A QUARTIC CONTACT INTERACTION WITH FANO–CSS STRUCTURE

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ABSTRACT. The exact identity $N(a)N(b) - N(ab) = 4$ for every zero-divisor-active pair in $\mathcal{A}_4$ ([6, Theorem 3.8]) is promoted to a quartic contact interaction $\mathcal{L}_{\text{int}} = \lambda \cdot \Delta N(\varphi, \psi)$ in a scalar field theory valued in $\mathcal{A}_4$. The interaction tensor T is computed in closed form; its non-zero entries decompose under the XOR block structure of $\mathcal{A}_4$ with a seven-fold Fano labelling indexed by $\mathrm{PG}(2, \mathbb{F}_2)$. The coupling λ is shown to be *unique*: the automorphism group $\mathrm{Aut}(\mathcal{A}_4) \cong C_2 \times \big((C_2^3) \,.\, \mathrm{PSL}(3,2)\big)$ of order 2688 acts transitively on the 168 unordered zero-divisor pairs, with stabiliser $C_2 \times D_8$ fixing the vertex structure. The CSS bridge of [8] identifies the seven Fano blocks with the seven Z-stabilisers of the Steane $[[7,1,3]]$ code. Physical consequences are drawn for the vertex amplitude structure and for the single logical qubit that survives the $\mathcal{A}_3 \to \mathcal{A}_4$ threshold.

Epistemic key. *[Derived]* — follows from Papers 1–8 and established mathematics, no additional assumptions. *[Conjecture]* — physically motivated, mathematically precise, not yet proved. *[Speculation]* — structurally motivated, not yet falsifiable. QPU measurements on `ibm_kingston` are labelled *[Conjecture] | QPU measurement* and are not mathematical proofs.

1. The Quartic Interaction Term

1.1. **Setup and conventions.** Let $\mathcal{A}_4$ denote the sedenion algebra over $\mathbb{R}$, constructed by four applications of the Cayley–Dickson doubling from $\mathbb{R}$, with the convention

$$(a,b)(c,d) = (ac - \bar{d}b,\ da + b\bar{c}). \tag{1}$$

The canonical basis $\{e_0, e_1, \ldots, e_{15}\}$ satisfies $N(e_k) = 1$ and $e_0 = 1$ (identity). For all $i, j \geq 0$:

$$e_i \cdot e_j = \varepsilon(i,j)\, e_{i \oplus j}, \tag{2}$$

where $\oplus$ denotes bitwise XOR of four-bit indices and $\varepsilon(i,j) \in \{\pm 1\}$ is the Cayley–Dickson sign function. The norm is $N(x) = \sum_k (x^k)^2$.

The *candidate set* is $C^*(4) = \{e_i \pm e_j : 1 \leq i < j \leq 15\}$, consisting of index-pure norm-2 elements. The zero-divisor set $\mathrm{ZD}(4) \subset C^*(4) \times C^*(4)$ consists of all ordered pairs (a,b) with $ab = 0$.

Remark 1.1 (Convention)**.** The pair $(e_1 + e_{10}, e_4 - e_7)$ cited in [6] employs a sign-reordered variant of (1); in the convention above the canonical ZD partner of $e_1 + e_{10}$ is $e_4 - e_{15}$. All structural results ($|\mathrm{ZD}(4)| = 336$, $\Delta N = 4$, Fano labelling) are convention-independent.

1.2. **The norm-gap operator.**

Definition 1.2. For $\mathcal{A}_4$-valued scalar fields φ, ψ, define the *norm-gap operator*

$$\Delta N(\varphi, \psi) := N(\varphi)N(\psi) - N(\varphi\psi).$$

This is a degree-4 polynomial in the field components: quartic, bilinear in each of the pairs (φ, φ) and (ψ, ψ).

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Proposition 1.3 (*[Derived]*, from [6]). *Theorem 3.8* [6]*] For every pair* $(a, b) \in \mathrm{ZD}(4)$*:*

$$\Delta N(a, b) = N(a)N(b) - N(ab) = 2 \cdot 2 - 0 = 4.$$

The value 4 is exact and unconditional; it depends only on $N(a) = N(b) = 2$ *and* $N(ab) = 0$.

Proposition 1.4 (Trichotomy, *[Derived]* [6]). *For all* $(a, b) \in C^*(4) \times C^*(4)$, $N(ab) \in \{0, 4, 8\}$, *giving*

$$\Delta N(a, b) \in \{+4,\ 0,\ -4\},$$

with exactly 336 *pairs in each of the* $\Delta N = +4$ *and* $\Delta N = -4$ *classes, and* 43428 *pairs in the* $\Delta N = 0$ *class. The trichotomy is exhaustive (verified over all* $210^2 = 44100$ *candidate pairs).*

Definition 1.5 (Contact Lagrangian). The natural candidate for a contact interaction arising from the norm gap is

$$\mathcal{L}_{\text{int}}[\varphi, \psi] := \lambda \cdot \Delta N(\varphi, \psi),$$

with $\lambda \in \mathbb{R}$ a single coupling constant. The tree-level four-point amplitude on ZD-active external states is

$$i\mathcal{M}_4 = -4i\lambda.$$

The value -4 is not a free parameter: it is fixed by the algebra.

1.3. **The interaction tensor in closed form.** Writing $\Delta N(\varphi, \psi) = \sum_{a,b,c,d} T[a, b, c, d]\, \varphi^a \varphi^c \psi^b \psi^d$, the tensor T is computed from the sedenion multiplication table.

Theorem 1.6 (*[Derived]*). *The interaction tensor is*

$$T[a, b, c, d] = \delta_{ac}\delta_{bd} - \varepsilon(a, b)\, \varepsilon(c, d)\, \delta_{a\oplus b,\, c\oplus d}. \tag{3}$$

Proof. Expanding $N(\varphi\psi) = \sum_k (\varphi\psi)_k^2$ using $(\varphi\psi)_k = \sum_{a\oplus b=k} \varepsilon(a, b)\varphi^a\psi^b$, and subtracting from $N(\varphi)N(\psi) = \sum_{a,b}(\varphi^a)^2(\psi^b)^2$, gives the stated form directly. □

Properties of T (all *[Derived]*, verified computationally):

- $T[a, b, c, d] \in \{-1, 0, +1\}$ for all indices.
- $T[a, b, a, b] = 0$ identically (the diagonal vanishes: $1 - \varepsilon(a, b)^2 = 0$).
- Non-zero entries: 3840 total (1920 at $+1$, 1920 at -1).
- T is non-zero iff $a \oplus b = c \oplus d$ and $(a, b) \neq (c, d)$.

The last property gives a block decomposition indexed by $m = a \oplus b$:

$$T = \sum_{m=0}^{15} T^{(m)}, \qquad T^{(m)}[a, b, c, d] = -\varepsilon(a, b)\varepsilon(c, d) \cdot \delta_{a\oplus b,\, m} \cdot \delta_{c\oplus d,\, m} \cdot (1 - \delta_{(a,b),(c,d)}).$$

Each block $T^{(m)}$ has 240 non-zero entries (16×15 off-diagonal). Block $m = 0$ corresponds to self-products ($a = b$) and is never used by ZD-active fields.

Remark 1.7 (Incidence count). Each ZD pair uses exactly 4 non-zero entries of T; each such entry is shared by exactly 2 ZD pairs. The exact incidence identity is:

$$|\,\mathrm{ZD}(4)| \times 4 = 2 \times \#\{T\text{-entries used by ZD sector}\} \implies 336 \times 4 = 2 \times 672.$$

The ratio $3840/336 \approx 11.4$ is not expected to be an integer, as it counts non-commensurate objects.

1.4. Fano labelling of ZD-active elements.

Proposition 1.8 (*[Derived]* [1])**.** *Every ZD-active element has the form*

$$a = e_p + \sigma e_{8+k}, \qquad p, k \in \{1, \ldots, 7\},\ \sigma \in \{\pm 1\},\ p \neq k.$$

There are $7 \times 6 \times 2 = 84$ *such elements in total. The* Fano label *of* $a = e_p + \sigma e_{8+k}$ *is* $m(a) := p \oplus k \in \{1, \ldots, 7\}$*, assigning each active element to one of the* 7 *points of* $\mathrm{PG}(2, \mathbb{F}_2)$*. There are exactly* 12 *elements per Fano point (*6 *index pairs* $\times$ 2 *signs).*

Proposition 1.9 (Fano conservation, *[Derived]* [1])**.** *Every ZD pair* (a, b) *satisfies* $m(a) = m(b)$*: zero-divisors pair only within the same Fano class. The* 336 *ZD pairs split as* 48 *per Fano point (*$7 \times 48 = 336$ ✓*).*

Proposition 1.10 (Fano line structure, *[Derived]*)**.** *Within each Fano class* m*, the six index pairs* $\{p, k\}$ *with* $p \oplus k = m$ *form exactly three Fano triples* $\{p, k, m\}$ *satisfying* $p \oplus k \oplus m = 0$*. These are precisely the three Fano lines through the point* $m \in \mathrm{PG}(2, \mathbb{F}_2)$*.*
Example. *For* $m = 3$*: pairs* $\{1, 2\}, \{4, 7\}, \{5, 6\}$ $\rightarrow$ *Fano lines* $\{1, 2, 3\}, \{3, 4, 7\}, \{3, 5, 6\}$ ✓*.*

Remark 1.11 (CSS bridge)**.** By [8, Theorem 3.1], each Fano line $\{p, r, t\}$ corresponds to a Z-stabiliser of the Steane $[[7, 1, 3]]$ code, and the projective duality of $\mathrm{PG}(2, \mathbb{F}_2)$ exchanges Fano points and Fano lines. The seven Fano labels $m = 1, \ldots, 7$ of the active elements thus correspond to the seven Z-stabilisers of the $[[7, 1, 3]]$ code via projective duality.

1.5. The $2 + 2$ splitting of the norm gap.

Theorem 1.12 (*[Derived]* – analytic proof)**.** *For every ZD pair* $(a, b) = (e_p + \sigma e_{8+k},\ e_q + \tau e_{8+l})$ *with Fano label* $m = p \oplus k = q \oplus l$*, setting* $m_1 = p \oplus q$ *and* $m_2 = 8 + (m_1 \oplus m)$*:*

$$\Delta N(a, b) = F_{m_1}(a, b) + F_{m_2}(a, b) = 2 + 2 = 4,$$

where $F_n(\varphi, \psi) := \sum_\alpha (\varphi^\alpha)^2 (\psi^{\alpha \oplus n})^2 - \left(\sum_\alpha \varepsilon(\alpha, \alpha \oplus n) \varphi^\alpha \psi^{\alpha \oplus n}\right)^2$.

Proof. Expanding the product ab using the XOR structure $e_p \cdot e_q = \varepsilon(p, q) e_{p \oplus q}$:

$$ab = \varepsilon(p, q)\, e_{m_1} + \tau \varepsilon(p, 8 + l)\, e_{m_2} + \sigma \varepsilon(8 + k, q)\, e_{m_2} + \sigma \tau \varepsilon(8 + k, 8 + l)\, e_{m_1}.$$

Since $m_1 \in \{1, \ldots, 7\}$ and $m_2 \in \{8, \ldots, 15\}$ are distinct, $ab = 0$ requires two independent sign constraints:

$$\text{(S1)} \quad \varepsilon(p, q) + \sigma\tau\, \varepsilon(8 + k, 8 + l) = 0 \quad [\text{cancellation at } e_{m_1}] \tag{4}$$

$$\text{(S2)} \quad \tau\, \varepsilon(p, 8 + l) + \sigma\, \varepsilon(8 + k, q) = 0 \quad [\text{cancellation at } e_{m_2}] \tag{5}$$

Both verified on all 336 ZD pairs.

$F_{m_1} = 2$*:* Index arithmetic gives $p \oplus m_1 = q$ and $(8 + k) \oplus m_1 = 8 + l$. First term: $(\varphi^p)^2 (\psi^q)^2 + (\varphi^{8+k})^2 (\psi^{8+l})^2 = 1 + \tau^2 = 2$. Second term: $\varepsilon(p, q) \cdot 1 \cdot 1 + \varepsilon(8+k, 8+l) \cdot \sigma \cdot \tau = \varepsilon(p, q) + \sigma\tau\varepsilon(8+k, 8+l) = 0$ by (S1). Therefore $F_{m_1} = 2 - 0 = 2$.

$F_{m_2} = 2$*:* Index arithmetic gives $p \oplus m_2 = 8 + l$ and $(8 + k) \oplus m_2 = q$. First term: $(\varphi^p)^2 (\psi^{8+l})^2 + (\varphi^{8+k})^2 (\psi^q)^2 = \tau^2 + 1 = 2$. Second term: $\varepsilon(p, 8+l) \cdot 1 \cdot \tau + \varepsilon(8+k, q) \cdot \sigma \cdot 1 = \tau\varepsilon(p, 8+l) + \sigma\varepsilon(8+k, q) = 0$ by (S2). Therefore $F_{m_2} = 2 - 0 = 2$. Hence $\Delta N = 4$. □

Remark 1.13 (Physical reading)**.** The ZD condition $ab = 0$ imposes exactly two sign constraints (S1),(S2). Each constraint kills the interference term in one of the two XOR blocks, leaving exactly the norm contribution (first term $= N(a)/2 + N(b)/2 = 2$). The $2 + 2$ splitting is a direct algebraic signature of the ZD condition: the norm of the pair is distributed equally across the low and high sectors.

2. The CSS Invariant and the Logical Qubit

2.1. **The Steane $[[7,1,3]]$ code: minimal review.** The Steane $[[7,1,3]]$ code [9, 10] is the $\mathrm{CSS}(C,C)$ code with $C=[7,4,3]$ the binary Hamming code. It encodes $k=1$ logical qubit into $n=7$ physical qubits with distance $d=3$. The stabiliser group $\mathcal{S}=\mathcal{S}_X\times\mathcal{S}_Z$ has 6 independent generators (3 X-type, 3 Z-type) with identical supports:

$$\begin{aligned} &g_1^X=X_1X_3X_5X_7, \quad g_2^X=X_2X_3X_6X_7, \quad g_3^X=X_4X_5X_6X_7,\\ &g_1^Z=Z_1Z_3Z_5Z_7, \quad\ \ g_2^Z=Z_2Z_3Z_6Z_7, \quad\ \ g_3^Z=Z_4Z_5Z_6Z_7.\end{aligned}$$

The code space has dimension $2^{7-6}=2$ (exactly one logical qubit). The 7 non-identity elements of $\mathcal{S}_X$ (resp. $\mathcal{S}_Z$) have supports equal to the 7 *lo-flats* of ZD(4): the weight-4 subsets $\{a,b,c,d\}\subset\{1,\dots,7\}$ with $a\oplus b\oplus c\oplus d=0$, each equal to the complement $\{1,\dots,7\}\setminus\{p,r,t\}$ of a Fano line [5].

Remark 2.1 (Experimental stabiliser measurement)**.** On 26 May 2026, the logical state $|0_L\rangle$ was prepared on the `ibm_kingston` 156-qubit QPU (job `d8amem5g7okc73eq33fg`, circuit depth 33) and all six stabilisers were measured, obtaining values in $[0.865, 0.996]$ — all in the $+1$ eigenspace within NISQ decoherence limits [2]. *These results confirm the stabiliser structure of the* $[[7,1,3]]$ *code on physical hardware. They do not verify any claim about sedenions or physics beyond the Standard Model.*

2.2. **The CSS bridge.**

Theorem 2.2 (CSS Bridge [8], *[Derived]*)**.** *Let* $\{p,r,t\}\subset\{1,\dots,7\}$ *be any Fano line and* $Q_{\{p,r,t\}}\in\{I,Z\}^{\otimes 3}$ *the* Z*-string defined by the Clifford triple product of* [7]*. Then:*

(i) (Projective orthogonality.) $Q_{\{p,r,t\}}=Z^{\otimes v}$, *where* $v\in\mathbb{F}_2^3\setminus\{0\}$ *is the unique vector orthogonal to* $\mathrm{bin}(p),\mathrm{bin}(r),\mathrm{bin}(t)$ *– the projective dual point.*

(ii) (CSS identification.) $Q_{\{p,r,t\}}$ *is the non-identity* Z*-stabiliser of* $[[7,1,3]]$ *whose support equals* $\{1,\dots,7\}\setminus\{p,r,t\}$.

(iii) ($\mathrm{GL}(3,\mathbb{F}_2)$-equivariance.) *For every* $g\in\mathrm{GL}(3,\mathbb{F}_2)$*:* $Q_{g(\{p,r,t\})}=Z^{\otimes(g^{-T}v)}$.

Corollary 2.3 (*[Derived]*)**.** *The map* $\{p,r,t\}\mapsto Q_{\{p,r,t\}}$ *is a* $\mathrm{GL}(3,\mathbb{F}_2)$*-equivariant bijection from the* 7 *Fano lines to the* 7 *non-identity* Z*-stabilisers of* $[[7,1,3]]$*. This is the projective duality of* $\mathrm{PG}(2,\mathbb{F}_2)$*.*

Remark 2.4 (Scope and failure for $n\geq 5$)**.** The CSS bridge is specific to $n=4$. For $n=5$, exhaustive computation shows 42 non-ZD pairs with $\{L_p^{\mathcal{A}_4},L_q^{\mathcal{A}_4}\}\neq 0$, breaking the Clifford structure [8]. The $n=4$ case is structurally isolated.

Remark 2.5 (Weil zeta connection)**.** The residue of the ZD zeta function [3] at its pole $x=1/2$ equals $-7=-|\mathrm{PG}(2,\mathbb{F}_2)|$. Via Theorem 2.2, the 7 counts (Fano lines, Z-stabilisers, lo-flats, Fano points) are all the same set. The CSS bridge is the algebraic explanation of this pole residue.

2.3. **The $\mathcal{A}_3\to\mathcal{A}_4$ threshold as a CSS transition.** At $n=3$ (octonions $\mathcal{A}_3$): N is multiplicative [12]. The left-multiplication operators $L_{e_k}^{\mathbb{O}}$ ($k=1,\dots,7$) are Clifford generators with $\{L_p^{\mathbb{O}},L_q^{\mathbb{O}}\}=0$ for $p\neq q$, generating $\mathrm{Cliff}(0,7)$ [7]. Their structure is encoded in the X-stabilisers of $[[7,1,3]]$ (each support = one lo-flat).

At $n=4$ (sedenions $\mathcal{A}_4$): N fails multiplicativity. Zero-divisor pairs appear. The CSS bridge maps the ZD structure to the Z-stabilisers: each Fano line $\{p,r,t\}\mapsto Z$-stabiliser with support = lo-flat complement.

Observation 2.6 (*[Derived]* (a); *[Conjecture]* (b))**.** (a) [Derived]. *The CSS bridge establishes:*

$$\begin{aligned} \textit{Fano line } \{p,r,t\} &\longleftrightarrow Z\textit{-stabiliser, support } =\{1,\dots,7\}\setminus\{p,r,t\}\\ \textit{Fano dual point } v &\longleftrightarrow Z^{\otimes v}\\ \mathrm{GL}(3,\mathbb{F}_2)\ \textit{on lines} &\longleftrightarrow \mathrm{GL}(3,\mathbb{F}_2)\ \textit{contragredient on } Z\textit{-stabilisers.}\end{aligned}$$

The X- and Z-stabilisers of $[[7,1,3]]$ share the same lo-flat supports by the $\mathrm{CSS}(C,C)$ *self-dual construction. The X/Z duality of the code is the projective duality of* $\mathrm{PG}(2,\mathbb{F}_2)$*.*

(b) [Conjecture]. *The identification of the X-stabiliser sector with $\mathcal{A}_3$ (octonionic structure) and the Z-stabiliser sector with $\mathcal{A}_4$ (ZD structure) is a physical interpretation requiring assumptions (A1) and (A2). The mathematical CSS bridge does not itself assert this physical assignment.*

Observation 2.7 (*[Derived]* (a); *[Conjecture]* (b))**.** (a) [Derived]. *The code space of $[[7,1,3]]$ is the unique subspace commuting with all 6 stabiliser generators, with dimension $2^{7-6}=2$. It encodes exactly 1 logical qubit, invariant under* $\mathrm{GL}(3,\mathbb{F}_2)$ *by Theorem 2.2(iii). These properties are mathematical consequences of the parameters of $[[7,1,3]]$ alone; they require no physical assumption.*

(b) [Conjecture]. *This qubit is the physical invariant of the $\mathcal{A}_3 \to \mathcal{A}_4$ transition: the information that neither the octonionic structure nor the ZD structure can describe separately. This identification requires the physical assumptions (A1) and (A2) and is not derivable from the algebra alone.*

Remark 2.8 (QPU evidence: Z_L and X_L, *[Conjecture]| QPU measurement*)**.** Both logical operators of the $[[7,1,3]]$ code were measured on `ibm_kingston`.

$Z_L = Z^{\otimes 7}$ (job `d8ctl5q4gq0s73apdjv0`, 2026-05-29, 8192 shots each):

$$\langle Z_L\rangle_{|0_L\rangle} = +0.7258 \pm 0.0076 \quad (96\sigma;\ \text{ideal } +1), \quad \text{fidelity } 83.1\%$$
$$\langle Z_L\rangle_{|1_L\rangle} = -0.7197 \pm 0.0077 \quad (94\sigma;\ \text{ideal } -1), \quad \text{fidelity } 82.0\%$$

Noise consistent with $\sim 2.3\%$ depolarising error per qubit at circuit depth 35–36.

$X_L = X^{\otimes 7}$ (job `d8cvisnd0j8c73f3frl0`, 2026-05-29, 8192 shots each):

$$\langle X_L\rangle_{|-_L\rangle} = -0.7825 \pm 0.0069 \quad (114\sigma;\ \text{ideal } -1), \quad \text{fidelity } 85.9\%$$
$$\langle X_L\rangle_{|+_L\rangle} = +0.7783 \pm 0.0069 \quad (112\sigma;\ \text{ideal } +1), \quad \text{fidelity } 85.5\%$$

The $|+_L\rangle$ circuit required a corrected transpilation (job `d8cvp2nd0j8c73f3g96g`): a `Qiskit` barrier had prevented $HH = I$ cancellation, giving a noise-dominated first result ($+0.096 \pm 0.011$, 9σ); removing the barrier restored the expected value.

All four expectation values carry the correct signs at $> 94\sigma$. Combined with the stabiliser measurement of Remark 2.1, the three experiments confirm: the $[[7,1,3]]$ code space is a non-trivial logical qubit on physical hardware, with both Z_L and X_L distinguishing the logical basis states at $> 94\sigma$. The identification of this qubit with the $\mathcal{A}_3 \to \mathcal{A}_4$ algebraic invariant remains *[Conjecture]* (Observation 2.6(b)).

Remark 2.9 (The $2+2$ splitting and CSS X/Z structure, *[Conjecture]*)**.** The splitting $\Delta N = F_{m_1} + F_{m_2} = 2+2$ is proved analytically (Theorem 1.12) from the ZD sign constraints (S1),(S2): $F_{m_1} = 2$ comes from (S1) (governing parallel products); $F_{m_2} = 2$ comes from (S2) (governing cross products). The structural parallel to the CSS X/Z split – whether (S1)$\leftrightarrow$ X-stabilisers and (S2)$\leftrightarrow$ Z-stabilisers in a precise sense – requires the physical identification of Observation 2.6(b), which is itself *[Conjecture]*. The algebraic proof is *[Derived]*; its CSS interpretation is *[Conjecture]*.

2.4. **The logical qubit conjecture.**

Conjecture 2.10 (Single logical qubit)**.** *Let $\mathcal{T}$ be any quantum field theory carrying an $\mathcal{A}_4$-valued scalar field φ with* $\mathrm{Aut}(\mathcal{A}_4)$ *symmetry and ZD-active matter content. Then $\mathcal{T}$ necessarily contains a physical subsystem isomorphic to the logical qubit of $[[7,1,3]]$. The physical claim requires assumptions (A1) (that $\mathcal{A}_4$-valued scalar fields are physically meaningful) and (A2) (that* $\mathrm{Aut}(\mathcal{A}_4)$ *symmetry is not spontaneously broken).*

Conjecture 2.11 (Threshold uniqueness). *The* $[[7,1,3]]$ *logical qubit is the unique quantum error-correcting code appearing as a structural invariant of the Cayley–Dickson tower at the ZD threshold. For* $n \geq 5$*, the Clifford structure fails and no analogous CSS code carries a single logical qubit in the same way.*

3. Physical Consequences

3.1. Uniqueness of the coupling constant.

Theorem 3.1 (Transitivity, *[Derived]*). *The algebra automorphism group* $\mathrm{Aut}(\mathcal{A}_4)$ *– the group of* $\mathbb{R}$*-linear bijections* $\phi\colon \mathcal{A}_4 \to \mathcal{A}_4$ *satisfying* $\phi(xy) = \phi(x)\phi(y)$ *– acts transitively on the set of* 168 *unordered ZD pairs.*

Proof. *All* 2688 *algebra automorphisms were constructed and represented as permutations of the* 168 *unordered ZD pairs (*`GAP` *4.15.1). The orbit of any single pair has size* 168*, exhausting the set. Orbit–stabiliser accounting:*

$$\textit{unordered} : |\,\mathrm{Aut}\,| = |\mathrm{orbit}| \times |\,\mathrm{Stab}\,| \;\Rightarrow\; 2688 = 168 \times 16 \checkmark \qquad \textit{ordered} : 2688 = 336 \times 8 \checkmark . \qquad \square$$

Remark 3.2 (Scope). The group computed here is $\mathrm{Aut}(\mathcal{A}_4)$ as an algebra. Whether it coincides with $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$ – the full automorphism group of the ZD pair structure – is not established in the current series. $\mathrm{Aut}(\mathcal{A}_4) \subseteq \mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$; equality is a separate open question (see §4, item 10).

Corollary 3.3 (*[Derived]*). *The coupling* λ *is unique. There is no consistent assignment of distinct coupling constants to sub-orbits of ZD pairs.* $\mathcal{L}_{\mathrm{int}} = \lambda \cdot \Delta N(\varphi, \psi)$ *has exactly one free parameter.*

Remark 3.4 (Group structure – two distinct groups, *[Derived]*). Two groups of order 2688 appear in this series. They are *not* isomorphic.

(A) $\mathrm{Aut}(\mathcal{A}_4)$ **as algebra:**

$$\mathrm{Aut}(\mathcal{A}_4) \cong C_2 \times \big((C_2^3)\,.\,\mathrm{PSL}(3,2)\big), \quad |\,\mathrm{Aut}(\mathcal{A}_4)| = 2688 \quad [\texttt{GAP}\ 4.15.1].$$

Extension class: **non-split** (confirmed by `GAP`: no complements to N_{16}, no subgroup isomorphic to $\mathrm{GL}(3,\mathbb{F}_2)$). Papers [1–8] use "$\rtimes$" (semidirect product) for $\mathrm{Aut}(\mathcal{A}_n)$; `GAP` 4.15.1 confirms the extension is non-split, so the correct notation is ".". This does not affect any result in the series that uses only the group order or its action on ZD pairs. [4, Remark 4.3] already acknowledges the non-split structure. Normal subgroups: $\{1, C_2, C_2^3, C_2^4, G', G\}$. Centre $= C_2$. $G/G' \cong C_2$.

Full cohomological determination. The extension $1 \to C_2^3 \to \mathrm{Aut}(\mathcal{A}_4) \to C_2 \times \mathrm{PSL}(3,2) \to 1$ is classified by the Künneth decomposition

$$H^2\big(C_2 \times \mathrm{PSL}(3,2),\, C_2^3\big) \;\cong\; \mathbb{F}_2^2,$$

since C_2 acts trivially on C_2^3. The two generators correspond to:

$$\mathrm{SmallGroup}(1344,\,814) = (C_2^3)\,.\,\mathrm{PSL}(3,2) \quad [\text{non-split, class } G'],$$

$$\mathrm{SmallGroup}(1344,\,11686) = (C_2^3) : \mathrm{PSL}(3,2) = \mathrm{AGL}(3,\mathbb{F}_2) \quad [\text{split}].$$

The extension class of $\mathrm{Aut}(\mathcal{A}_4)$ is $(1,0) \in \mathbb{F}_2^2$: the inflation of the unique non-zero element of $H^2(\mathrm{PSL}(3,2), C_2^3) \cong \mathbb{F}_2$. `GAP` commands: `IdGroup(DerivedSubgroup(G))`$\to$`[1344,814]`; `ComplementClassesRepresentatives`: 0 complements for $(1344, 814)$, 2 for $(1344, 11686)$.

Note on H^1. $H^1(\mathrm{PSL}(3,2), C_2^3) \cong \mathbb{F}_2$ (computed via bar resolution). This is independent of the extension class: for the non-split extension, $H^1 \neq 0$ implies the existence of non-inner automorphisms of G' acting trivially on the quotient. It does not affect the classification of extensions, which is governed by H^2.

(B) $\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$**:**

$$\mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4)) = S_7 \wr G_{\mathrm{class}}, \quad |G_{\mathrm{class}}| = 768, \quad |\,\mathrm{Aut}_{\mathrm{struct}}\,| = 7! \times 768^7 \approx 7.9 \times 10^{23} \quad [4].$$

$\mathrm{Aut}(\mathcal{A}_4) \le \mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$ is a tiny subgroup. *[Derived]* [4]. The non-isomorphism is immediate from the orders alone.
Physical consequence: unaffected. All results in §§3.2–3.5 use $\mathrm{Aut}(\mathcal{A}_4)$ as algebra.

3.2. **Vertex symmetry and amplitude ratios.**

Theorem 3.5 (*[Derived]*)**.** *The stabiliser of a ZD pair under* $\mathrm{Aut}(\mathcal{A}_4)$ *is*

$$\mathrm{Stab}(pair) \cong C_2 \times D_8, \qquad |\,\mathrm{Stab}\,| = 16.$$

Proof. *By GAP 4.15.1,* `StructureDescription(Stabilizer(G,1)) = "C2 x D8"`. □

The stabiliser $C_2 \times D_8$ is the *residual symmetry of the* 4*-point vertex*: D_8 acts on the four legs as the symmetry group of a square (Bose symmetry of a quartic scalar vertex); the additional C_2 factor is consistent with a sign conjugation or charge-parity operation.

Corollary 3.6 (Amplitude ratios, *[Derived]*)**.** *For any theory with symmetry* $\mathrm{Aut}(\mathcal{A}_4)$ *and ZD-active fields, the ratios of* 4*-point amplitudes in distinct kinematic channels are fixed by the representation theory of* $C_2 \times D_8$*, independently of the energy scale and of* λ*. By Proposition 3.7, the action of* $C_2 \times D_8$ *on* $\mathcal{A}_4$ *decomposes into all irreps; the amplitude ratios in each partial-wave channel are determined by the corresponding Clebsch–Gordan coefficients of* $C_2 \times D_8$*.*

Proposition 3.7 (Regular representation, *[Derived]*)**.** *The* 16*-dimensional representation of* $C_2 \times D_8$ *on* $\mathcal{A}_4$ *via the automorphism action is the* ***regular representation*** *of* $C_2 \times D_8$*:*

$$\chi(g) = 16 \cdot \delta_{g,e}.$$

Proof. *The* 16 *stabiliser elements act on* $\mathcal{A}_4 = \mathbb{R}^{16}$ *as orthogonal sign-permutation matrices (verified computationally). Traces:* $\chi(e) = 16$*,* $\chi(g \ne e) = 0$*, matching the regular representation character.* □

Corollary 3.8 (*[Derived]*)**.** $\mathcal{A}_4 \cong \bigoplus_\rho \rho^{\dim(\rho)}$ *as a* $C_2 \times D_8$ *representation, where* ρ *ranges over all* 10 *irreps (*8 *one-dimensional,* 2 *two-dimensional). The trivial irrep (multiplicity* 1*) spans the* e_0 *direction. ZD-active fields (index-pure, no* e_0 *component) lie in the* 15*-dimensional complement.*

Proposition 3.9 (Orbit structure on ZD pairs, *[Derived]*)**.** $\mathrm{Stab}(pair) \cong C_2 \times D_8$ *acts on the* 168 *unordered ZD pairs with:* 4 *fixed points (size* 1*),* 12 *orbits of size* 2*,* 35 *orbits of size* 4*; total* 51 *orbits with* $4 \times 1 + 12 \times 2 + 35 \times 4 = 168$ ✓*.*

The 4 *fixed points (Fano label* $m = 3$*, all sharing the label of the incoming pair):* $\{e_1+e_{10},\, e_4-e_{15}\}$, $\{e_1 - e_{10},\, e_4 + e_{15}\}$, $\{e_2 + e_9,\, e_7 - e_{12}\}$, $\{e_2 - e_9,\, e_7 + e_{12}\}$*.*

Corollary 3.10 (Isotropic angular distribution, *[Derived]*)**.** *For ZD-active* $2 \to 2$ *self-scattering,* $d\sigma/d\Omega = \mathrm{const}$ *(isotropic), for any* λ *and any energy scale* E*. This is the unique* E*-independent,* λ*-independent observable prediction of* $\mathcal{L}_{\mathrm{int}} = \lambda\Delta N$ *on ZD-active states.*

3.3. **Dark matter: new content from this paper.** [2] established Conjecture IIB-III: under (A1) and (A2), degrees of freedom in $\mathcal{A}_4 \setminus \mathcal{A}_3$ are gauge-invisible and gravitationally active. This paper adds an explicit Lagrangian.

Proposition 3.11 (Self-interaction Lagrangian for $\mathcal{A}_4 \setminus \mathcal{A}_3$ fields, *[Conjecture]*)**.** *Under assumption (A1) and Conjecture IIB-III, the leading self-interaction of* $\mathcal{A}_4 \setminus \mathcal{A}_3$ *scalar fields* φ, ψ *is*

$$\mathcal{L}_{\mathrm{int}} = \lambda \cdot \Delta N(\varphi, \psi) = \lambda \cdot [N(\varphi)N(\psi) - N(\varphi\psi)],$$

with λ *the unique coupling constant fixed by* $\mathrm{Aut}(\mathcal{A}_4)$*-transitivity (Corollary 3.3).*
Discriminating predictions:
(P3) [Conjecture]*. Direct-detection experiments yield null results – not because the coupling is small, but because the electroweak coupling does not exist.*

(P4) [Conjecture]*. Any violation of the equivalence principle for the dark sector falsifies IIB-III. Gravitational coupling is exactly universal.*
(P5) [Conjecture]*. The dark sector has a quartic self-interaction with vertex structure $C_2 \times D_8$, a single coupling λ, amplitude coefficient 4, and isotropic angular distribution. These are E-independent and λ-independent structural predictions.*

3.4. **The energy scale: what the algebra fixes and what it does not.**

Remark 3.12 (Free parameters, *[Speculation]*)**.** The coupling λ and the energy scale E are both free parameters not fixed by the algebra. From dimensional analysis: $\sigma_{\text{self}} \sim \lambda^2/E^2$. Observational constraints (Bullet Cluster, halo shapes) give $\sigma/M < \mathcal{O}(1)$ cm^2/g; for $\lambda \sim 1$ this constrains $E > \mathcal{O}(10\,\text{MeV})$ – a lower bound only.

The minimal falsifiable prediction: *[Conjecture].* Any theory with symmetry $\text{Aut}(\mathcal{A}_4)$ and ZD-active scalar matter has a quartic self-interaction with exactly one coupling constant λ, vertex symmetry $C_2 \times D_8$, on-shell amplitude coefficient 4, and amplitude ratios fixed by the representation theory of $C_2 \times D_8$, independently of E and of λ.

Remark 3.13 (SIDM observational constraints on λ and E/M, *[Speculation]*)**.** *Disclaimer.* These bounds assume $\mathcal{L}_{\text{int}} = \lambda \Delta N$ is the dominant self-interaction and that a QFT over $\mathcal{A}_4$ exists (Open Problem 4 of §4). Neither assumption is established.

Part 1 – Analytical constraint. From $|i\mathcal{M}_4|^2 = 16\lambda^2$ (Definition 1.5), the tree-level total cross-section for identical scalars in the CM frame is

$$\sigma_{\text{self}} = \frac{9\lambda^2}{\pi E^2} \qquad \text{(natural units, } E = \text{CM energy scale).}$$

The condition $\sigma/M < C$ (cm^2/g, M = dark matter mass) gives

$$\frac{E}{M} > \left(\frac{E}{M}\right)_{\text{min}} = \frac{2.237 \times 10^{-2}\,\lambda}{(M/\text{GeV})^{3/2}},$$

using $\hbar c = 197.3\,\text{MeV}\cdot\text{fm}$ and $1\,\text{GeV}/c^2 = 1.783 \times 10^{-24}$ g. For non-relativistic dark matter ($E/M \approx 1$), this becomes a lower bound on M:

$$M > M_{\text{min}}(\lambda) = (2.237 \times 10^{-2}\,\lambda)^{2/3}\ \text{GeV}.$$

Part 2 – Numerical table (Bullet Cluster: $\sigma/M < 1.25$ cm^2/g).

λ	$(E/M)_{\text{min}}$, $M = 1\,\text{GeV}$	$(E/M)_{\text{min}}$, $M = 100\,\text{MeV}$	M_{min} (NR)
0.1	2.24×10^{-3}	7.07×10^{-2}	17 MeV
0.5	1.12×10^{-2}	3.54×10^{-1}	50 MeV
1.0	2.24×10^{-2}	7.07×10^{-1}	79 MeV
4.0	8.95×10^{-2}	> 1 (excluded NR)	200 MeV

"Excluded NR": requires $E/M > 1$, incompatible with non-relativistic dark matter. M_{min}: minimum mass compatible with the Bullet Cluster bound for NR dark matter. For the SIDM-preferred range $0.1 \le \sigma/M \le 1\,\text{cm}^2/\text{g}$ (Peter et al. 2013), the mass window for NR dark matter is: $\lambda = 1$: $86\,\text{MeV} < M < 184\,\text{MeV}$; $\lambda = 4$: $216\,\text{MeV} < M < 464\,\text{MeV}$.

Part 3 – Angular distribution. Corollary 3.10 establishes $d\sigma/d\Omega$ = const for all scattering angles. For an isotropic distribution, the momentum-transfer cross-section satisfies $\sigma_T = \sigma_{\text{tot}}$ exactly (no angular-weighting correction needed). Standard WIMPs with t- and u-channel Yukawa mediators have forward-peaked distributions ($d\sigma/d\Omega \sim \sin^{-4}(\theta/2)$), requiring $\sigma_T \ll \sigma_{\text{tot}}$; the contact interaction of $\mathcal{L}_{\text{int}}$ has no such distinction, making the comparison with SIDM observational data uniquely direct.

3.5. The VEV expansion and absence of mass.

Theorem 3.14 (Decoupling of the VEV, *[Derived]*)**.** $\Delta N(v \cdot e_0, \psi) = 0$ *identically for all* $\psi \in \mathcal{A}_4$ *and all* $v \in \mathbb{R}$.

Proof. Since $e_0 = 1$ (the identity): $N(v \cdot e_0) \cdot N(\psi) = v^2 N(\psi)$ and $N(v \cdot e_0 \cdot \psi) = N(v\psi) = v^2 N(\psi)$. Hence $\Delta N(v \cdot e_0, \psi) = 0$. □

Theorem 3.15 (Absence of mass term, *[Derived]*)**.** *With* $\varphi = v \cdot e_0 + h$ *and* $\psi = w \cdot e_0 + \chi$ *(h, χ index-pure):*

$$\mathcal{L}_{\text{int}} = \lambda \cdot [\Delta N(h, \chi) - 2w\langle h, h\chi\rangle - 2v\langle \chi, h\chi\rangle].$$

There is no quadratic term in (h, χ)*: no* h^2*, no* χ^2*, no* $h \cdot \chi$ *mixing.*

Corollary 3.16 (ZD vertex is VEV-invariant, *[Derived]*)**.** *For ZD-active fields* h, χ *with* $h\chi = 0$*: the cubic corrections also vanish, and* $\mathcal{L}_{\text{int}} = \lambda \cdot \Delta N(h, \chi) = 4\lambda$*, exact, with no VEV correction. The amplitude* -4λ *on ZD-active states is VEV-independent.*

Remark 3.17 (Algebraic masslessness, *[Conjecture]*)**.** ZD-active modes h, χ are algebraically massless: their mass cannot be generated by any VEV in the e_0 direction through $\mathcal{L}_{\text{int}}$. A mass would require a separate quadratic term $-\mu^2 N(\varphi)$ not derivable from the norm-gap interaction alone. Under (A1), if no such mechanism exists in a complete QFT over $\mathcal{A}_4$, the $\mathcal{A}_4 \setminus \mathcal{A}_3$ sector behaves as *dark radiation*, not cold dark matter.

3.6. The tower structure: $\mathcal{A}_4$ is uniquely selected.

n	$\dim(\mathcal{A}_n)$	$\lvert \text{ZD}_{\text{unord}} \rvert$	$\lvert \text{Aut}(\mathcal{A}_n) \rvert$	Orbits	$\lvert$Stab$\rvert$	Reg. rep
4	16	168	2688	1	16	✓
5	32	2520	5376	15	32	✓

Observation 3.18 (Tower ratio, **VERIFIED** $n = 4, 5$; *[Conjecture]* $n \geq 6$)**.** $\lvert \text{Aut}(\mathcal{A}_4) \rvert = 2^4 \times 168$ *[`GAP`]*, $\lvert \text{Aut}(\mathcal{A}_5) \rvert = 2^5 \times 168$ *[GPU computation]. The pattern* $\lvert \text{Aut}(\mathcal{A}_n) \rvert = 2^n \times 168$ *for* $n \geq 6$ *is* [Conjecture] *– not established by two data points alone.*

Theorem 3.19 (Orbit structure, *[Derived]* for $n = 4, 5$)**.** $n = 4$*:* 1 *orbit of size* 168 *(transitive).* $n = 5$*:* 15 *orbits of size* 168 *(*$\to$ 15 *independent couplings* $\lambda_1, \dots, \lambda_{15}$*). The* 15 *orbits are in bijection with* $\text{PG}(3, \mathbb{F}_2)$ *(the* 15 *non-zero elements of* $\mathbb{F}_2^4$*, indexing the imaginary basis of* $\mathcal{A}_4 \subset \mathcal{A}_5$*).*

Corollary 3.20 (Uniqueness of $\mathcal{A}_4$ in the tower, *[Derived]*)**.** $\mathcal{A}_4$ *is the* ***unique*** *level of the tower at which the ZD-active contact interaction has a single free coupling constant. For* $n \leq 3$*:* $\text{ZD}(n) = \emptyset$*, no contact term. For* $n = 4$*:* 1 *orbit* $\to \lambda$ *unique. For* $n = 5$*:* 15 *orbits* $\to$ 15 *independent couplings.*

4. Open Problems

(1) **The 2+2 splitting – RESOLVED** (Theorem 1.12): analytic proof from (S1),(S2). Closed.
(2) **Extension class – FULLY RESOLVED** (Remark 3.4): non-split confirmed by `GAP`. $H^2(C_2 \times \text{PSL}(3,2), C_2^3) \cong \mathbb{F}_2^2$; the extension class of $\text{Aut}(\mathcal{A}_4)$ is $(1,0) \in \mathbb{F}_2^2$ – the inflation of the unique non-zero element of $H^2(\text{PSL}(3,2), C_2^3) \cong \mathbb{F}_2$. `SmallGroup`$(1344, 814)$ = non-split; `SmallGroup`$(1344, 11686)$ = split = $\text{AGL}(3, \mathbb{F}_2)$. `[DERIVED, GAP 4.15.1]` *Sub-problem remaining*: write the 2-cocycle $c\colon \text{PSL}(3,2) \times \text{PSL}(3,2) \to C_2^3$ representing the non-zero class explicitly.
(3) **The logical qubit as physical subsystem** (Conjecture 2.10): formalise the qubit of $[[7,1,3]]$ as an $\text{Aut}(\mathcal{A}_4)$-invariant subsystem in a QFT over $\mathcal{A}_4$. Requires item 4.
(4) **QFT over $\mathcal{A}_4$**: construct states, observables, and $T_{\mu\nu}$ for a field theory with $\mathcal{A}_4$-valued fields. Prerequisite for all numerical predictions.

(5) **Amplitude ratios – RESOLVED** (Corollary 3.6 and Proposition 3.9): $C_2 \times D_8$ representation is the regular representation; angular distribution is isotropic. Closed.
(6) **Regular representation from first principles** (Remark after Proposition 3.7): prove analytically that $|\operatorname{Stab}(\text{ZD pair})| = \dim(\mathcal{A}_4) = 16$ implies the regular representation.
(7) **Measurement of Z_L and X_L on QPU – RESOLVED** *[Conjecture]| QPU measurement.* Z_L: `ibm_kingston`, job `d8ctl5q4gq0s73apdjv0`, 2026-05-29, 8192 shots each:

$$\langle Z_L\rangle_{|0_L\rangle} = +0.7258 \pm 0.0076\ (96\sigma), \qquad \langle Z_L\rangle_{|1_L\rangle} = -0.7197 \pm 0.0077\ (94\sigma).$$

Fidelity: 83.1% and 82.0%.
X_L: `ibm_kingston`, job `d8cvisnd0j8c73f3frl0`, 2026-05-29, 8192 shots each:

$$\langle X_L\rangle_{|-_L\rangle} = -0.7825 \pm 0.0069 \quad (114\sigma;\ \text{ideal } -1), \quad \text{fidelity } 85.9\%$$
$$\langle X_L\rangle_{|+_L\rangle} = +0.7783 \pm 0.0069 \quad (112\sigma;\ \text{ideal } +1), \quad \text{fidelity } 85.5\%$$

($|+_L\rangle$ circuit: corrected transpilation, job `d8cvp2nd0j8c73f3g96g`; first attempt noise-dominated due to `Qiskit` barrier blocking $HH = I$ cancellation.)
Both logical operators distinguish their respective eigenstates at $> 94\sigma$. The identification with the $\mathcal{A}_3 \to \mathcal{A}_4$ invariant remains *[Conjecture]. All experimental sub-problems closed.*
(8) **Orbit count formula for the tower**: verify whether the number of $\operatorname{Aut}(\mathcal{A}_n)$-orbits on ZD pairs follows a pattern indexed by projective geometry, using $n = 6$ computation.
(9) **Tower ratio formula**: prove $|\operatorname{Aut}(\mathcal{A}_n)| = 2^n \times 168$ for all $n \geq 4$ from the recursive Cayley–Dickson structure.
(10) $\operatorname{Aut}(\mathcal{A}_n)$ **vs** $\operatorname{Aut}_{\text{struct}}(\text{ZD}(n))$ **– PARTIALLY RESOLVED**: for $n = 4$ the two groups are non-isomorphic (orders differ by ~ 20 orders of magnitude; see Remark 3.4). All results in this paper use $\operatorname{Aut}(\mathcal{A}_4)$ as algebra. Open for $n \geq 5$.

5. Conclusions

5.1. **Summary of derived results. The interaction tensor** $T[a,b,c,d] = \delta_{ac}\delta_{bd} - \varepsilon(a,b)\varepsilon(c,d)\delta_{a\oplus b,\,c\oplus d}$ determines a quartic contact Lagrangian $\mathcal{L}_{\text{int}} = \lambda\Delta N(\varphi,\psi)$ with a single free coupling λ. The coupling is unique by transitivity of $\operatorname{Aut}(\mathcal{A}_4)$ on the 168 unordered ZD pairs (`GAP`-certified). The norm gap $\Delta N = 4$ on ZD-active states is an exact algebraic identity.

The $2+2$ **splitting** $\Delta N = F_{m_1} + F_{m_2} = 2 + 2$ is proved analytically: the ZD condition $ab = 0$ is equivalent to two sign constraints (S1),(S2), each annihilating the interference term in one block.

The vertex symmetry is $C_2 \times D_8$. It acts on $\mathcal{A}_4$ via the regular representation ($\chi(e) = 16$, $\chi(g \neq e) = 0$). The angular distribution of ZD-active self-scattering is isotropic for any λ and E.

The VEV decouples: no mass is generated for ZD-active fluctuations by any VEV in the e_0 direction. The vertex -4λ is VEV-independent.

The tower structure: $\mathcal{A}_4$ is the unique level of the Cayley–Dickson tower with a single ZD-active coupling constant. At $n = 5$, 15 independent couplings appear. Verified for $n = 4$ (`GAP`) and $n = 5$ (GPU computation).

The CSS invariant: the $[[7,1,3]]$ Steane logical qubit has dimension 2, is $\mathrm{GL}(3,\mathbb{F}_2)$-invariant, and is the unique subspace of the 7-qubit Hilbert space commuting with all stabilisers. X- and Z-stabilisers share the same lo-flat supports by the $\mathrm{CSS}(C,C)$ self-dual construction *[Derived]*; the physical assignment $X \leftrightarrow \mathcal{A}_3$, $Z \leftrightarrow \mathcal{A}_4$ is *[Conjecture]* (Observation 2.6(b)).

The automorphism groups: $\operatorname{Aut}(\mathcal{A}_4) \cong C_2 \times ((C_2^3) \,.\, \mathrm{PSL}(3,2))$ – non-split, `GAP`-confirmed. Papers [1–8] use $\rtimes$; the correct notation is ".". No result using only group order or action on ZD pairs is affected. $\operatorname{Aut}_{\text{struct}}(\text{ZD}(4)) = S_7 \wr G_{\text{class}}$ of order $\approx 7.9 \times 10^{23}$ is an entirely different group [4].

5.2. **Summary of physical conjectures.** Under assumptions (A1) and (A2):
(P3) Direct-detection experiments yield null results – algebraically, not because the coupling is small, but because the electroweak coupling does not exist for $\mathcal{A}_4 \setminus \mathcal{A}_3$ fields.

(P4) The gravitational coupling of $\mathcal{A}_4 \setminus \mathcal{A}_3$ fields is exactly universal; any equivalence principle violation falsifies IIB-III.
(P5) The dark-sector self-interaction has vertex symmetry $C_2 \times D_8$, a single coupling λ, amplitude coefficient 4, and isotropic angular distribution. These are E-independent and λ-independent structural predictions.
New: ZD-active fluctuations are algebraically massless. If no external mass mechanism exists, the $\mathcal{A}_4 \setminus \mathcal{A}_3$ sector behaves as dark radiation, not cold dark matter.

5.3. **The gap between algebra and observation.** The logical chain required to make the predictions of this paper numerically testable:

(1) Construct a QFT over $\mathcal{A}_4$ (Open Problem 4 – not yet done).
(2) Verify $\mathcal{L}_{\text{int}}$ appears in this QFT (conditional on Step 1).
(3) Fix λ and E from renormalisation (conditional on Step 2).
(4) Derive testable cross-sections and compare to observation.

Until Step 1 is complete, This paper establishes the *necessary algebraic form* of the interaction and its symmetry structure – a prerequisite for any future QFT construction, but not sufficient alone.

Appendix A. Epistemic map

Result	Label	Depends on
$T[a,b,c,d]$ closed form	[DERIVED]	[1,6]
Trichotomy $\Delta N \in \{+4,0,-4\}$	[DERIVED]	[6]
Fano labelling $m = p \oplus k$	[DERIVED]	[1,4]
$2+2$ splitting – analytic	[DERIVED]	Thm. 1.12
CSS Bridge (Thm. 2.2)	[DERIVED]	[8]
Qubit: dim $= 2$, $\mathrm{GL}(3,\mathbb{F}_2)$-inv	[DERIVED]	[8] + QEC
$\mathrm{Aut}(\mathcal{A}_4)$ transitive on 168 pairs	[DERIVED]	`GAP` 4.15.1
Stab $\cong C_2 \times D_8$	[DERIVED]	`GAP` 4.15.1
$\mathrm{Aut}(\mathcal{A}_4)$ non-split; $H^2 \cong \mathbb{F}_2^2$; class $(1,0)$	[DERIVED]	`GAP` 4.15.1
$\mathrm{Aut}(\mathcal{A}_4) \neq \mathrm{Aut}_{\mathrm{struct}}(\mathrm{ZD}(4))$	[DERIVED]	[4, Thm. 7.4]
Z_L distinguishes $\lvert 0_L\rangle$, $\lvert 1_L\rangle$	[CONJECTURE — QPU]	`ibm_kingston`, `d8ctl5q4`, 2026-05-29
X_L on $\lvert -_L\rangle$: -0.7825 ± 0.0069 (114σ)	[CONJECTURE — QPU]	`ibm_kingston`, `d8cvisnd`, 2026-05-29
X_L on $\lvert +_L\rangle$: $+0.7783 \pm 0.0069$ (112σ)	[CONJECTURE — QPU]	`ibm_kingston`, `d8cvp2nd`, 2026-05-29
λ unique	[DERIVED]	transitivity
Regular rep of $C_2 \times D_8$ on $\mathcal{A}_4$	[DERIVED]	Prop. 3.7
Isotropic $d\sigma/d\Omega$	[DERIVED]	Cor. 3.10
51 amplitude channel orbits	[DERIVED]	Prop. 3.9
$X \leftrightarrow \mathcal{A}_3$, $Z \leftrightarrow \mathcal{A}_4$ id.	[CONJECTURE]	(A1),(A2)
Qubit as threshold invariant	[CONJECTURE]	(A1),(A2)
$2+2 \leftrightarrow X/Z$ parallel	[CONJECTURE]	open problem
$\mathcal{L}_{\mathrm{int}}$ for $\mathcal{A}_4 \setminus \mathcal{A}_3$ dark matter	[CONJECTURE]	(A1),(A2)+IIB-III
P3: null direct detection	[CONJECTURE]	IIB-III
P4: equivalence principle	[CONJECTURE]	IIB-III
P5: $C_2 \times D_8$ vertex structure	[CONJECTURE]	(A1)+(A2)+Thm. 3.5
E, λ numerical values	*free parameters*	not fixed by algebra
SIDM: $E/M > (2.24 \times 10^{-2}\lambda)/M_{\mathrm{GeV}}^{3/2}$	[SPECULATION]	Rem. 3.13; OP 4
IIB-IV (CCC residue)	[SPECULATION]	(A1),(A2),(A3)
$n=5$: 15 orbits, 15 couplings	[DERIVED]	GPU computation
$\lvert \mathrm{Aut}(\mathcal{A}_n)\rvert = 2^n \times 168$ $(n = 4,5)$	[DERIVED]	`GAP`+GPU
$\mathcal{A}_4$ unique single coupling	[DERIVED]	Cor. 3.20

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