FANO DUALITY, PROJECTIVE ORTHOGONALITY, AND THE CSS BRIDGE

GIOVANNI LEVRATTI

ABSTRACT. The companion paper [5] constructed an explicit map $\rho\colon \mathrm{ZD}^*(4) \to \mathrm{Cliff}(0,7)$ and observed, via exhaustive computation, that the triple-product identity $\tilde{\rho}(e_p)\tilde{\rho}(e_r)\tilde{\rho}(e_t) = \varphi_{p,r,t} \cdot Q_{\{p,r,t\}}$ defines a bijection from the 7 Fano lines to the 7 non-zero elements of $\{I, Z\}^{\otimes 3}$, which coincides with the projective duality of $\mathrm{PG}(2, \mathbb{F}_2)$. The paper left open whether this duality is the X/Z duality of the Steane $[[7, 1, 3]]$ code studied in [4].

We close this gap, prove the analogous result for $n = 5$ (Theorem 5.1), and establish that $M = I_{n-1}$ for all $n \geq 2$ under the MSB-first generator convention (Proposition 5.2). The main theorem (Theorem 3.1) states that the map $\{p, r, t\} \mapsto Q_{\{p,r,t\}}$ equals the projective orthogonal complement in $\mathbb{F}_2^3$, and that $Q_{\{p,r,t\}}$ is exactly the Z-stabilizer of the Steane code associated to the dual point. The proof proceeds in three steps: (1) an explicit verification that $Q_{\{p,r,t\}} = Z^{\otimes \mathrm{bin}^{-1}(\{p,r,t\}^{\perp})}$ for all seven Fano lines; (2) identification of the dual point with the unique Z-stabilizer support in the Steane code; (3) a $\mathrm{GL}(3, \mathbb{F}_2)$-equivariance argument showing the correspondence is canonical.

As a corollary, the sign-parity constraint $\pi_A \pi_B \pi_C = -1$ (Theorem 7.18 of [1]) acquires a unified interpretation: it is simultaneously the Clifford phase of a balanced ZD triple ([4], Theorem 4.2) and the projective orthogonality relation between Fano lines and their dual points in $\mathrm{PG}(2, \mathbb{F}_2)$.

1. INTRODUCTION

The Cayley–Dickson tower $A_0 = \mathbb{R} \subset A_1 = \mathbb{C} \subset A_2 = \mathbb{H} \subset A_3 = \mathbb{O} \subset A_4 = \mathbb{S} \subset \cdots$ has been the subject of a series of papers establishing a complete classification of zero-divisors. The counting formula

$$|\mathrm{ZD}(n)| = 336 \cdot \binom{n-1}{3}_2, \qquad n \geq 4, \tag{1}$$

was proved in [1], and its generating function was identified as a Weil zeta function in [3]. The active stabiliser construction of [4] established bijections between zero-divisor structure and stabiliser codes. The Clifford representation of [5] constructed an explicit map $\rho\colon \mathrm{ZD}^*(4) \to \mathrm{Cliff}(0,7)$ and computed, for every Fano line $\{p, r, t\}$, the triple product

$$\tilde{\rho}(e_p)\tilde{\rho}(e_r)\tilde{\rho}(e_t) = \varphi_{p,r,t} \cdot Q_{\{p,r,t\}}, \quad \varphi_{p,r,t} \in \{+i, -i\}, \tag{2}$$

where $Q_{\{p,r,t\}} \in \{I, Z\}^{\otimes 3}$ is a non-identity Z-type Pauli string, the same for all eight valid single-Pauli selections. [5] observed that $\{p, r, t\} \mapsto Q_{\{p,r,t\}}$ is a bijection from Fano lines to $\{I, Z\}^{\otimes 3} \setminus \{I^{\otimes 3}\}$, consistent with projective duality, and stated:

> *"The Fano duality $\{p, r, t\} \mapsto Q_{\{p,r,t\}}$ identified above is the duality between X-type and Z-type stabilizers in the CSS codes of* [4]*. Establishing this correspondence as a theorem ... is the natural problem for a subsequent paper."*

Date: May 29, 2026.
2020 *Mathematics Subject Classification.* 81P70, 94B05, 17A35, 51E20.

The present paper establishes this as Theorem 3.1 ($n = 4$) and proves the analogue for $n = 5$ (Theorem 5.1), along with a general result that $M = I_{n-1}$ for all $n \geq 2$ under the MSB-first generator convention (Proposition 5.2).

Overview of the argument. Let $\mathrm{bin}\colon \{1, \ldots, 7\} \xrightarrow{\sim} \mathbb{F}_2^3 \setminus \{0\}$ be the standard bijection $k \mapsto$ binary representation of k. For a set $S \subset \mathbb{F}_2^3$ we write $S^\perp$ for its orthogonal complement under the standard inner product on $\mathbb{F}_2^3$.

Definition 1.1 (Dual Pauli string)**.** For a Fano line $L = \{p, r, t\} \subset \{1, \ldots, 7\}$ (i.e. $\mathrm{bin}(p) \oplus \mathrm{bin}(r) \oplus \mathrm{bin}(t) = 0$), the *dual Z-string* is

$$Z_L := Z^{\otimes \mathrm{bin}^{-1}(v)}, \qquad v := \mathrm{span}_{\mathbb{F}_2}\{\mathrm{bin}(p), \mathrm{bin}(r), \mathrm{bin}(t)\}^\perp \cap (\mathbb{F}_2^3 \setminus \{0\}),$$

i.e. $v \in \mathbb{F}_2^3 \setminus \{0\}$ is the unique non-zero vector orthogonal to all three binary representations. Here $Z^{\otimes w}$ for $w = (w_1, w_2, w_3) \in \mathbb{F}_2^3$ denotes the Pauli string $\bigotimes_{j=1}^3 Z^{w_j}$.

Remark 1.2. Since $\mathrm{bin}(p), \mathrm{bin}(r), \mathrm{bin}(t) \in \mathbb{F}_2^3$ are the three non-zero vertices of a Fano line, they span a 2-dimensional subspace $W \subset \mathbb{F}_2^3$, and $W^\perp$ is 1-dimensional (as $\dim \mathbb{F}_2^3 = 3$), so v is uniquely determined. There are exactly seven Fano lines and seven non-zero vectors, so $L \mapsto v$ is a bijection.

The main theorem asserts two things: that $Q_{\{p,r,t\}} = Z_L$ (the computational duality of [5] equals projective orthogonality), and that Z_L is the Z-stabiliser of the Steane code associated to the dual point.

2. Background

We recall the notation used throughout.

2.1. **Fano plane and lo-flats.** The Fano plane $\mathrm{PG}(2, \mathbb{F}_2)$ has seven points $\{1, \ldots, 7\} = \mathbb{F}_2^3 \setminus \{0\}$ and seven lines. Under the bijection bin, a Fano line is a set $\{p, r, t\}$ with $\mathrm{bin}(p) + \mathrm{bin}(r) + \mathrm{bin}(t) = 0 \in \mathbb{F}_2^3$.

A *lo-flat* of ZD(4) is a weight-4 subset $F = \{a, b, c, d\} \subset \{1, \ldots, 7\}$ with $a \oplus b \oplus c \oplus d = 0$. There are exactly seven lo-flats; they are the complements of the seven Fano lines [3, Theorem 3.2].

2.2. **Steane code and stabilisers.** The Steane $[[7, 1, 3]]$ code is $\mathrm{CSS}(C, C)$ [7] with $C = [7, 4, 3]$ the binary Hamming code. Its X-stabiliser group $S_X \cong (\mathbb{Z}/2\mathbb{Z})^3$ is generated by g_1^X, g_2^X, g_3^X with supports equal to the rows of the parity-check matrix $H_{7,4,3}$:

$$\begin{aligned} g_1^X &= X_1 X_3 X_5 X_7, \\ g_2^X &= X_2 X_3 X_6 X_7, \\ g_3^X &= X_4 X_5 X_6 X_7. \end{aligned}$$

The Z-stabilisers g_1^Z, g_2^Z, g_3^Z have the same supports. The seven non-identity elements of S_X (resp. S_Z) have supports exactly equal to the seven lo-flats [4, Theorem 3.1].

2.3. **The map $\tilde{\rho}$ and Pauli decompositions.** The map ρ of [5] is defined by $\rho\colon \mathrm{ZD}^*(4) \to \mathrm{Cliff}(0, 7)$ by $\rho(e_p + \sigma e_{8+k}) := L_{e_p}^{\mathbb{O}}$, the left-multiplication operator of e_p in the octonions. The Clifford generators $L_{e_k}^{\mathbb{O}}$ decompose as

$$L_{e_k}^{\mathbb{O}} = \frac{-i}{2} \sum_{j=1}^{4} s_j^{(k)} P_j^{(k)},$$

with all four coefficients of equal magnitude $\frac{1}{2}$. The map $\tilde{\rho}$ selects one Pauli string per generator from its four-element Pauli coset; there are exactly eight valid selections satisfying $\{\tilde{\rho}(e_i), \tilde{\rho}(e_j)\} = 0$ for all $i \neq j$.

For every valid selection and every Fano line $\{p, r, t\}$:

$$\tilde{\rho}(e_p)\tilde{\rho}(e_r)\tilde{\rho}(e_t) = \varphi_{p,r,t} \cdot Q_{\{p,r,t\}}, \tag{3}$$

where $Q_{\{p,r,t\}} \in \{I, Z\}^{\otimes 3}$ is independent of the selection [5, Remark 6.2].

3. Main Theorem

Theorem 3.1 (CSS Bridge). *Let $\{p, r, t\} \subset \{1, \ldots, 7\}$ be any Fano line and let $Q_{\{p,r,t\}} \in \{I, Z\}^{\otimes 3}$ be as in* (3)*. Then:*

(i) (Projective orthogonality) $Q_{\{p,r,t\}} = Z^{\otimes v}$, *where $v \in \mathbb{F}_2^3 \setminus \{0\}$ is the unique vector orthogonal to* $\mathrm{bin}(p), \mathrm{bin}(r), \mathrm{bin}(t)$, *i.e. $Q_{\{p,r,t\}} = Z_L$ in the notation of Definition 1.1.*

(ii) (CSS identification) *Let $M \in \mathrm{GL}(3, \mathbb{F}_2)$ be the bit-reversal matrix $M = \left(\begin{smallmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{smallmatrix}\right)$. The non-identity Z-stabiliser of the Steane* $[[7, 1, 3]]$ *code* [6] *whose support equals the lo-flat $\{1, \ldots, 7\} \setminus \{p, r, t\}$ corresponds, under the generator-combination labelling of $S_Z \cong (\mathbb{Z}/2\mathbb{Z})^3$, to the vector $Mv_L \in \mathbb{F}_2^3 \setminus \{0\}$. Equivalently, $S_Z(Mv_L) = \{1, \ldots, 7\} \setminus \{p, r, t\}$, where $S_Z(w)$ denotes the support of the Z-stabiliser labelled w.*

(iii) (Contragredient equivariance) *For every $g \in \mathrm{GL}(3, \mathbb{F}_2)$ and every Fano line $L = \{p, r, t\}$:*

$$Q_{g(L)} = Z^{\otimes g^{-T} v_L},$$

where $g(L) = \{g(p), g(r), g(t)\}$ and $g^{-T} = (g^{-1})^T$. In other words, $L \mapsto Q_L$ is equivariant for the contragredient *(dual) action of* $\mathrm{GL}(3, \mathbb{F}_2)$ *on* $\mathbb{F}_2^3$.

Proof. We prove parts (i)–(iii) in order.

Part (i): Projective orthogonality.

The seven Fano lines and the explicit Q-values from [5, Remark 6.2] are given in Table 1. For each line $L = \{p, r, t\}$, we compute $W_L := \mathrm{span}_{\mathbb{F}_2}\{\mathrm{bin}(p), \mathrm{bin}(r), \mathrm{bin}(t)\}$ and its orthogonal complement $W_L^{\perp} \subset \mathbb{F}_2^3$. Since $\mathrm{bin}(p) + \mathrm{bin}(r) + \mathrm{bin}(t) = 0$, the three vectors span a 2-dimensional subspace, so $W_L^{\perp}$ is 1-dimensional. Let v_L be its unique non-zero element.

Table 1. Fano duality: explicit verification for all seven Fano lines. Columns: Fano line; projective orthogonal v_L; dual index $q = \mathrm{bin}^{-1}(v_L)$; Z-string $Z_L = Q_{\{p,r,t\}}$ (Part i); stabiliser label Mv_L (Part ii); lo-flat $= S_Z(Mv_L)$.

Line $\{p, r, t\}$	v_L	q	$Q_{\{p,r,t\}}$	Mv_L	$S_Z(Mv_L)$
$\{1,2,3\}$	$(1,0,0)^T$	4	ZII	$(0,0,1)^T$	$\{4,5,6,7\}$
$\{1,4,5\}$	$(0,1,0)^T$	2	IZI	$(0,1,0)^T$	$\{2,3,6,7\}$
$\{1,6,7\}$	$(1,1,0)^T$	6	ZZI	$(0,1,1)^T$	$\{2,3,4,5\}$
$\{2,4,6\}$	$(0,0,1)^T$	1	IIZ	$(1,0,0)^T$	$\{1,3,5,7\}$
$\{2,5,7\}$	$(1,0,1)^T$	5	ZIZ	$(1,0,1)^T$	$\{1,3,4,6\}$
$\{3,4,7\}$	$(0,1,1)^T$	3	IZZ	$(1,1,0)^T$	$\{1,2,5,6\}$
$\{3,5,6\}$	$(1,1,1)^T$	7	ZZZ	$(1,1,1)^T$	$\{1,2,4,7\}$

Inspection of Table 1 shows that in every row, $Q_{\{p,r,t\}} = Z^{\otimes v_L}$: the Z-factors appear in exactly the bit-positions where v_L is 1. This is an explicit verification for all seven Fano lines; the bijectivity of $L \mapsto v_L$ (projective duality of $\mathrm{PG}(2, \mathbb{F}_2)$) is a standard fact.

Part (ii): CSS identification.

Let $M = \left(\begin{smallmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{smallmatrix}\right) \in \mathrm{GL}(3, \mathbb{F}_2)$ be the bit-reversal matrix ($M^2 = I$, $\det M = 1$).

The Steane Z-generators have supports $\mathrm{supp}(g_1^Z) = \{1, 3, 5, 7\}$, $\mathrm{supp}(g_2^Z) = \{2, 3, 6, 7\}$, $\mathrm{supp}(g_3^Z) = \{4, 5, 6, 7\}$. For $w = (w_1, w_2, w_3) \in \mathbb{F}_2^3 \setminus \{0\}$, write $S_Z(w)$ for the support of the corresponding non-identity element of $S_Z \cong (\mathbb{Z}/2\mathbb{Z})^3$:

$$S_Z(w) := \bigoplus_{j:\, w_j = 1} \mathrm{supp}(g_j^Z) \quad \text{(symmetric difference)}.$$

By [4, Theorem 3.1], $\{S_Z(w) : w \neq 0\}$ is exactly the set of seven lo-flats.

Claim: $S_Z(Mv_L) = \{1, \ldots, 7\} \setminus \{p, r, t\}$ for every Fano line $\{p, r, t\}$.

Proof of claim: exhaustive verification over all seven Fano lines, recorded in Table 1. Abstractly, this is a finite check that the composition $L \overset{\perp}{\longmapsto} v_L \overset{M}{\longmapsto} Mv_L \overset{S_Z}{\longmapsto} S_Z(Mv_L)$ equals $\{1, \ldots, 7\} \setminus L$ for all seven lines L.

The matrix M is the unique element of $\mathrm{GL}(3, \mathbb{F}_2)$ that reconciles the bit-ordering convention of bin (MSB first) with the generator-index ordering of the parity-check matrix $H_{7,4,3}$.

Part (iii): Contragredient equivariance.

The orthogonal complement map $W \mapsto W^\perp$ in $\mathbb{F}_2^3$ satisfies $(gW)^\perp = g^{-T}(W^\perp)$ for all $g \in \mathrm{GL}(3, \mathbb{F}_2)$ and all subspaces W. Indeed: $v \in (gW)^\perp \Leftrightarrow \langle v, gw \rangle = 0$ for all $w \in W \Leftrightarrow \langle g^T v, w \rangle = 0$ for all $w \in W \Leftrightarrow g^T v \in W^\perp \Leftrightarrow v \in g^{-T}(W^\perp)$.

For a Fano line $L = \{p, r, t\}$, this gives $v_{g(L)} = g^{-T} v_L$ and hence $Q_{g(L)} = Z^{\otimes v_{g(L)}} = Z^{\otimes g^{-T} v_L}$.

Verified exhaustively for the three generators $\{\mathrm{swap}(0, 1), \mathrm{swap}(1, 2), \mathrm{add}(0 \leftarrow 1)\}$ of $\mathrm{GL}(3, \mathbb{F}_2)$, over all seven Fano lines by direct computation.

Remark 3.2 (Contragredient vs. covariant action)**.** The contragredient action is intrinsic to projective duality: under $\mathrm{GL}(3, \mathbb{F}_2)$, points transform covariantly and hyperplanes (lines in $\mathrm{PG}(2, \mathbb{F}_2)$) transform contravariantly. That Q_L transforms via g^{-T} confirms it encodes the Z-type (dual) information to the X-type line L.

□

4. Corollaries

Corollary 4.1 (Unified interpretation of $\pi_A \pi_B \pi_C = -1$)**.** *For every 2-flat of A_4, the sign-parity constraint $\pi_A \pi_B \pi_C = -1$* (Theorem 7.18 of [1]) *admits three equivalent interpretations:*

(a) (Sign-parity, algebraic): *Exactly one of the three splits of every 2-flat is inactive ($\pi = -1$) and two are active ($\pi = +1$).*

(b) (Clifford, [4]): *The product of three balanced half-operators from the three splits carries global phase -1 and produces $-Y_\alpha Y_\beta$.*

(c) (Projective, Theorem 3.1): *The product $\tilde\rho(e_p)\tilde\rho(e_r)\tilde\rho(e_t)$ equals $\varphi \cdot Z_L$, where Z_L is the projective dual of the Fano line $\{p, r, t\}$; the phase $\varphi \in \{+i, -i\}$ records the residual sign, and Z_L records the geometric obstruction to associativity.*

Proof. Part (a) is [1, Theorem 7.3 and 7.18]. Part (b) is [4, Theorem 4.2]. Part (c) follows from Theorem 3.1(i) and (3): the triple product $\tilde\rho(e_p)\tilde\rho(e_r)\tilde\rho(e_t)$ is $\varphi \cdot Q_{\{p,r,t\}}$ with $Q_{\{p,r,t\}} = Z_L$ by part (i). □

Corollary 4.2 (Non-Fano obstruction). *The fundamental obstruction identified in* [5] *(that no single Pauli selection can satisfy the octonion algebra relations) has a code-theoretic interpretation: the 28 non-collinear triples of* $\mathrm{PG}(2, \mathbb{F}_2)$ *are exactly the triples* $\{p, r, t\}$ *for which* $\mathrm{bin}(p) + \mathrm{bin}(r) + \mathrm{bin}(t) \neq 0$ *in* $\mathbb{F}_2^3$*, i.e. the triples that do not define a Z-type stabiliser dual. The cocycle condition for* c_{oct} *fails on precisely these triples.*

Proof. A triple $\{p, r, t\}$ is collinear in $\mathrm{PG}(2, \mathbb{F}_2)$ iff $\mathrm{bin}(p) + \mathrm{bin}(r) + \mathrm{bin}(t) = 0$. There are $\binom{7}{3} = 35$ unordered triples total, 7 collinear (Fano lines) and 28 non-collinear. By [5, Remark 6.2], the cocycle identity for c_{oct} holds iff the triple is collinear. For non-collinear triples, no Z-type dual exists, so $Q_{\{p,r,t\}}$ is not a pure Z-string; the cocycle failure is the algebraic shadow of the absence of a projective dual. □

Remark 4.3 (Dependence of M on the generator basis). The bit-reversal matrix M is not an intrinsic feature of the projective duality; it is an artefact of two independent conventions. The map bin encodes $k \in \{1, \ldots, 7\}$ with the most significant bit first, while the standard parity-check matrix $H_{7,4,3}$ orders the generators so that g_j^Z detects the j-th bit least-significant-first. If one instead labels the Z-generators so that $\mathrm{supp}(g_j^Z) = \{k : \mathrm{bit}_{j-1}(k) = 1\}$ with bit-positions indexed MSB-first (i.e., matching the bin convention), then $S_Z(v_L) = \{1, \ldots, 7\} \setminus \{p, r, t\}$ directly and Part (ii) holds with $M = I_3$. The appearance of M in the theorem is a convention artefact, not a geometric obstruction.

5. Extension to $n = 5$: the $[[15, 7, 3]]$ CSS Bridge

We establish the analogue of Theorem 3.1 for $n = 5$. The ambient geometry is now $\mathrm{PG}(3, \mathbb{F}_2)$ (with 15 points), the role of Fano lines is played by Fano planes (2-dimensional projective subspaces, 7-element sets), and the code is $[[15, 7, 3]]$.

Theorem 5.1 ($n = 5$ CSS Bridge). *Let* $W \subset \{1, \ldots, 15\}$ *be a Fano plane in* $\mathrm{PG}(3, \mathbb{F}_2)$*, i.e. a 7-element set closed under bitwise XOR. Let* $v_W \in \mathbb{F}_2^4 \setminus \{0\}$ *be the unique nonzero vector orthogonal to every element of* W*:*

$$W = \{k \in \{1, \ldots, 15\} : \mathrm{bin}_4(k) \cdot v_W = 0\}.$$

Let the Z-generators of the $[[15, 7, 3]]$ *code have supports* $\mathrm{supp}(g_j^Z) = \{k : \mathrm{bin}_4(k)_{j-1} = 1\}$ *for* $j = 1, 2, 3, 4$ *(MSB-first convention), and write* $S_Z(w)$ *for the support of the Z-stabiliser labelled* $w \in \mathbb{F}_2^4 \setminus \{0\}$*. Then:*

- (i) (Projective duality) *The assignment* $W \mapsto v_W$ *is the projective duality of* $\mathrm{PG}(3, \mathbb{F}_2)$*: it is a bijection from the* 15 *Fano planes to* $\mathbb{F}_2^4 \setminus \{0\}$*, sending each hyperplane to its unique orthogonal point.*
- (ii) (CSS identification, $M = I_4$) $S_Z(v_W) = \{1, \ldots, 15\} \setminus W$ *for every Fano plane* W*. Equivalently, the Z-stabiliser labelled* v_W *has support equal to the complement of* W*, which by* [4, Theorem 5.1] *is also the support of the X-stabiliser corresponding to* W*.*
- (iii) (Contragredient equivariance) *For every* $g \in \mathrm{GL}(4, \mathbb{F}_2)$*:* $v_{g(W)} = g^{-T} v_W$*.*

Proof. Parts (i) and (iii) are standard facts of projective geometry over $\mathbb{F}_2$; part (i) is the definition of projective duality and (iii) follows from the contravariance of the orthogonal complement map $(gW)^{\perp} = g^{-T}(W^{\perp})$ (same argument as in the proof of Theorem 3.1(iii)).

Part (ii) is verified exhaustively for all 15 Fano planes by direct computation (the sign table of A_4, which determines the lo-flat structure of $\mathrm{ZD}(5)$ via [1]). The explicit data are in Table 2. □

Proposition 5.2 ($M = I_{n-1}$ for the MSB-first PCM convention)**.** *For all $n \geq 2$, let the $[[2^{n-1}-1,\, 2^{n-1}-n,\, 3]]$ Hamming CSS code have Z-generators g_i^Z with support*

$$\mathrm{supp}(g_i^Z) = \{k \in \{1, \dots, 2^{n-1}-1\} : \mathrm{bit}_{i-1}(k) = 1\}, \quad i = 1, \dots, n-1,$$

where $\mathrm{bit}_{i-1}(k)$ is the $(i-1)$-th bit of $\mathrm{bin}_{n-1}(k)$ counted MSB-first. Then for every hyperplane W of $\mathrm{PG}(n-2, \mathbb{F}_2)$ with normal v_W,

$$S_Z(v_W) = \{1, \dots, 2^{n-1}-1\} \setminus W.$$

Equivalently, $M = I_{n-1}$ for all $n \geq 2$ with the MSB-first PCM convention.

Proof. By the structure of the Hamming parity-check matrix:

$$S_Z(w) = \{k : \langle \mathrm{bin}_{n-1}(k), w\rangle = 1 \pmod 2\}.$$

By the definition of the hyperplane W_{v_W}:

$$\{1, \dots, 2^{n-1}-1\} \setminus W_{v_W} = \{k : \langle \mathrm{bin}_{n-1}(k), \mathrm{bin}_{n-1}(v_W)\rangle = 1 \pmod 2\}.$$

Setting $w = \mathrm{bin}_{n-1}(v_W)$ gives $S_Z(v_W) = \{1, \dots, 2^{n-1}-1\} \setminus W_{v_W}$. □

Remark 5.3 ($M = I_4$ vs $M \neq I_3$ at $n = 4$)**.** In contrast to $n = 4$ (where the bit-reversal $M \neq I_3$ was needed), Theorem 5.1(ii) holds with $M = I_4$. This confirms the prediction of Remark 4.3: the bit-reversal at $n = 4$ is a convention artefact that vanishes when the stabiliser-generator ordering matches the MSB-first convention of bin. For $n = 5$, the standard parity-check matrix $H_{15,11,3}$ already uses MSB-first bit-ordering, so no correction is needed.

Remark 5.4 (Computational diagnosis of the Clifford obstruction at $n = 5$)**.** The precise reason why the Clifford triple-product formula of Theorem 3.1(i) does not extend to $n \geq 5$ is the loss of alternativity. For $n = 4$, $A_3 = \mathbb{O}$ is alternative, giving $L_p^{\mathbb{O}} L_q^{\mathbb{O}} = \varepsilon(p,q)\, L_{p\oplus q}^{\mathbb{O}}$; antisymmetry of ε then forces $\{L_p^{\mathbb{O}}, L_q^{\mathbb{O}}\} = 0$ for $p \neq q$, making the seven operators Clifford generators.

For $n = 5$, $A_4 = \mathbb{S}$ is not alternative. Exhaustive computation over all $\binom{15}{2} = 105$ pairs (p, q) with $1 \leq p < q \leq 15$ gives:

- **63 pairs** (ZD-index pairs): $\{L_p^{A_4}, L_q^{A_4}\} = 0$, consistent with $\pi_A \pi_B \pi_C = -1$.
- **42 pairs** (non-ZD, alternativity-breaking pairs): $\{L_p^{A_4}, L_q^{A_4}\} \neq 0$ and $\neq -2I$.

For the 42 non-ZD pairs, $L_p^{A_4} L_q^{A_4}$ is a linear combination of multiple operators $L_r^{A_4}$, because the associator $[e_p, e_q, e_j] \neq 0$ for some j. This is intrinsic to A_4: no change of basis removes it.

6. Discussion

6.1. Summary of the bridge.

The following table summarises the chain of equivalences established across [1, 4, 5] and the present paper.

ZD algebra	$\longleftrightarrow$	**Projective geometry**	$\longleftrightarrow$	**Stabiliser codes**
$\pi_A \pi_B \pi_C = -1$ ([1, Thm. 7.18])		Fano line $\leftrightarrow$ dual point (Thm. 3.1(i))		X-stab $\leftrightarrow$ Z-stab ([4, Thm. 3.1])
$-Y_\alpha Y_\beta$ phase ([4, Thm. 4.2])		$Q_{\{p,r,t\}} = Z_L$ (Table 1)		Z-stabiliser support = lo-flat complement

6.2. Relation to the Weil zeta function.

The residue at the pole $x = \frac{1}{2}$ of the ZD zeta function [2] equals $-7 = -|\mathrm{PG}(2, \mathbb{F}_2)|$ [3, Theorem 8.2] and now acquires a structural interpretation: 7 counts the Fano lines, each of which corresponds (via Theorem 3.1) to a Z-stabiliser of the Steane code. The sign -1 reflects the residue of $Z(x)$ at the pole $x = 1/2$, which equals $-|\mathrm{PG}(2, \mathbb{F}_2)| = -7$ by the Weil identity ([3, Theorem 8.2]).

TABLE 2. $n = 5$ CSS bridge: all 15 Fano planes W with $M = I_4$. Columns: normal v_W; first four elements of W (full W has 7 elements); $S_Z(v_W) = \{1, \ldots, 15\} \setminus W$ (8-element complement). All 15 verified by direct computation.

v_W	W (Fano plane, 7 elements)	$S_Z(v_W) = \{1, \ldots, 15\} \setminus W$
$(0,0,0,1)^T$	$\{2,4,6,8,10,12,14\}$	$\{1,3,5,7,9,11,13,15\}$
$(0,0,1,0)^T$	$\{1,4,5,8,9,12,13\}$	$\{2,3,6,7,10,11,14,15\}$
$(0,0,1,1)^T$	$\{3,4,7,8,11,12,15\}$	$\{1,2,5,6,9,10,13,14\}$
$(0,1,0,0)^T$	$\{1,2,3,8,9,10,11\}$	$\{4,5,6,7,12,13,14,15\}$
$(0,1,0,1)^T$	$\{2,5,7,8,10,13,15\}$	$\{1,3,4,6,9,11,12,14\}$
$(0,1,1,0)^T$	$\{1,6,7,8,9,14,15\}$	$\{2,3,4,5,10,11,12,13\}$
$(0,1,1,1)^T$	$\{3,5,6,8,11,13,14\}$	$\{1,2,4,7,9,10,12,15\}$
$(1,0,0,0)^T$	$\{1,2,3,4,5,6,7\}$	$\{8,9,10,11,12,13,14,15\}$
$(1,0,0,1)^T$	$\{2,4,6,9,11,13,15\}$	$\{1,3,5,7,8,10,12,14\}$
$(1,0,1,0)^T$	$\{1,4,5,10,11,14,15\}$	$\{2,3,6,7,8,9,12,13\}$
$(1,0,1,1)^T$	$\{3,4,7,9,10,13,14\}$	$\{1,2,5,6,8,11,12,15\}$
$(1,1,0,0)^T$	$\{1,2,3,12,13,14,15\}$	$\{4,5,6,7,8,9,10,11\}$
$(1,1,0,1)^T$	$\{2,5,7,9,11,12,14\}$	$\{1,3,4,6,8,10,13,15\}$
$(1,1,1,0)^T$	$\{1,6,7,10,11,12,13\}$	$\{2,3,4,5,8,9,14,15\}$
$(1,1,1,1)^T$	$\{3,5,6,9,10,12,15\}$	$\{1,2,4,7,8,11,13,14\}$

6.3. **Obstruction to a full group-algebra lifting.** Corollary 4.2 makes precise why no Clifford algebra representation can lift the full octonion multiplication table to a group algebra: the 28 non-Fano triples do not satisfy the cocycle condition, and there is no Z-type dual to assign to them. The projective geometry of $\mathrm{PG}(2, \mathbb{F}_2)$ thus encodes exactly the boundary of what the CSS framework can see.

7. Open Problems

(1) **Extension to $n \geq 6$: Clifford structure.** The CSS identification (Parts (i)–(ii) of Theorems 3.1 and 5.1) now extends to all $n \geq 2$ by Proposition 5.2: the bridge $S_Z(v_W) = \{1, \ldots, 2^{n-1} - 1\} \setminus W$ is proved analytically, with $M = I_{n-1}$ for $n \geq 5$ and $M =$ bit-reversal for $n = 4$. The remaining open problem is the *Clifford* part: for $n \geq 5$, there is no $\tilde{\rho}$-triple-product formula for Q_W (Remark 5.4). The open question is whether there exists a natural map $\sigma\colon \mathrm{ZD}^*(n) \to \mathrm{Cliff}(0, 2^{n-1} - 1)$ extending ρ from $n = 4$ to all n, and whether triple products under σ recover $Z^{\otimes v_W}$ for the appropriate geometric objects.

(2) **Twisted cocycle interpretation.** The map c_{oct} is not a 2-cocycle, but it satisfies the cocycle condition on Fano triples. Does it define a *twisted* 2-cocycle in the sense of group cohomology of $(\mathbb{Z}/2\mathbb{Z})^3$, with twisting data determined by the projective duality of $\mathrm{PG}(2, \mathbb{F}_2)$? Relate this to the semidirect product structure $\mathrm{Aut}(A_4) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathrm{GL}(3, \mathbb{F}_2)$ of [3, §7].

(3) **Physical interpretation.** The map $\{p, r, t\} \mapsto Q_{\{p,r,t\}}$ sends each Fano line to a Z-type logical operator in the Steane code. Does the CSS bridge of Theorem 3.1 have a physical interpretation in terms of fault-tolerant gates or transversal operations on the Steane code?

Acknowledgements

The author thanks the mathematical physics community, and in particular the body of work on octonions and Cayley–Dickson algebras by C. Furey, G. M. Dixon, and

J. C. Baez [8]. Computational assistance was provided by Claude (Anthropic) for algebraic verification, structural analysis, and proof scaffolding. All mathematical results have been independently verified numerically; the author takes full responsibility for any errors.

References

[1] G. Levratti, *Sign-Parity Classification of Zero Divisors in Cayley–Dickson Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20315751. Under review at *Communications in Algebra*.

[2] G. Levratti, *Sign Structure, Automorphism Groups, and the ZD Zeta Function of Cayley–Dickson Algebras*, Zenodo preprint (2026), doi:10.5281/zenodo.20397587.

[3] G. Levratti, *Zero-Divisors in Cayley–Dickson Algebras: Fano Geometry, Automorphisms, and the Weil Zeta Function*, Zenodo preprint (2026), doi:10.5281/zenodo.20417859.

[4] G. Levratti, *Active Stabilizers in Reed–Muller CSS Codes from Cayley–Dickson Zero-Divisors*, Zenodo preprint (2026), doi:10.5281/zenodo.20431754.

[5] G. Levratti, *A Clifford Representation of the Fano Zero-Divisors*, Zenodo preprint (2026), doi:10.5281/zenodo.20445208.

[6] A. M. Steane, Multiple-particle interference and quantum error correction, *Proc. R. Soc. Lond. A* **452** (1996), 2551–2577.

[7] D. Gottesman, *Stabilizer Codes and Quantum Error Correction*, Ph.D. thesis, California Institute of Technology, 1997.

[8] J. C. Baez, The octonions, *Bull. Amer. Math. Soc.* **39** (2002), no. 2, 145–205.

Independent Researcher, Modena, Italy
Email address: `giovanni.levratti.research@gmail.com`