	Total	Expected	Time
4	16	336	—	—	—	336	336	<0.1 s
5	32	3,696	336	336	672	5,040	5,040	<1 s
6	64	31,920	5,040	5,040	10,080	52,080	52,080	1 s
7	128	260,400	52,080	52,080	104,160	468,720	468,720	58 s

Table 2. ZD pair counts by block type for A_n, $n = 4, \ldots, 7$. EF: one index in each block (new pairs); EE/FF: both indices in the same block (inherited from A_{n-1}); Cross: pairs involving one inherited and one new EF element (= EE+FF). Expected total: $|\mathrm{ZD}(n)| = 336\binom{n-1}{3}_2$.

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